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Rough Hurst function estimation

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ABSTRACT

The fractional Brownian motion (fBm) is parameterized by the Hurst exponent $H \in (0,1)$, which determines the dependence structure and regularity of sample paths. Empirical findings suggest that the Hurst exponent may be non-constant in time, giving rise to the so-called multifractional Brownian motion (mBm). The Itô-mBm is an alternative to the classical mBm, and has been shown to admit more intuitive sample path properties in case the Hurst function is rough. In this paper, we show that the Itô-mBm also allows for a simplified statistical treatment compared to the classical mBm. In particular, estimation of the local Hurst exponent $H(t)$ with Hölder exponent $\eta > 0$ achieves rates of convergence which are standard in nonparametric regression, whereas similar results for the classical mBm only hold for the smoother regime $\eta > 1$. Furthermore, we derive an estimator of the integrated Hurst function $\int_0^t H(s)\,ds$ which achieves a parametric rate of convergence, and use it to construct goodness-of-fit tests.

1. Introduction

Fractional Brownian motion (fBm) is a centered Gaussian process B_t^H with covariance function $\mathrm{Cov}(X_s, X_t) = \frac{1}{2}(|t|^{2h} + |s|^{2H} - |t-s|^{2H})$, for a scale parameter $\sigma^2 > 0$ and a so-called Hurst exponent $H \in (0,1)$. It admits the Mandelbrot-van Ness representation

$$B_t^H = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \int_{-\infty}^t \sigma \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right] dB_s,$$

where B_s is a standard Brownian motion, and $(x)_+ = x \cdot \mathbb{1}_{x>0}$. The fBm and related processes have been widely applied in diverse fields, e.g. in modeling turbulent flows [1,2] or stochastic volatility in financial markets [3]. The Hurst exponent H is of major interest, as it governs the temporal dependence of the process, the regularity of sample paths, and the self-similarity.

In some applications, e.g. when analyzing stock market efficiency [4], sun-spot data [5], or network traffic [6], empirical evidence suggests that a single Hurst exponent H is not sufficient, and a temporally-varying Hurst function H_t should be considered. An inhomogeneous generalization of the fBm, the multifractional Brownian motion (mBm), has been introduced by [7] as

$$X_t = \int_{-\infty}^t \sigma \left[(t-s)_+^{H_t-\frac{1}{2}} - (-s)_+^{H_t-\frac{1}{2}} \right] dB_s. \tag{1}$$

See also [8] for related definitions of mBm. Recently, [9] suggested an alternative generalization of the fBm, given by

$$X_t = \int_{-\infty}^t \sigma_s \left[(t-s)_+^{H_s-\frac{1}{2}} - (-s)_+^{H_s-\frac{1}{2}} \right] dB_s. \tag{2}$$

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This process has been referred to as a Multifractional Process with Random Exponent (MPRE) [9,10], or as a Moving Average Multifractional Process with Random Exponent (MAMPRE) [11]. Following [12], we refer to (1) as the classical mBm, and to (2) as Itô-mBm, since the latter is well-defined by standard Itô calculus. Both processes (1) and (2) behave locally like a (homogeneous) fBm, i.e. $h^{-H_t}(X_{t+h} - X_t) \Rightarrow B_h^{H_t}$ as $h \to 0$. Thus, they are both valid candidate models for time-varying extensions of the fBm. Probabilistic and statistical analysis of the classical mBm is facilitated by the fact that its covariance function admits an explicit expression [8, Thm. 4.1]. On the other hand, the Itô-mBm is also well-defined as Itô integral if the Hurst function H_t is an (adapted) stochastic process rather than a deterministic function. Moreover, the latter process has been shown in [12] to have attractive analytical features, see also [10]. In particular, the local Hölder coefficient at time t is $\min(\eta, H_t)$ for the classical mBm [13, Prop. 13], and simply H_t for the Itô-mBm. The goal of this paper is to demonstrate that the Itô-mBm is not only preferable from an analytical point of view, but also allows for improved statistical inference compared to the classical mBm. In particular, we want to perform nonparametric statistical inference about the function $t \mapsto H_t$ for the Itô-mBm. Note that, although the latter process is also well-defined for random Hurst functions, we require H_t and σ_t to be deterministic throughout this article.

Estimators for fBm from low-frequency observations $X_1, \dots, X_n$ are reviewed and compared by [14]. Optimality of the MLE has been established by [15], and by [16] for the high-frequency regime $X_{1/n}, \dots, X_1$. Local nonparametric estimators for the classical mBm (1) have be introduced by [17,18], and [19]. More recently, [20] developed rate-optimal local nonparametric estimators of H_t exploiting higher-order smoothness. [21] estimate the global regularity $\min_t H_t$, and [22] describe a goodness-of-fit test for the Hurst function. Further works on estimation of the mBm include, among others, [23–25]. Estimation for some alternative extensions of fBm allowing for irregular H_t is studied by [26] and [27,28].

The limitation of all works on the classical mBm (1) is that they require $t \mapsto H_t$ to be Hölder continuous with exponent $\eta > \sup_t H_t$. This restriction is necessary to ensure a good local approximation of the process by a fBm, see e.g. [19, eqn. 7.14]. For the Itô-mBm, the only statistical treatment to date is due to [11], showing uniform consistency of a localized Hurst estimator. Their estimator is analogous to the estimator we propose below, and they but the rate derived therein is slower, indicating that their asymptotic analysis is suboptimal. Here, we study two estimators $\widehat{H}_n(u)$ and $\widehat{H}_n^\dagger(u)$ based on local polynomial regression, which achieve the rate $n^{-\eta/(2\eta+1)}$ for $\eta \le 1$ (exactly for $\widehat{H}_n^\dagger$, and up to logarithmic factors for $\widehat{H}_n$). This rate is standard in nonparametric estimation, and our results show that estimation of the Itô-mBm works *just as expected* – in stark contrast to estimation of the classical mBm, where estimators in the regime $\eta < 1$ admit non-standard rates of convergence [19]. In view of its elegant analytical and statistical properties, we propose the Itô-mBm as canonical inhomogeneous variant of the fBm.

Beyond the local estimation of H_t, we derive an estimator $\widehat{\mathcal{H}}(u)$ of the integrated parameter $\mathcal{H}(u) = \int_0^u H_v \, dv$, and the error $\widehat{\mathcal{H}} - \mathcal{H}$ converges to a Gaussian process at rate $\sqrt{n}$. We use this estimator to construct a changepoint test for constant Hurst exponent, and a goodness-of-fit test for the function $t \mapsto H_t$. An important feature of both tests is that they are robust to a non-constant volatility σ_t, which constitutes a nuisance under the null hypothesis. Our mathematical results are based on a functional central limit theorem for locally stationary time series established in [29], and the proof technique linking methods for stochastic processes in discrete and continuous time might be of independent interest.

The pointwise nonparametric estimator of the Hurst function is studied in Section 2. The integrated parameter estimator is introduced in Section 3, including the changepoint test and the goodness-of-fit test. All technical proofs are gathered in the Appendix.

2. Local nonparametric estimation

Performing high-frequency inference for the Itô-mBm (2), we want to estimate the exponent H_s nonparametrically, using discrete observations $X_{\frac{1}{n}}, \dots, X_{\frac{n}{n}}$. The volatility σ_s is considered a statistical nuisance. In the sequel, we consider H_s and σ_s as non-random functions. For simplicity of exposition, we suppose that $X_{-\frac{k}{n}}$ for $k = 1, 2, 3$ are also observable, and we define the second order increments as

$$\chi_{i,n} = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}, \qquad i = 1, \dots, n,$$
$$\widetilde{\chi}_{i,n} = X_{\frac{i}{n}} - 2X_{\frac{i-2}{n}} + X_{\frac{i-4}{n}}, \qquad i = 1, \dots, n.$$

Our estimation procedure is based on the change-of-frequency principle introduced by [30], which is a special case of the quadratic variation estimator of [31], and has also been used by [32]: If $H_s = H$ and $\sigma_s = \sigma$ are constant, the self-similarity of the fBm yields $\mathbb{E}(\widetilde{\chi}_{i,n}^2)/\mathbb{E}(\chi_{i,n}^2) = 2^{2H}$, such that the moment estimator given by $\widehat{H} = \frac{1}{2}\log_2\left(\sum_i \widetilde{\chi}_{i,n}^2 / \sum_i \chi_{i,n}^2\right)$ is consistent at rate $\sqrt{n}$. For estimation of the Itô-mBm, we use the same idea, but localize the moment estimator around time $u \in (0,1)$. To this end, we use a kernel function $K : \mathbb{R} \to [0, \infty)$ satisfying

$$\int K(x)\,dx = 1, \qquad K(x) = 0 \text{ for } |x| > 1, \tag{K}$$
$$\{x : K(x) > 0\} \text{ has positive Lebesgue measure.}$$

For a bandwidth $b = b_n > 0$, denote $K_b(x) = \frac{1}{b}K(\frac{x}{b})$, and for any $u \in (0,1)$, consider the locally weighted polynomial regression [33,34]

$$\min_q \sum_{i=1}^n K_{b_n}(\tfrac{i}{n} - u) \left\| \begin{pmatrix} \chi_{i,n}^2 \\ \widetilde{\chi}_{i,n}^2 \end{pmatrix} - q(\tfrac{i}{n} - u) \right\|^2,$$

where the minimum q^* is determined among all polynomials $q : \mathbb{R} \to \mathbb{R}^2$ of degree l. We use $\widehat{\phi}_n(u) = q^*(0)$ as moment estimator at location u. It is well known [34, Lem. 1.3 & Prop. 1.12] that the estimator admits the linear representation

$$\widehat{\phi}_n(u) = \begin{pmatrix} \widehat{\phi}_{n,1}(u) \\ \widehat{\phi}_{n,2}(u) \end{pmatrix} = \sum_{i=1}^n w_{i,n}(u) \begin{pmatrix} \chi^2_{i,n} \\ \widetilde{\chi}^2_{i,n} \end{pmatrix},$$

and the weights satisfy, for some universal $C > 0$, and for all $u \in [b_n, 1 - b_n]$,

(i) $\sup_i |w_{i,n}(u)| \le \frac{C}{nb_n}$,
(ii) $\sum_{i=1}^n |w_{i,n}(u)| \le C$,
(iii) $w_{i,n}(u) = 0$ for $|\frac{i}{n} - u| > b_n$,
(iv) $\sum_{i=1}^n w_{i,n}(u) = 1$ and $\sum_{i=1}^n (\frac{i}{n} - u)^k w_{i,n}(u) = 0$ for $k = 1, \dots, l$,
(v) $|w_{i+1,n}(u) - w_{i,n}(u)| \le \frac{C}{(nb_n)^2}$,
(vi) $nb_n \sum_{i=1}^n w_{i,n}(u)^2 \to w^* > 0$, defined in (B.2).

The latter two properties are derived in Proposition 9.

Remark 1. If K is supported on $[-1, 0]$ such that still $\{x \in [-1, 0] : K(x) > 0\}$ has positive Lebesgue measure, then the latter properties (i)–(iv) hold for all $u \in [b_n, 1]$. Similarly, if K is supported on $[0, 1]$, then (i)–(iv) hold for all $u \in [0, 1 - b_n]$. Moreover, if $K(x) > K_0 \mathbb{1}(|x| < \tau)$ for some small $\tau > 0$ and $K_0 > 0$, then properties (i)–(iv) hold for all $u \in [0, 1]$.

We estimate H_u via the log-ratio estimator

$$\widehat{H}_n(u) = \left[\left(\frac{1}{2} \log_2 \left(\frac{\widehat{\phi}_{n,2}(u)}{\widehat{\phi}_{n,1}(u)} \right) \right) \vee 0 \right] \wedge 1.$$

Theorem 1. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0, l + 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all v, and $0 < \underline{\sigma}^2 \le \sigma_v^2 \le \overline{\sigma}^2 < \infty$. If $b_n \ll n^{q-1}$ for some $q \in (0, 1)$, then for any $p \ge 2$,*

$$\widehat{H}_n(u) = H_u + \mathcal{O}_{L_p}\left(\log(n)^{\lceil \eta \rceil} b_n^\eta + \frac{1}{\sqrt{n\, b_n}} \right) + \mathcal{O}\big(\log(n) n^{-(1 \wedge \eta)}\big).$$

The implicit constants are uniform for $u \in [b_n, 1 - b_n]$. The upper bound is minimized for $b_n \asymp n^{-\frac{1}{2\eta+1}} \log(n)^{-\frac{2\lceil \eta \rceil}{2\eta+1}}$, so that

$$\widehat{H}_n(u) = H_u + \mathcal{O}_{L_p}\left(\log(n)^{\frac{\lceil \eta \rceil}{2\eta+1}} n^{-\frac{\eta}{2\eta+1}} \right).$$

If b_n is chosen slightly smaller such that

$$n^{-1} \ll b_n \ll \min\left\{ \frac{n^{2(\eta \wedge 1)} - 1}{\log(n)^2},\quad n^{-\frac{\eta}{2\eta+1}} \log(n)^{\frac{2\eta\lceil \eta \rceil}{2\eta+1}} \right\},$$

then

$$\sqrt{n\, b_n}\Big[\widehat{H}_n(u) - H(u) \Big] \quad \Rightarrow \quad \mathcal{N}\big(0, w^* \tau^2(H_u)\big),$$

where $\tau^2(H)$ is defined in (B.19), and w^* corresponds to the kernel via (vi).

Remark 2. Under the conditions described in Remark 1, the bounds of Theorem 1 and Proposition 10 in the appendix can be extended to $u \in [b_n, 1]$, $u \in [0, 1 - b_n]$, and $u \in [0, 1]$, respectively.

For the locally constant variant, i.e. $l = 1$, this estimator is the same as the one considered by [17] for the classical mBm. Central to this approach is the scaling relation $\mathbb{E}(\widetilde{\chi}^2_{i,n})/\mathbb{E}(\chi^2_{i,n}) \approx 2^{2H_{i/n}}$, and the estimator $\widehat{H}(u)$ utilizes this by estimating the mean first, and then taking the ratio. An alternative approach pursued by [20] is to take the ratio first, and smooth second. This leads to the local polynomial estimator

$$\widehat{H}^\dagger_n(u) = \frac{1}{2} \sum_{i=1}^n w_{i,n}(u) \log_2 \left(\frac{\widetilde{\chi}^2_{i,n}}{\chi^2_{i,n}} \right).$$

Again, we may constrain the estimator manually to the interval $[0, 1]$. It turns out that this estimator leads to an improvement of the rate of convergence by a logarithmic factor. Compared to the results of [20] for the classical mBm, we show that this estimation approach, when applied to the Itô-mBm, also works for less smooth Hurst functions with Hölder exponent $\eta < 1$.

Theorem 2. *Under the conditions of Theorem 1, the estimator $\widehat{H}^\dagger_n$ satisfies*

$$\widehat{H}^\dagger_n(u) = H_u + \mathcal{O}\big(b_n^\eta\big) + \mathcal{O}_{L_2}\left(\frac{1}{\sqrt{n\, b_n}} \right) + \mathcal{O}\big(\log(n) n^{-(1 \wedge \eta)}\big).$$

The upper bound is minimized for $b_n \asymp n^{-\frac{1}{2\eta+1}}$, so that

$$\widehat{H}^\dagger_n(u) = H_u + \mathcal{O}_{L_2}\Big(n^{-\frac{\eta}{2\eta+1}} \Big).$$

Remark 3. The estimator of [20] achieves the rate $\mathcal{O}((n\log(n)^2)^{-\frac{\eta}{2\eta+1}})$. The logarithmic advantage can be attributed to the fact that they suppose σ to be constant in time. It may thus be estimated at a faster rate, and is essentially known for the local estimation of H_u, which allows for slightly better estimation of the latter. The same phenomenon of logarithmically faster rates if σ is known has been shown for the homogeneous case of the fBm [16].

Remark 4. The slower rate of the estimator $\widehat{H}_u$ is due to a logarithmically larger bias of the estimator $\widehat{\phi}_n(u)$. In particular, Lemma 4 in the appendix shows that $\mathbb{E}\chi^2_{i,n} \approx n^{2H_{i/n}}\sigma^2_{i/n}\Gamma_{H_{i/n}}(0)$, and since H appears in the exponent, differentiating with respect to H to control the bias yields additional terms of order $\log(n)$.

3. Integrated parameter estimation

Another approach to treat the Hurst exponent nonparametrically is via the integrated parameter

$$\mathcal{H}(u) = \int_0^u H_v\, dv.$$

Interest in this functional arises because many hypotheses about H_u may be formulated in terms of $\mathcal{H}(u)$. For instance, the no-change hypothesis $H_v = H_0$ for all v is equivalent to $\mathcal{H}(u) = u\mathcal{H}(1)$; monotonicity of H_u is equivalent to convexity of $\mathcal{H}(u)$; and $H_v \geq \underline{H}$ is equivalent to $\mathcal{H}(u_2) - \mathcal{H}(u_1) \geq \underline{H}(u_2 - u_1)$ for all $u_2 \geq u_1$. We will discuss these examples in detail below.

A naive estimator for the integrated Hurst exponent $\mathcal{H}(u)$ can be obtained by integrating the local estimator, i.e. $\widetilde{\mathcal{H}}(u) = \int_0^u \widehat{H}_n(v)\, dv$. However, as discussed in [35], this estimator will in general be asymptotically bias-driven. Instead, we employ a generic linearization procedure which has been introduced in [35] in the context of locally-stationary time series. To this end, we propose the estimator

$$\widehat{\mathcal{H}}(u) = \frac{1}{n}\sum_{t=2L}^{\lfloor un\rfloor}\left\{\widehat{H}_n(\tfrac{t-L}{n}) + \frac{\widetilde{\chi}_{t,n}}{2\widehat{\phi}_{n,2}(\frac{t-L}{n})} - \frac{\chi_{t,n}}{2\widehat{\phi}_{n,1}(\frac{t-L}{n})}\right\},$$

where $\widehat{H}_n$ uses a kernel which is supported on $[-1,0]$. That is, $\widehat{H}_n(\frac{t}{n})$ only uses data up to time $\frac{t}{n}$.

Remark 5. If $t \leq \lceil nb_n\rceil$, the one-sided estimators $\hat{\phi}_n(\frac{t}{n})$ and $\widehat{H}_n(\frac{t}{n})$ effectively use the bandwidth $\frac{t}{n} \leq b_n$. This is due to the boundary adaptation of the local polynomial regression.

Theorem 3. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (\frac{1}{2}, 1]$, and that $0 < \underline{H} \leq H_v \leq \overline{H} < 1$ for all v, and $0 < \underline{\sigma}^2 \leq \sigma_v^2 \leq \overline{\sigma}^2 < \infty$. Suppose furthermore that b_n, L_n are chosen such that*

$$n^{-\frac{1}{2}+r} \ll b_n \ll n^{-\frac{1+2r}{4\eta}} \quad \text{for some } r \in (0, \tfrac{1}{2}),$$
$$n^{\frac{1}{2}-r}\log(n)^2 \ll L_n \ll n^{\frac{1}{2}-\delta} \quad \text{for some } \delta \in (0, \tfrac{1}{2}).$$

Then, as $n \to \infty$

$$\sqrt{n}(\widehat{\mathcal{H}}(u) - \mathcal{H}(u)) \quad \Rightarrow \quad \int_0^u \tau(H_v)\, dW_v \overset{d}{=} W\left(\int_0^u \tau^2(H_v)\, dv\right),$$

in the Skorokhod space $D[0,1]$. The local asymptotic variance $\tau^2(H)$ is defined in (B.19).

Remark 6. The local variance of the integrated estimator, $\tau^2(H_v)$, is identical to the asymptotic variance of the local estimator $\widehat{H}_n(u)$ derived in Theorem 1, except for the factor w^* due to kernel smoothing. This should not be surprising, as the integrated estimator is based on debiasing and summing the local estimator.

Remark 7. The condition on L_n is rather weak, and any choice $L_n \asymp n^a$ for some $a \in (\frac{1}{6}, \frac{1}{2})$ will satisfy the conditions. The choice of b_n basically needs to ensure that the local estimator $\widehat{\phi}_n(u)$ and hence $\widehat{H}_n(u)$, admits a sufficiently fast rate of convergence of order $\mathcal{O}(n^{-\frac{1}{4}-\frac{r}{2}})$. The feasible range of choices for b_n allows for this rate to be driven by bias or variance, and may always be satisfied for $\eta > \frac{1}{2}$.

Remark 8. Throughout our derivations, we require H_v to be a non-random function. In contrast, [11] analyze a very similar estimator for an adapted process H_v, showing uniform consistency. As seen from Theorem 3, the limit distribution depends on the unknown Hurst function. If the latter is random, one would expect a Gaussian variance mixture as limit distribution, as commonly observed in high-frequency econometrics [36]. Thus, we conjecture that the limit distribution of $\widehat{\mathcal{H}}(u)$ could be derived in the framework of stable convergence in distribution, see Podolskij and Vetter [37], using the classical result of Jacod [38].

Denote the limiting variance process by $\Sigma(u) = \int_0^u \tau^2(H_v)\, dv$. To perform statistical inference, we may estimate this process via the plug-in method as $\widehat{\Sigma}(u) = \frac{1}{n}\sum_{t=2L_n}^{\lfloor un\rfloor} \tau^2(\widehat{H}_n(\frac{t}{n}))$.

Theorem 4. *Under the conditions of Theorem 3,* $\sup_{u\in[0,1]}\left|\Sigma(u) - \widehat{\Sigma}(u)\right| \xrightarrow{\mathbb{P}} 0.$

That is, the limiting process $W(\Sigma(u))$ of Theorem 3 is approximated by $W(\widehat{\Sigma}(u))$, which allows for feasible statistical inference.

It is also possible to extend our asymptotic result to allow for sequences of Hurst functions $H_{u,n} = H_u + r_n\Delta_u$ for a bounded Hölder smooth deviation Δ_u, and a rate sequence $r_n \to 0$. Of particular interest is $r_n = 1/\sqrt{n}$, for which we obtain the following result.

Corollary 1. *Let the conditions of Theorem 3 hold, and suppose that $v \mapsto \Delta_u$ is bounded and Hölder continuous with exponent $\eta \in (\frac{1}{2}, 1]$. Suppose that $X_t = X_{t,n}$ is an Itô-mBm with parameter $v \mapsto \theta_{v,n} = (\sigma_v, H_v + \Delta_v/\sqrt{n})$. Then, as $n \to \infty$*

$$\sqrt{n}(\widehat{\mathcal{H}}(u) - \mathcal{H}(u)) \quad \Rightarrow \quad \int_0^u \Delta_v \, dv + \int_0^u \tau(H_v) \, dW_v,$$

in the Skorokhod space $D[0,1]$. Moreover, the claim of Theorem 4 remains valid.

The proof is analogous to Theorem 3 and sketched in the Appendix.

In the next two subsections, we proceed to describe two specific hypothesis tests based on the estimator for the integrated Hurst exponent.

3.1. Testing for constant Hurst function

The estimator for the integrated Hurst function can be used to test for a constancy, that is, to treat the statistical problem

$$\mathbb{H}_0 : H_v \text{ constant} \quad \leftrightarrow \quad \mathbb{H}_1 : H_v \text{ not constant.}$$

Rejecting $\mathbb{H}_0$ is interpreted as evidence that a model based on fBm is not sufficient, and multifractional extensions need to be considered. This problem has been studied by [5]. Therein, the specific multifractional model, i.e. Itô-mBm (2) vs classical mBm (1), does not matter, because the Hurst function is constant under the null. [5] employs a CUSUM statistic based on the squared increments $\chi^2_{t,n}$. The test is applied to sunspot data, finding evidence for an inhomogeneous Hurst exponent. A methodological limitation of the referenced method is that it will detect both changes in σ_v and in H_v, and a post-hoc analysis is necessary to distinguish both types of changes. In contrast, we would like to design a test which is only sensitive to changes in H_v, while being robust against changes in σ_v. That is, we treat the volatility as a nuisance. The relevance of allowing for time-varying nuisance quantities under the no-change null hypothesis was first recognized by [39], and further developed by Dette et al. [40], Górecki et al. [41], Pesta and Wendler [42], Demetrescu and Wied [43], Schmidt et al. [44], Cui et al. [45]. To the best of our knowledge, changepoint testing with time-varying nuisance quantities has not yet been considered for continuous-time processes.

To test for a change, we suggest the CUSUM-type statistic

$$T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) = \sup_{u \in [0,1]} \left| \widehat{\mathcal{H}}(u) - u\widehat{\mathcal{H}}(1) \right|.$$

Under $\mathbb{H}_0$, and if the volatility function satisfies the conditions of Theorem 3, the statistic $\sqrt{n} T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}})$ will converge in distribution to $\sup_{u \in [0,1]} |W(\Sigma(u)) - uW(\Sigma(1))|$. This can be used to derive critical values.

Proposition 2. *Suppose that the conditions of Theorem 3 hold. Let W be a Brownian motion independent of the data X_T and let $q_n(\alpha)$ be the $1-\alpha$, quantile of the random variable $Y_n = \sup_{u \in [0,1]} \left| W\left(\widehat{\Sigma}(u)\right) - uW(\widehat{\Sigma}(1)) \right|$, conditionally on X. If $v \mapsto H_v$ is constant, then*

$$\limsup_{n \to \infty} P\left(\sqrt{n} \, T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha) \right) \leq \alpha.$$

Alternatively, if $v \mapsto H_v$ is not constant, then

$$\lim_{n \to \infty} P\left(\sqrt{n} \, T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha) \right) = 1.$$

Lastly, consider the smooth local alternative $H_{v,n} = H + c\Delta_v/\sqrt{n}$ for some bounded function $v \mapsto \Delta_v$ which is Hölder continuous with exponent $\eta \in (\frac{1}{2}, 1]$. If $v \mapsto \Delta_v$ is not constant, then

$$\begin{aligned} &\lim_{n \to \infty} P\left(\sqrt{n} \, T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha) \right) \\ &= P\left(Y_c := \sup_u \left| W^*(u) - uW^*(1) + c \int_0^u \Delta_v \, dv - u \int_0^1 \Delta_v \, dv \right| > q(\alpha) \right), \end{aligned}$$

where $q(\alpha)$ is the $1-\alpha$ quantile of $\sup_{u \in [0,1]} |W(\Sigma(u)) - uW(\Sigma(1))|$. In particular,

$$\lim_{c \to \infty} \lim_{n \to \infty} P\left(\sqrt{n} \, T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha) \right) = 1.$$

3.2. Application to goodness-of-fit testing

Let $\mathcal{G}_0 \subset \{H : [0,1] \to (0,1)\}$ be a class of candidate functions for the Hurst exponent, and we want to test the null hypothesis

$$\mathbb{H}_0 : H \in \mathcal{G}_0 \quad \leftrightarrow \quad \mathbb{H}_1 : H \notin \mathcal{G}_0.$$

For example, setting $\mathcal{G}_0 = \{H_v = av + b \mid b \in (0,1), a + b \in (0,1)\}$ yields a test for linearity, and setting $\mathcal{G}_0 = \{H \mid \exists v_0 \text{ s.t. } H \text{ is increasing on } [0,v] \text{ and decreasing on } [v,1]\}$ tests for unimodality. We suggest to apply the test statistic

$$\widehat{T}(\mathcal{G}_0) = \inf_{\tilde{H} \in \mathcal{G}_0} \widehat{T}(\tilde{H}), \quad \text{where } \widehat{T}(\tilde{H}) = \sup_{u \in [0,1]} \left| \widehat{\mathcal{H}}(u) - \int_0^u \tilde{H}(v) \, dv \right|.$$

Under $\mathbb{H}_0$, we clearly have $\sqrt{n}\widehat{T}(\mathcal{G}_0) \leq \sqrt{n}\widehat{T}(H) \Rightarrow \sup_{u\in[0,1]} \left|\int_0^u \tau(H_v)\, dW_v\right|$, and we can use the quantiles of the latter as critical values. Importantly, we can estimate the asymptotic variance function as in Theorem 4, which is also consistent under the alternative. As a result, we obtain a consistent test.

Proposition 3. *Suppose that the conditions of Theorem 3 hold. Let W be a Brownian motion independent of the data X_T and let $q_n(\alpha)$ be the $1-\alpha$, quantile of the random variable $Z_n = \sup_{u\in[0,1]} |W(\widehat{\Sigma}(u))|$ conditionally on X. If $H \in \mathcal{G}_0$, then*

$$\limsup_{n\to\infty} P\Big(\sqrt{n}\,\widehat{T}(\mathcal{G}_0) > q_n(\alpha)\Big) \leq \alpha.$$

Alternatively, if $H \notin \mathcal{G}_0$ and $\mathcal{G}_0$ is closed with respect to the uniform norm, then

$$\lim_{n\to\infty} P\Big(\sqrt{n}\,\widehat{T}(\mathcal{G}_0) > q_n(\alpha)\Big) = 1.$$

An alternative goodness-of-fit test for the classical mBm and a singleton null $\mathcal{G}_0 = \{\check{H}\}$ has been suggested by [22], based on a Cramer-von-Mises type test statistic of the form $\sum_{l=1}^{L} |\widehat{H}_n(\frac{l}{l}) - \check{H}(\frac{l}{L})|^2$ which they show to be asymptotically normal if $L = L_n \to \infty$ suitably as $n \to \infty$. However, they do not specify the constraints on L_n, making their test practically infeasible. Moreover, they suppose σ to be constant.

4. Discussion

We constructed two local nonparametric estimators for the Hurst function of the Itô-mBm which achieve standard nonparametric rates of convergence even in the smoothness regime $\eta < 1$. In contrast, existing estimators for the classical mBm require $\eta > 1$. This suggests that the statistical treatment of the Itô-mBm is in some sense simpler, as the smoothness of the Hurst function is less critical, which is in analogy to previous findings on its path regularity. However, it is currently not clear whether the difficulties of estimating the classical mBm with a rougher Hurst function, i.e. $\eta < 1$, are intrinsic to the problem, or could be overcome by improved estimation procedures. In particular, the optimality result [20, Thm. 3.1] only covers the case $\eta > 1$, while optimality for the classical mBm in the rough regime $\eta < 1$ remains open.

Although we argue that the Itô-mBm is superior to the classical mBm for statistical modeling, a big advantage of the latter is its explicit covariance function, which enables exact likelihood inference and efficient simulation. Indeed, we attempted to simulate the Itô-mBm by discretizing the stochastic integral, but could not reach satisfactory accuracy for a statistical simulation study.

Lastly, we advocate for the use of Itô-mBm over the classical mBm on the basis of its attractive analytical and statistical properties. However, this is a purely mathematical argument and not based on empirical evidence. Obviously, the looming open questions are: (i) Can we distinguish the two models based on data? (ii) Do the two models lead to different conclusions for practical questions, e.g. forecasting or asset pricing? We formulate these open questions as promising directions for future research, especially in view of the recent success of fractional models for stochastic volatility [3,46].

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Appendix A. Multiplier invariance principle

For the proof of Theorem 3, we make use of a functional central limit theorem for nonstationary time series, developed in prior work [29]. To make this article self-contained, we repeat the essential assumptions and the result in this section.

The main result of this section is a functional central limit theorem for the partial sum process $\frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} \hat{g}_{t,n}[X_{t,n} - \mathbb{E}(X_{t,n})]$, where $X_{t,n}$ is an array of potentially nonlinear and nonstationary time series, and $\hat{g}_{t,n}$ are data-dependent multipliers. This invariance principle will be used to proof Theorem 3, where $X_{t,n}$ corresponds to the random vectors $(\widetilde{\chi}_{t,n}, \chi_{t,n})$, and the multipliers $\hat{g}_{t,n}$ correspond to the weights $1/\widehat{\phi}_{n,1}$ and $1/\widehat{\phi}_{n,2}$. In the sequel, we present the general framework for nonstationary time series, and specialize it to our case in the course of proving Theorem 3.

For iid random seeds $\epsilon_i \sim U(0,1)$, and functions $G_{t,n} : \mathbb{R}^\infty \to \mathbb{R}^d$, $t = 1, \dots, n$, define the nonstationary array of time series $X_{t,n}$ as

$$X_{t,n} = G_{t,n}(\boldsymbol{\epsilon}_t), \quad t = 1, \dots, n,$$
$$\boldsymbol{\epsilon}_t = (\epsilon_t, \epsilon_{t-1}, \dots) \in \mathbb{R}^\infty.$$

Throughout this section, we assume that $\mathbb{E}(X_{t,n}) = 0$. For an independent copy $\tilde{\epsilon}_i \sim U(0,1)$, define also

$$\tilde{\boldsymbol{\epsilon}}_{t,h} = (\epsilon_t, \dots, \epsilon_{t-h+1}, \tilde{\epsilon}_{t-h}, \epsilon_{t-h-1} \dots) \in \mathbb{R}^\infty,$$
$$\boldsymbol{\epsilon}_{t,h} = (\epsilon_t, \dots, \epsilon_{t-h+1}, \tilde{\epsilon}_{t-h}, \tilde{\epsilon}_{t-h-1} \dots) \in \mathbb{R}^\infty.$$

We first impose the following set of assumptions, for some $\Gamma_n \geq 1$, $q > 2$, and $\beta > 1$:

$$\|G_{1,n}(\boldsymbol{\epsilon}_0)\|_{L_2} + \sum_{t=2}^{n} \|G_{t,n}(\boldsymbol{\epsilon}_0) - G_{t-1,n}(\boldsymbol{\epsilon}_0)\|_{L_2} \leq \Theta_n \Gamma_n, \tag{A.1}$$

$$\max_{t=1,\dots,n} \|G_{t,n}(\epsilon_0) - G_{t,n}(\tilde{\epsilon}_{0,h})\|_{L_q} \le \Theta_n (h+1)^{-\beta}, \tag{A.2}$$

$$\int_0^1 \|G_{\lfloor vn \rfloor,n}(\epsilon_0) - G_v(\epsilon_0)\|_{L_2}\, dv \to 0. \tag{A.3}$$

Assumption (A.2) is a bound on the physical dependence measure, as introduced by Wu [47].
For non-random matrices $g_{t,n}, g_u \in \mathbb{R}^{m\times d}$, and random matrices $\hat{g}_{t,n} \in \mathbb{R}^{m\times d}$, define

$$\Lambda_n = \sqrt{\sum_{t=1}^n \|\hat{g}_{t,n} - g_{t,n}\|_{L_2}^2},$$

$$\Psi_n = \sum_{t=2}^n \|g_{t,n} - g_{t-1,n}\|,$$

$$\Phi_n = \max_{t=1,\dots,n} \|g_{t,n}\| + \sup_{u\in[0,1]} \|g_u\|,$$

and formulate the assumption

$$\int_0^1 \|g_{\lfloor vn \rfloor,n} - g_v\|\, dv \to 0. \tag{A.4}$$

Define the rate

$$\xi(q,\beta) = \begin{cases} \frac{q-2}{6q-4}, & \beta \ge \frac{3}{2},\ \beta > \frac{2q}{q+2}, \\ \frac{\beta-1}{4\beta-2}, & \beta \ge \frac{3}{2},\ \beta \le \frac{2q}{q+2}, \\ \frac{(\beta-1)(q-2)}{4q\beta-3q-2}, & \beta < \frac{3}{2},\ \beta > \frac{2q}{q+2}, \\ \frac{(\beta-1)^2}{2\beta^2-1-\beta}, & \beta < \frac{3}{2},\ \beta \le \frac{2q}{q+2}, \end{cases} \qquad q > 2,\ \beta > 1.$$

and the local long run covariance matrices

$$\Sigma_{t,n} = \sum_{h=-\infty}^{\infty} g_{t,n} \operatorname{Cov}[G_{t,n}(\epsilon_0), G_{t,n}(\epsilon_h)] g_{t,n}^T,$$

$$\Sigma_u = \sum_{h=-\infty}^{\infty} g_u \operatorname{Cov}[G_u(\epsilon_0), G_u(\epsilon_h)] g_u^T.$$

Theorem 5. *Suppose that $\hat{g}_{t,n}$ is ϵ_{t-L}-measurable for $L = L_n$, and let (A.1), (A.2), (A.3), (A.4) hold with $\Theta_n, \Phi_n = \mathcal{O}(1)$ such that*

$$n^{-\frac{1}{2}} \Lambda_n^2 + \Lambda_n L_n^{1-\beta} + n^{\frac{1}{q}-\frac{1}{2}} L_n \to 0$$

$$\left(\Gamma_n + \Psi_n\right)^{\frac{\beta-1}{2\beta}} \sqrt{\log(n)}\, n^{-\xi(q,\beta)} \to 0.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} \hat{g}_{t,n}[X_{t,n} - \mathbb{E}(X_{t,n})] \Rightarrow \int_0^u \Sigma_v^{\frac{1}{2}}\, dW_v.$$

Appendix B. Proofs

B.1. Preliminaries

For $H \in (0,1)$, define

$$\gamma_H(h) = \frac{b_H^2}{2}\left[|h+1|^{2H} - 2|h|^{2H} + |h-1|^{2H}\right], h \in \mathbb{Z},$$

$$\Gamma_H(h) = -\gamma_H(h+1) + 2\gamma_H(h) - \gamma_H(h-1),$$

$$b_H^2 = \frac{\Gamma(H+\frac{1}{2})^2}{2H \sin(\pi H)\Gamma(2H)}.$$

Due to the functional properties of the Gamma function Γ, there exist various formulations of the normalization factor b_H^2 in the literature, see [48, Thm. 1.3.1] for this factor. Note that, up to the scaling factor b_H, $\gamma_H(h)$ is the autocovariance function of the increments of a fractional Brownian motion with Hurst exponent H, and $\Gamma_H(h)$ is the autocovariance of the corresponding second order increments. In particular, if $(\sigma_s, H_s) \equiv (\sigma, H)$, then $\operatorname{Cov}(\chi_{i,n}, \chi_{j,n}) = n^{-2H}\sigma^2\Gamma_H(i-j)$. Moreover, we note that $\Gamma_H(h) \asymp |h|^{2H-4}$ as $|h| \to \infty$.

We proceed to give quantitative bounds on the approximation error between the inhomogeneous process and its homogenous tangent process. The following Lemma is in analogy to [17, Lemma 1] for the classical mBm. The notable difference is that we do not require $\eta > \sup_v H_v$, which is an advantage of the Itô-mBm model.

Lemma 4. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0,1]$ on some open interval $I = (a,b] \subset (-\infty, 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all $v \in (-\infty, 1]$, and $\sigma_v^2 \le \overline{\sigma}^2 < \infty$. Then, uniformly for all $\frac{i}{n} \in I$, it holds that*

$$n^{2H_{i/n}} \mathbb{E}\chi_{i,n}^2 = \sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}(\log(n) n^{-\eta}).$$

Proof of Lemma 4. Denote $t_i = \frac{i}{n}$, and $\alpha_s = H_s - \frac{1}{2}$. Then, by Itô's isometry,

$$\begin{aligned}
\mathbb{E}\chi_{i,n}^2 &= \int_{-\infty}^{t_i} \sigma_s^2 \Big[(t_i - s)_+^{\alpha_s} - 2(t_{i-1} - s)_+^{\alpha_s} + (t_{i-2} - s)_+^{\alpha_s} \Big]^2 ds \\
&= n^{-1} \int_{-\infty}^{n \cdot t_i} \sigma_{s/n}^2 \Big[\Big(t_i - \tfrac{s}{n}\Big)_+^{\alpha_{s/n}} - 2\Big(t_{i-1} - \tfrac{s}{n}\Big)_+^{\alpha_{s/n}} + \Big(t_{i-2} - \tfrac{s}{n}\Big)_+^{\alpha_{s/n}} \Big]^2 ds \\
&= \int_{-\infty}^{i} n^{-2H_{s/n}} \sigma_{s/n}^2 \Big[(i-s)_+^{\alpha_{s/n}} - 2(i-1-s)_+^{\alpha_{s/n}} + (i-2-s)_+^{\alpha_{s/n}} \Big]^2 ds \\
&= \int_0^\infty g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big) dv, \qquad \theta_v = \big(H_v, \sigma_v^2\big),
\end{aligned}$$

$$g(v,\theta) = n^{-2H} \sigma^2 \Big[v^{H-\frac{1}{2}} - 2(v-1)_+^{H-\frac{1}{2}} + (v-2)_+^{H-\frac{1}{2}} \Big]^2.$$

Now observe that, for some finite C,

$$\begin{aligned}
&|\partial_{\sigma^2} g(v,\theta)| \le C n^{-2H} \big[v^{2H-1} \wedge v^{2H-5} \big], \\
&|\partial_H g(v,\theta)| \le C n^{-2H} \big(16\sigma^2\big) \big[v^{2H-1} \wedge v^{2H-5} \big] \big[\log(n) + |\log(v)| \big], \\
\rightsquigarrow \quad &\|D_\theta g(v,\theta)\| \le C \log(n) n^{-2H} \big[v^{2H-5} \wedge v^{2H-1} \big] (|\log(v)| + 1)\big(1 + \sigma^2\big).
\end{aligned}$$

We now bound, for a small $q \in (0,1)$ to be chosen later,

$$\int_0^\infty |g\Big(v, \theta_{\frac{i}{n}}\Big) - g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big)| \, dv = \int_0^{n^{1-q}} |g\Big(v, \theta_{\frac{i}{n}}\Big) - g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big)| \, dv + \int_{n^{1-q}}^\infty |g\Big(v, \theta_{\frac{i}{n}}\Big) - g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big)| \, dv.$$

Using the local Hölder continuity of $v \mapsto \theta_v$, we find that

$$\begin{aligned}
&\int_0^{n^{1-q}} |g\Big(v, \theta_{\frac{i}{n}}\Big) - g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big)| \, dv \\
&\le \int_0^{n^{1-q}} \|\theta_{\frac{i}{n}} - \theta_{\frac{i}{n} - \frac{v}{n}}\| \cdot \sup_{u \in [0, n^{-q}]} \|D_\theta g(u, \theta_u)\| \, dv \\
&\le C \log(n) \max_{s \in [0, n^{-q}]} n^{-2H_{\frac{i}{n} - s}} \int_0^{n^{1-q}} \Big(\frac{v}{n}\Big)^\eta |v^{2\underline{H}-1} \wedge v^{2\overline{H}-5}| |\log(v)| \, dv \\
&\le C \log(n) n^{-2H} n^{-\eta},
\end{aligned}$$

because $n^{n^{-\eta q}} \to 1$ implies $n^{2(H_{\frac{i}{n}} - H_{\frac{i}{n} - s})} \to 1$ for $s \in [0, n^{-q}]$. Moreover,

$$\begin{aligned}
\int_{n^{1-q}}^\infty |g\Big(v, \theta_{\frac{i}{n}}\Big) - g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big)| \, dv &\le 2 \int_{n^{1-q}}^\infty \sup_u |g\big(v, \theta_u\big)| \, dv \\
&\le C \int_{n^{1-q}}^\infty \sup_{H \in [\underline{H}, \overline{H}]} n^{-2H} v^{2H-5} \, dv \\
&= C \int_{n^{1-q}}^n \Big(\frac{v}{n}\Big)^{2\underline{H}} v^{-5} \, dv + C \int_n^\infty \Big(\frac{v}{n}\Big)^{2\overline{H}} v^{-5} \, dv \\
&\le C\Big(n^{(1-q)(2\underline{H}-4) - 2\underline{H}} + n^{-4} \Big) \\
&\le C n^{-4+\epsilon},
\end{aligned}$$

for any $\epsilon > 0$, by making $q > 0$ sufficiently small. Since $\eta \in (0,1]$ and $H \in (0,1)$, the choice $\epsilon = \frac{1}{2}$ yields $n^{-4+\epsilon} \ll \log(n) n^{-2H-\eta}$.

Hence, we have shown that

$$\int_0^\infty g\Big(v, \theta_{\frac{i}{n} - \frac{v}{n}}\Big) dv = \int_0^\infty g\Big(v, \theta_{\frac{i}{n}}\Big) dv + \mathcal{O}\big(\log(n) n^{-2H-\eta}\big).$$

To complete the proof, we observe that $\int_0^\infty g(v,\theta) \, dv = n^{-2H} \sigma^2 \Gamma_H(0)$. □

Lemma 5. *Suppose that $v \mapsto \theta_v = (\sigma_v^2, H_v)$ is Hölder continuous with exponent $\eta \in (0,1]$ on some interval $I = (a,b] \subset (-\infty, 1]$, and that $0 < \underline{H} \le H_v \le \overline{H} < 1$ for all v, and $\sigma_v^2 \le \overline{\sigma}^2 < \infty$. Then uniformly for all $1 \le i \le j \le n$, such that $\frac{i}{n} \in I$,*

$$\begin{aligned}
\mathrm{Cov}\left(\chi_{i,n}, \chi_{j,n}\right) &= n^{-2H_{i/n}} \sigma_{i/n}^2 \Gamma_{H_{i/n}}(i-j) \\
&\quad + \mathcal{O}\Big(\log(n) n^{-2H_{i/n} - \left(\eta \wedge \frac{1}{2}\right)} (|i-j| \vee 1)^{H_{i/n} - \frac{5}{2}} \Big).
\end{aligned}$$

Proof of Lemma 5. Denote $t_i = \frac{i}{n}$, and $\alpha_s = H_s - \frac{1}{2}$. By Itô's isometry,

$$\begin{aligned}
\mathrm{Cov}(\chi_{i,n}, \chi_{j,n}) &= \int_{-\infty}^{\frac{i}{n}} \sigma_s^2 \Big[(t_i - s)^{\alpha_s} - 2\big(t_{i-1} - s\big)_+^{\alpha_s} + \big(t_{i-2} - s\big)^{\alpha_s}\Big] \\
&\qquad \cdot \Big[(t_j - s)^{\alpha_s} - 2\big(t_{j-1} - s\big)_+^{\alpha_s} + \big(t_{j-2} - s\big)^{\alpha_s}\Big]\, ds \\
&= \int_{-\infty}^{i} n^{-1} \sigma_{\frac{s}{n}}^2 \Big[\big(\tfrac{i}{n} - \tfrac{s}{n}\big)^{\alpha_{\frac{s}{n}}} - 2\Big(\tfrac{i-1}{n} - \tfrac{s}{n}\Big)_+^{\alpha_{\frac{s}{n}}} + \Big(\tfrac{i-2}{n} - \tfrac{s}{n}\Big)^{\alpha_{\frac{s}{n}}}\Big] \\
&\qquad \cdot \Big[\big(\tfrac{j}{n} - \tfrac{s}{n}\big)^{\alpha_{\frac{s}{n}}} - 2\Big(\tfrac{j-1}{n} - \tfrac{s}{n}\Big)_+^{\alpha_s} + \Big(\tfrac{j-2}{n} - \tfrac{s}{n}\Big)^{\alpha_{\frac{s}{n}}}\Big]\, ds \\
&= \int_0^{\infty} n^{-2H_{\frac{i-v}{n}}} \sigma_{\frac{i-v}{n}}^2 \Big[v^{\alpha_{\frac{i-v}{n}}} - 2(v-1)_+^{\alpha_{\frac{i-v}{n}}} + (v-2)^{\alpha_{\frac{i-v}{n}}}\Big] \\
&\qquad \cdot \Big[(v+\delta)^{\alpha_{\frac{i-v}{n}}} - 2(v+\delta-1)_+^{\alpha_{\frac{i-v}{n}}} + (v+\delta-2)^{\alpha_{\frac{i-v}{n}}}\Big]\, dv \\
&=: \int_0^{\infty} f\Big(v, \delta, \theta_{\frac{i-v}{n}}\Big)\, dv,
\end{aligned}$$

for $\delta = j - i$. We observe that, for $\delta \geq 1$,

$$\begin{aligned}
\|D_\theta f(v, \delta, \theta)\| &\leq C \log(n) n^{-2H} \Big[v^{H-\frac{1}{2}} \wedge v^{H-\frac{5}{2}}\Big] \cdot (v+\delta)^{H-\frac{5}{2}} (1 + |\log(v)|) \\
&\leq C \log(n) n^{-2H} (\delta \vee v)^{H-\frac{5}{2}} \Big[v^{H-\frac{1}{2}} \wedge v^{H-\frac{5}{2}}\Big] =: C \log(n) \overline{f}(v, \delta, H).
\end{aligned}$$

Using the η-Hölder continuity of $v \mapsto \theta_v$, with constant c, say, we find that

$$\begin{aligned}
&\Big|\mathrm{Cov}\left(\chi_{i,n}, \chi_{j,n}\right) - \int_0^{\infty} f\Big(v, \delta, \theta_{\frac{i}{n}}\Big)\, dv\Big| \\
&\leq C \log(n) \int_0^{\infty} \Big[\Big(\tfrac{v}{n}\Big)^{\eta} \wedge 1\Big] \sup_{\substack{H \in \left[\underline{H}, \overline{H}\right] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv.
\end{aligned} \tag{B.1}$$

We split the domain of integration into the three segments $[0, 1]$, $[1, n^{1-q}]$, and $[n^{1-q}, \infty)$, for some small $q \in (0, 1)$ to be specified later. The first portion of the integral may be bounded as

$$\begin{aligned}
&\int_0^1 \Big[\Big(\tfrac{v}{n}\Big)^{\eta} \wedge 1\Big] \sup_{\substack{H \in \left[\underline{H}, \overline{H}\right] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv \\
&\leq n^{-\eta - 2H_{i/n} + cn^{-\eta}} \int_0^1 v^{\underline{H} + \eta - \frac{1}{2}} \delta^{H_{i/n} + cn^{-\eta} - \frac{5}{2}} (1 + |\log(v)|)\, dv \\
&\leq C n^{-\eta - 2H_{i/n}} \delta^{H_{i/n} + cn^{-\eta} - \frac{5}{2}}.
\end{aligned}$$

The second portion of the integral may be bounded as

$$\begin{aligned}
&\int_1^{n^{1-q}} \Big[\Big(\tfrac{v}{n}\Big)^{\eta} \wedge 1\Big] \sup_{\substack{H \in [\underline{H}, \overline{H}] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv \\
&\leq n^{-\eta - 2H_{i/n} + cn^{-q\eta}} \int_1^{n^{1-q}} v^{H_{i/n} + \eta - \frac{5}{2} + cn^{-q\eta}} \delta^{H_{i/n} - \frac{5}{2} + cn^{-q\eta}} (1 + |\log(v)|)\, dv \\
&\leq n^{-\eta - 2H_{i/n}} \delta^{H_{i/n} + cn^{-q\eta} - \frac{5}{2}} \Big[1 + n^{(1-q)(H_{i/n} + \eta - \frac{3}{2} + cn^{-q\eta})}\Big] \\
&\leq n^{-\eta - 2H_{i/n}} \delta^{H_{i/n} + cn^{-q\eta} - \frac{5}{2}} \Big[1 + n^{(1-q)(H_{i/n} + \eta - \frac{3}{2})}\Big].
\end{aligned}$$

The third portion of the integral may be bounded as

$$\begin{aligned}
&\int_{n^{1-q}}^{\infty} \sup_{\substack{H \in [\underline{H}, \overline{H}] \\ |H - H_{i/n}| \leq c(v/n)^{\eta}}} \overline{f}(v, \delta, H)\, dv \\
&\leq \int_{n^{1-q}}^{\infty} \sup_{H \in [\underline{H}, \overline{H}]} \overline{f}(v, \delta, H)\, dv \\
&= \int_{n^{1-q}}^{\infty} \sup_{H \in [\underline{H}, \overline{H}]} n^{-2H} v^{H - \frac{5}{2}} v^{H - \frac{5}{2}}\, dv
\end{aligned}$$

$$
\begin{aligned}
&= \int_{n^{1-q}}^{\infty} \sup_{H\in[\underline{H},\overline{H}]} (\tfrac{v}{n})^{2H} v^{-5}\, dv \\
&\le \int_{n^{1-q}}^{\infty} (\tfrac{v}{n})^{2\underline{H}} v^{-5}\, dv \;+\; \int_{n^{1-q}}^{\infty} (\tfrac{v}{n})^{2\overline{H}} v^{-5}\, dv \\
&\le C n^{(1-q)(2\underline{H}-4)-2\underline{H}} + C n^{(1-q)(2\overline{H}-4)-2\overline{H}} \\
&\le C n^{-4(1-q)} \qquad \le C n^{-4(1-q)} \left(\tfrac{\delta}{n}\right)^{H_{i/n}-\frac{5}{2}} \qquad \le C\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}-\frac{1}{2}}.
\end{aligned}
$$

The last inequality holds for $q > 0$ sufficiently small. We have thus established that

$$
\begin{aligned}
&\left|\mathrm{Cov}(\chi_{i,n}, \chi_{j,n}) - \int_0^{\infty} f(v,\delta,\theta_{\frac{i}{n}})\, dv\right| \\
&\le C\log(n) n^{-\eta-2H_{i/n}} \delta^{H_{i/n}-\frac{5}{2}+cn^{-q\eta}} \left[1 + n^{(1-q)(H_{i/n}+\eta-\frac{3}{2})}\right] + C\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}-\frac{1}{2}} \\
&\le C\log(n)\delta^{H_{i/n}-\frac{5}{2}} n^{-2H_{i/n}} n^{-(\eta\wedge\frac{1}{2})}.
\end{aligned}
$$

To conclude the proof, we observe that $\int_0^{\infty} f(v,\delta,\theta)\, dv$ is the lag-δ autocovariance of the second order increments at frequency $\frac{1}{n}$ of a fractional Brownian motion with parameters $\theta = (\sigma^2, H)$. Thus,

$$
\int_0^{\infty} f(v,\delta,\theta)\, dv = n^{-2H}\sigma^2 \Gamma_H(i-j).
$$

In the derivations above, we assumed that $i > j$. For $i = j$, the claim is a direct consequence of Lemma 4, with $\theta = \theta_{i/n}$. □

Lemma 6. *Let the conditions of Lemma 5 hold, and define* $Z_{i,n} = (\chi_{i,n}^2, \tilde{\chi}_{i,n}^2)^T$. *Then*

$$
\begin{aligned}
\mathrm{Cov}(Z_{i,j}, Z_{j,n}) &= n^{-4H_{i/n}} \sigma_{i/n}^4 \Sigma_{H_{i/n}}(i-j) + \mathcal{O}\Big(\log(n) n^{-4H_{i/n}-(\eta\wedge\frac{1}{2})} (|i-j|\vee 1)^{2H_{i/n}-5}\Big) \\
&= \mathcal{O}\Big(n^{-4H_{i/n}} (|i-j|\vee 1)^{-3}\Big),
\end{aligned}
$$

where

$$
\Sigma_H(h) := 2\begin{pmatrix} \Gamma_H(h)^2 & (\Gamma_H(h)+\Gamma_H(h+1))^2 \\ (\Gamma_H(h)+\Gamma_H(h-1))^2 & (2\Gamma_H(h)+\Gamma_H(h-1)+\Gamma_H(h+1))^2 \end{pmatrix}, \quad h \in \mathbb{Z}.
$$

Proof of Lemma 6. Define $Y_{i,n} = (\chi_{i,n},\ \chi_{i,n} + 2\chi_{i-1,n} + \chi_{i-2,n})^T = (\chi_{i,n},\ \tilde{\chi}_{i,n})^T$, and introduce the matrix.

$$
\overline{\Gamma}_H(h) := \begin{pmatrix} \Gamma_H(h) & \Gamma_H(h)+\Gamma_H(h+1) \\ \Gamma_H(h)+\Gamma_H(h-1) & 2\Gamma_H(h)+\Gamma_H(h-1)+\Gamma_H(h+1) \end{pmatrix}.
$$

Via Lemma 5, we find that

$$
\mathrm{Cov}(Y_{i,n}, Y_{j,n}) = n^{-2H_{i/n}} \sigma_{i/n}^2 \overline{\Gamma}_{H_{i/n}}(i-j) + \mathcal{O}\Big(\log(n) n^{-2H_{i/n}-(\eta\wedge\frac{1}{2})} (|i-j|\vee 1)^{H_{i/n}-\frac{5}{2}}\Big).
$$

Note that the matrix $\Sigma_H(h)$ is twice the entry-wise square of $\overline{\Gamma}_H(h)$, and that $|\overline{\Gamma}_H(h)| \asymp h^{2H-4}$. Hence, Lemma 7 yields

$$
\mathrm{Cov}(Z_{i,j}, Z_{j,n}) = n^{-4H_{i/n}} \sigma_{i/n}^4 \Sigma_{H_{i_n}}(i-j) + \mathcal{O}\Big(\log(n) n^{-4H_{i/n}-(\eta\wedge\frac{1}{2})} (|i-j|\vee 1)^{2H_{i/n}-5}\Big).
$$

□

Lemma 7. *For two centered, jointly Gaussian random variables* X, Y, *it holds* $\mathrm{Cov}(X^2, Y^2) = \mathrm{Cov}(X,Y)^2 \frac{\mathrm{Var}(X^2)}{\mathrm{Var}(X)^2} = 2\,\mathrm{Cov}(X,Y)^2$.

Proof. This is a special case of Isserlis' formula [49] for product moments of multivariate Gaussians, and we give a self-contained proof.

Write $(X,Y) = (X, aX+Z)$ for $a = \mathrm{Cov}(X,Y)/\mathrm{Var}(X)$, and Z centered Gaussian, independent form X. Observe that for independent centered Gaussian random variables X, Z, and $a \in \mathbb{R}$, we have $\mathrm{Cov}(X^2, (aX+Z)^2) = a^2\,\mathrm{Var}(X^2) + 2a\mathbb{E}(X^3 Z) + \mathrm{Cov}(X^2, Z^2) = a^2\,\mathrm{Var}(X^2)$. □

Lemma 8. *There exists a universal* K *such that for any two centered, jointly Gaussian random variables* X, Y, *with* $\mathrm{Cor}(X,Y) = \rho$,

$$
\left|\mathrm{Cov}(\log(X^2), \log(Y^2))\right| \le K|\rho|.
$$

Proof. Since this specific covariance is invariant to rescaling of X and Y, we may assume both are standard normal, with correlation ρ. We proceed similar to the proof of [20, Lemma 8.5].

Denote by $H_l : \mathbb{R} \to \mathbb{R}$ the l-th Hermite polynomial (i.e. of degree l), and decompose $\log(x^2) = \sum_{l=0}^{\infty} c_l H_l(x)$. Because $\log(X^2) \in L_2(P)$, the sequence c_l is square-summable. Moreover,

$$
\mathrm{Cov}(\log(X^2), \log(Y^2)) = \sum_{k,l=0}^{\infty} c_l c_k\, \mathrm{Cov}(H_l(X), H_k(Y)).
$$

Now use $\mathrm{Cov}(H_l(X), H_k(Y)) = \rho^k k! \mathbb{1}(l = k)$ [20, S.3.4] to find that

$$\mathrm{Cov}(\log(X^2), \log(Y^2)) = \sum_{k=1}^{\infty} c_k^2 \rho^k k!.$$

Because $\mathrm{Cov}(\log(X^2), \log(X^2)) < \infty$, corresponding to $\rho = 1$, we conclude that $\sum_k c_k^2 k! < \infty$. This yields $|\mathrm{Cov}(\log(X^2), \log(Y^2))| \leq |\rho| \sum_k c_k^2 k!$. □

B.2. Local nonparametric estimation

First, we derive the properties (v) and (vi) of the local polynomial weights.

Proposition 9. *The kernel weights $w_{i,n}(u)$ of the local polynomial regression estimator satisfy*

(v) $|w_{i+1,n}(u) - w_{i,n}(u)| \leq \frac{C}{(nb_n)^2}$,
(vi) $nb_n \sum_{i=1}^n w_{i,n}(u)^2 \to w^* > 0$, *defined in* (B.2).

Proof. As derived, for example, in [50], the weights may be represented as

$$\begin{aligned}
w_{i,n}(u) &= \frac{1}{n} e_1 S_n(u)^{-1} T_{b_n}(\tfrac{i}{n} - u), \\
S_n(u)_{p,q} &= \frac{1}{n} \sum_{j=1}^{n} K_{b_n}\Big(\tfrac{j}{n} - u\Big)\left(\frac{\frac{j}{n} - u}{b_n}\right)^{p+q} \xrightarrow{n\to\infty} S^*_{p,q} = \int_{-1}^{1} K(v) v^{p+q}\, dv, \\
T_{b_n}(v) &= [(v/b_n)^0, \dots, (v/b_n)^l]^T K_{b_n}\Big(\tfrac{i}{n} - u\Big), \\
e_1 &= (1, 0, \dots, 0).
\end{aligned}$$

Since S_n converges and the limit S^* is regular, and since K is compactly supported and Lipschitz, we readily find that $|w_{i+1,n} - w_{i,n}| \leq C/(nb_n)^2$. This establishes (v). Moreover,

$$\begin{aligned}
&(nb_n) \sum_{i=1}^{n} w_{i,n}(u)^2 \\
&= \frac{1}{nb_n} \sum_{i=1}^{n} \left\{ e_1 S_n(u)^{-1} \left[\left(\frac{\frac{i}{n} - u}{b_n}\right)^0, \dots, \left(\frac{\frac{i}{n} - u}{b_n}\right)^l \right]^T \right\}^2 K\left(\frac{\frac{i}{n} - u}{b_n}\right)^2 \\
&\xrightarrow{n\to\infty} w^* := \int_{-1}^{1} \left\{ e_1 (S^*)^{-1} [v^0, \dots, v^l]^T \right\}^2 K(v)^2\, dv
\end{aligned} \tag{B.2}$$

as a Riemann sum, establishing (vi). □

The error of $\widehat{\phi}_n(u)$ admits the following asymptotic representation.

Proposition 10. *Suppose that $v \mapsto \theta_v = (\sigma_v, H_v)$ is Hölder continuous with exponent $\eta \in (0, l+1]$, and that $0 < \underline{H} \leq H_v \leq \overline{H} < 1$ for all v, and $\sigma_v^2 \leq \overline{\sigma}^2 < \infty$. If $b_n \ll n^{q-1}$ for some $q \in (0,1)$, then for any $p \geq 2$,*

$$\begin{aligned}
n^{2H_u} \widehat{\phi}_n(u) &= \phi(u) + \mathcal{O}\Big(\log(n)^{\lceil \eta \rceil} b_n^{\eta}\Big) + \mathcal{O}_{L_p}\left(\frac{1}{\sqrt{n\, b_n}}\right) + \mathcal{O}\big(\log(n) n^{-(1\wedge\eta)}\big), \\
\phi(u) &= \sigma_u^2 \Gamma_{H_u}(0) \cdot \begin{pmatrix} 1 \\ 2^{2H_u} \end{pmatrix}.
\end{aligned}$$

For any p, the bound holds uniformly for all (b_n, u) such that $u \in [b_n, 1 - b_n]$. If moreover b_n satisfies

$$n^{-1} \ll b_n \ll \min\left\{ \frac{n^{2(\eta\wedge 1)-1}}{\log(n)^2},\quad n^{-\frac{\eta}{2\eta+1}} \log(n)^{\frac{2\eta\lceil\eta\rceil}{2\eta+1}} \right\},$$

then

$$\sqrt{nb_n}\Big[n^{2H_u} \widehat{\phi}_n(u) - \phi(u) \Big] \Rightarrow \mathcal{N}(0, w^* \overline{\tau}(u)^2),$$

with asymptotic covariance matrix $\overline{\tau}(u)^2 \in \mathbb{R}^{2\times 2}$ given by (B.5)*, and w^* depending on the kernel K via* (B.2).

Proof of Proposition 10. Lemma 4 yields

$$\mathbb{E}\chi_{i,n}^2 = n^{-2H_{i/n}} \sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}\big(\log(n) n^{-(1\wedge\eta)-2H_u}\big),$$

where the latter bounds holds uniformly in i. Thus, since $\sum_{i=1}^n w_{i,n} = 1$,

$$\mathbb{E}\Big(\widehat{\phi}_{n,1}(u)\Big) = \sum_{i=1}^n w_{i,n}\mathbb{E}\chi_{i,n}^2 = \sum_{i=1}^n w_{i,n} n^{-2H_{i/n}}\sigma_{i/n}^2\Gamma_{H_{i/n}}(0) + \mathcal{O}\big(\log(n)n^{-(1\wedge\eta)-2H_u}\big),$$

$$\rightsquigarrow \quad \mathbb{E}\Big(n^{2H_u}\widehat{\phi}_{n,1}(u)\Big) = \sum_{i=1}^n w_{i,n} n^{2(H_u-H_{i/n})}\sigma_{i/n}^2\Gamma_{H_{i/n}}(0) + \mathcal{O}\big(\log(n)n^{-(1\wedge\eta)}\big).$$

A Taylor expansion of the function $\mu_n(s) = n^{2(H_u-H_s)}\sigma_s^2\Gamma_{H_s}(0)$, together with property (iv) of the weights, yields,

$$\mathbb{E}\Big(n^{2H_u}\widehat{\phi}_{n,1}(u)\Big) = \mu_n(u) + \mathcal{O}\Big(b_n^\eta \log(n)^{\lceil\eta\rceil}\Big) + \mathcal{O}\big(\log(n)n^{-(1\wedge\eta)}\big)$$

The same bound holds for $\widehat{\phi}_{n,2}$, hence

$$\mathbb{E}\Big(n^{2H_u}\widehat{\phi}_n(u)\Big) = \sigma_u^2\Gamma_{H_u}(0)\cdot\begin{pmatrix}1\\ 2^{2H_u}\end{pmatrix} + \mathcal{O}\big(\log(n)n^{-(1\wedge\eta)}\big) + \mathcal{O}\Big(b_n^\eta \log(n)^{\lceil\eta\rceil}\Big), \tag{B.3}$$

and the bound holds uniformly in $u \in (0,1)$.

To bound the stochastic term, denote $Z_{i,n} = (\chi_{i,n}^2, \tilde{\chi}_{i,n}^2)^T$ and observe

$$\begin{aligned}\left\|\mathrm{Cov}\left(n^{2H_u}\widehat{\phi}_n(u)\right)\right\| &\le n^{4H_u}\sum_{i,j=1}^n w_{i,n}(u)w_{j,n}(u)\left\|\mathrm{Cov}(Z_{i,n}, Z_{j,n})\right\| \\ &\le \frac{C^2 n^{4H_u}}{n^2 b_n^2}\sum_{i=1}^n\sum_{j=1}^n \mathbb{1}_{|\frac{i}{n}-u|\le b_n}\mathbb{1}_{|\frac{j}{n}-u|\le b_n}\|\mathrm{Cov}(Z_{i,n}, Z_{j,n})\|.\end{aligned}$$

For $|\frac{i}{n}-u| \le b_n \ll n^{1-q}$, and $|\frac{j}{n}-u| \ll n^{1-q}$, by virtue of Lemma 6,

$$\begin{aligned}\|\mathrm{Cov}(Z_{i,n}, Z_{j,n})\| &\le C(|i-j|\vee 1)^{-3}n^{-4H_u+Cn^{-q}} \\ &\le C(|i-j|\vee 1)^{-3}n^{-4H_u}.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|\mathrm{Cov}\left(n^{2H_u}\widehat{\phi}_n(u)\right)\right\| &\le C\frac{1}{n^2 b_n^2}\sum_{i,j=1}^n \mathbb{1}_{|\frac{i}{n}-u|\le b_n}\mathbb{1}_{|\frac{j}{n}-u|\le b_n}(|i-j|\vee 1)^{-3} \\ &\le C\frac{1}{n^2 b_n^2}\sum_{i=\lfloor n(u-b_n)\rfloor\vee 1}^{\lceil n(u+b_n)\rceil\wedge n}\sum_{j=-\infty}^{\infty}(|i-j|\vee 1)^{-3} \qquad \le \frac{C}{n\,b_n}.\end{aligned}$$

This establishes the upper bound on the stochastic term of $\widehat{\phi}_n(u)$ in L_2, that is,

$$\left\|n^{2H_u}[\widehat{\phi}_n(u) - \mathbb{E}\widehat{\phi}_n(u)]\right\|_{L_2} \le \frac{C}{nb_n} \tag{B.4}$$

To obtain the bound in L_p, we observe that $\widehat{\phi}_n(u)$ belongs to the second Wiener chaos, and all L_p norms are equivalent up to a universal factor [51, 2.8.14]. Hence, (B.4) also holds for the L_p norm. This establishes the first part of the Proposition.

In order to derive a central limit theorem, we first compute the exact asymptotic covariance as

$$(nb_n)\cdot\mathrm{Cov}\left(n^{2H_u}\widehat{\phi}_n(u)\right) = nb_n\sum_{i,j=1}^n w_{i,n}(u)w_{j,n}(u)\,\mathrm{Cov}(n^{H_u}Z_{i,n}, n^{H_u}Z_{j,n}).$$

By Lemma 6,

$$\max_{i,j=\lfloor n(u-b_n)\rfloor,\dots,\lfloor n\rfloor(u+b_n)\rfloor}\left\|\mathrm{Cov}(n^{H_u}Z_{i,n}, n^{H_u}Z_{j,n}) - \sigma_u^4\Sigma_{H_u}(i-j)\right\| \overset{n\to\infty}{\longrightarrow} 0,$$

$$\|\mathrm{Cov}(n^{H_u}Z_{i,n}, n^{H_u}Z_{j,n})\| \le C(|i-j|\vee 1)^{-3}.$$

Thus, using dominated convergence and the fact that $w_{i,n} \le 1/(nb_n)$,

$$\begin{aligned}(nb_n)\cdot\mathrm{Cov}\left(n^{2H_u}\widehat{\phi}_n(u)\right) &= nb_n\sum_{i=1}^n\sum_{h\in\mathbb{Z}} w_{i,n}(u)w_{i+h,n}(u)\,\mathrm{Cov}(n^{H_u}Z_{i,n}, n^{H_u}Z_{i+h,n}) \\ &= \sigma_u^4 nb_n\sum_{i=1}^n\sum_{h\in\mathbb{Z}} w_{i,n}(u)w_{i+h,n}(u)\Sigma_{H_u}(h) \quad + o(1) \\ &= \sigma_u^4 nb_n\sum_{i=1}^n\sum_{h\in\mathbb{Z}} w_{i,n}(u)^2\Sigma_{H_u}(h) \quad + o(1).\end{aligned}$$

In the last step, we used that $|w_{i,n}(u) - w_{i+h,n}(u)| \leq C\frac{h}{(nb_n)^2}$ by virtue of property (v), and again dominated convergence. Via property (vi) of the local polynomial weights, we may conclude that

$$\lim_{n\to\infty}(nb_n)\cdot \mathrm{Cov}\left(n^{2H_u}\widehat{\phi}_n(u)\right) = \sigma_u^4\, w^* \sum_{h\in\mathbb{Z}} \Sigma_{H_u}(h) \;=:\; w^*\overline{\tau}(u)^2, \tag{B.5}$$

for w^* as in (B.2).

Based on (B.3), we find that the bias of $\sqrt{nb_n}n^{2H_u}\widehat{\phi}_n(u)$ is negligible under the sharper rate conditions imposed on b_n. Thus, it only remains to establish the asymptotic normality of $\sqrt{nb_n}n^{2H_u}\widehat{\phi}_n(u)$. As established in (B.11) and (B.12) below, we may represent the squared increments as

$$\begin{aligned} n^{2H_{i/n}}(\widetilde{\chi}^2_{i,n}, \chi^2_{i,n}) &= G_{i,n}(\epsilon_i, \epsilon_{i-1}, \ldots) \\ \sqrt{nb_n}w_{i,n}(u)\, n^{2H_{i/n}}(\widetilde{\chi}^2_{i,n}, \chi^2_{i,n}) &= \sqrt{nb_n}w_{i,n}(u)G_{i,n}(\epsilon_i, \epsilon_{i-1}, \ldots) \\ &=:\; \overline{G}_{i,n}(\epsilon_i, \epsilon_{i-1}, \ldots), \end{aligned}$$

for $\epsilon_i \overset{iid}{\sim} U(0,1)$, and the physical dependence measure is bounded as

$$\|\overline{G}_{i,n}(\epsilon_0) - \overline{G}_{i,n}(\widetilde{\epsilon}_{0,h})\|_{L_q} \leq C\sqrt{nb_n}w_{i,n}(u)(h+1)^{-\frac{3}{2}} \leq \tfrac{C}{\sqrt{nb_n}}(h+1)^{-\frac{3}{2}},$$

for any $q \geq 1$. Moreover, (B.14) below and property (v) of the weights $w_{i,n}$ yield

$$\begin{aligned} &\left\|\overline{G}_{i,n}(\epsilon_0) - \overline{G}_{i-1,n}(\epsilon_0)\right\|_{L_2} \\ &\leq \sqrt{nb_n}|w_{i,n} - w_{i-1,n}|\cdot\|G_{i,n}(\epsilon_0)\|_{L_2} + \sqrt{nb_n}|w_{i-1,n}|\cdot\|G_{i,n}(\epsilon_0) - G_{i-1,n}(\epsilon_0)\|_{L_2} \\ &\leq C(nb_n)^{-3/2} + (nb_n)^{-1/2}n^{-\eta\wedge 1}. \end{aligned} \tag{B.6}$$

We may hence use the Gaussian approximation result [29, Thm. 2], to find a sequence of bivariate Gaussian random vectors Y_n such that

$$\begin{aligned} &\left\|\sqrt{nb_n}n^{2H_u}\left[\widehat{\phi}_n(u) - \mathbb{E}\widehat{\phi}_n(u)\right] - Y_n\right\|_{L_2} \\ &= \left\|\sum_{i=\lfloor n(u-b_n)\rfloor}^{\lfloor n(u+b_n)\rfloor}\left[\overline{G}_{i,n}(\epsilon_i) - \mathbb{E}\overline{G}_{i,n}(\epsilon_i)\right] - Y_n\right\|_{L_2} + \mathcal{O}\big(\log(n)b_n^{-\eta\wedge 1}\big) \\ &\leq C\Theta_n\Gamma_n^{\frac{\beta-1}{2\beta}}\sqrt{\log(nb_n)}\cdot(nb_n)^{\frac{1}{2}-\xi(q,\beta)} + \mathcal{O}\big(\log(n)b_n^{-\eta\wedge 1}\big), \end{aligned}$$

where the $\mathcal{O}(\ldots)$ term is due to the difference between n^{2H_u} and $n^{2H_{i/n}}$ for $|\frac{i}{n} - u| \leq b_n$, and tends to zero. From the physical dependence bound, we find $\beta = 3/2$ and $\Theta_n = C/\sqrt{nb_n}$, and (B.6) yields $\Gamma_n = 1 + (nb_n)\cdot n^{-\eta\wedge 1}$. As we may choose q arbitrarily large, the exponent is $\xi(q,\beta) = (\beta-1)/(4\beta-2) = 1/8$, see [29]. Thus,

$$\begin{aligned} \left\|\sqrt{nb_n}n^{2H_u}\left[\widehat{\phi}_n(u) - \mathbb{E}\widehat{\phi}_n(u)\right] - Y_n\right\|_{L_2} &\leq C\sqrt{\log(nb_n)}(nb_n)^{-\frac{1}{8}}\Gamma_n^{\frac{1}{6}} + o(1) \\ &\leq C\sqrt{\log(nb_n)}(nb_n)^{-\frac{1}{8}}\left[1 + (nb_n)^{\frac{1}{6}}n^{-\frac{\eta\wedge 1}{6}}\right] + o(1). \end{aligned}$$

Under the imposed conditions on b_n, this tends to zero. In view of (B.5) and the L_2 convergence above, this also yields $Y_n \Rightarrow \mathcal{N}(0, w^*\overline{\tau}(u)^2)$, and Slutsky's Lemma completes the proof of asymptotic normality.

□

Proof of Theorem 1. Proposition 10 yields

$$\begin{aligned} \widehat{H}_n(u) &= \frac{1}{2}\log_2\left(\frac{2^{2H_u}\sigma_u^2\Gamma_{H_u}(0) + \mathcal{O}(\log(n)^{\lceil\eta\rceil}b_n^\eta) + \mathcal{O}_{L_p}(1/\sqrt{nb_n}) + \mathcal{O}(\log(n)n^{-(1\wedge\eta)})}{\sigma_u^2\Gamma_{H_u}(0) + \mathcal{O}(\log(n)^{\lceil\eta\rceil}b_n^\eta) + \mathcal{O}_{L_p}(1/\sqrt{nb_n}) + \mathcal{O}(\log(n)n^{-(1\wedge\eta)})}\right) \\ &= \frac{1}{2}\log_2\left(2^{2H_u} + \mathcal{O}(\log(n)^{\lceil\eta\rceil}b_n^\eta) + \mathcal{O}_{L_p}(1/\sqrt{nb_n}) + \mathcal{O}(\log(n)n^{-(1\wedge\eta)})\right) \\ &= H_u + \mathcal{O}(\log(n)^{\lceil\eta\rceil}b_n^\eta) + \mathcal{O}_{L_p}(1/\sqrt{nb_n}) + \mathcal{O}(\log(n)n^{-(1\wedge\eta)}) \end{aligned}$$

and for the optimal choice of b_n,

$$= H_u + \mathcal{O}_{L_p}\left(\log(n)^{\frac{\lceil\eta\rceil}{2\eta+1}}n^{-\frac{\eta}{2\eta+1}}\right).$$

The asymptotic normality of $\widehat{H}_n(u)$ is derived from the asymptotic normality of $n^{2H_u}\widehat{\phi}_n(u)$ via the delta method, and its asymptotic variance is $w^*\tau(u)^2$, with

$$\tau(u)^2 = v^T\overline{\tau}(u)^2 v,$$

$$\upsilon = \frac{1}{2}\begin{pmatrix} -\frac{1}{\phi_1(u)} \\ \frac{1}{\phi_2(u)} \end{pmatrix} = \frac{1}{2\sigma_u^2 \Gamma_{H_u}(0)} \cdot \begin{pmatrix} -1 \\ 2^{-2H_u} \end{pmatrix}.$$

An explicit formula for this is given in (B.19) below.

□

Proof of Theorem 2. Denote $a_{i,n} = \mathbb{E}\chi_{i,n}^2$ and $\tilde{a}_{i,n} = \mathbb{E}\widetilde{\chi}_{i,n}^2$. Then $\chi_{i,n}^2 \overset{d}{=} a_{i,n} Z^2$ for $Z \sim \mathcal{N}(0,1)$, and $\widetilde{\chi}_{i,n}^2 \overset{d}{=} \tilde{a}_{i,n} Z^2$, so that Lemma 4 yields

$$\begin{aligned}
\mathbb{E}\log_2\left(\frac{\widetilde{\chi}_{i,n}^2}{\chi_{i,n}^2}\right) &= \log_2\left(\frac{\tilde{a}_{i,n}}{a_{i,n}}\right) + \left[\mathbb{E}\log_2(Z^2) - \mathbb{E}\log_2(Z^2)\right] = \log_2\left(\frac{\tilde{a}_{i,n}}{a_{i,n}}\right) \\
&= 2H_{i/n} + \log_2\left(\frac{(n/2)^{2H_{i/n}}\tilde{a}_{i,n}}{n^{2H_{i/n}} a_{i,n}}\right) \\
&= 2H_{i/n} + \log_2\left(\frac{\sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}(\log(n) n^{-(\eta\wedge 1)})}{\sigma_{i/n}^2 \Gamma_{H_{i/n}}(0) + \mathcal{O}(\log(n) n^{-(\eta\wedge 1)})}\right) \\
&= 2H_{i/n} + \log_2\left(1 + \mathcal{O}(\log(n) n^{-(\eta\wedge 1)})\right) \\
&= 2H_{i/n} + \mathcal{O}\big(\log(n) n^{-(\eta\wedge 1)}\big).
\end{aligned}$$

Thus, by standard bias bounds for local polynomial estimators, we obtain the bias bound $\mathbb{E}(\widehat{H}_n^\dagger(u)) = H(u) + \mathcal{O}(b_n^\eta + \log(n) n^{-(\eta\wedge 1)})$.

To control the variance, we use Lemma 8 to find that

$$\mathrm{Var}\left(\sum_{i=1}^n w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) \le K \sum_{i,j=1}^n |w_{i,n}(u) w_{j,n}(u)| \cdot \left|\mathrm{Cor}(\chi_{i,n}, \chi_{j,n})\right|.$$

In the following, K is a generic constant which may vary from line to line. Lemma 5 yields

$$\left|\mathrm{Cor}(\chi_{i,n}, \chi_{j,n})\right| \le K|i-j|^{2\overline{H}-4} + K|i-j|^{\overline{H}-\frac{5}{2}} \quad \le K|i-j|^{-\frac{3}{2}},$$

which is summable. In combination with the boundedness and finite support of the weights $w_{i,n}(u)$, we find

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^n w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) &\le K \sum_{i,j=\lfloor n(u-b_n)\rfloor}^{\lceil n(u+b_n)\rceil} w_{i,n}(u) w_{j,n}(u)(|i-j|+1)^{-\frac{3}{2}} \\
&\le \frac{K}{nb_n} \sum_{h=-\infty}^{\infty} (|h|+1)^{-\frac{3}{2}} \quad = \mathcal{O}(1/(nb_n)).
\end{aligned}$$

Similarly, $\mathrm{Var}\left(\sum_{i=1}^n w_{i,n}(u)\log_2(\chi_{i,n}^2)\right) = \mathcal{O}(1/(nb_n))$, which in particular yields

$$\mathrm{Var}\left(\widehat{H}_n^\dagger(u)\right) = \mathcal{O}\Big(\tfrac{1}{nb_n}\Big).$$

□

B.3. Integrated parameter estimation

Proof of Theorem 3. Step (i): Denote $t_0 = 2L_n$. We may write the estimator $\widehat{\mathcal{H}}(u)$ equivalently as

$$\widehat{\mathcal{H}}(u) = \frac{1}{n}\sum_{t=2L}^{\lfloor un\rfloor} \left\{m(\hat{\psi}_{t,n}) + Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \hat{\psi}_{t,n})\right\},$$

with

$$\begin{aligned}
&m : (0,\infty)^2 \to \mathbb{R},\ (x,y) \mapsto \left[\tfrac{1}{2}\log(\tfrac{y}{x})\right] \vee 0 \wedge 1, \\
&Z_{t,n} = \frac{n^{2H_{t/n}}}{\sigma_{t/n}^2}\begin{pmatrix} \chi_{t,n}^2 \\ \tilde{\chi}_{t,n}^2 \end{pmatrix}, \qquad \psi_u = \Gamma_{H_u}(0) \cdot \begin{pmatrix} 1 \\ 2^{2H_u} \end{pmatrix}, \qquad \psi_{t,n} = \psi_{t/n} \\
&\hat{\psi}_{t,n} = \frac{n^{2H_{t/n}}}{\sigma_{t/n}^2}\hat{\phi}_n(\tfrac{t-L_n}{n}) \text{ for } t = 2L,\dots,n, \text{ and } \hat{\psi}_{t,n} = \psi_{t,n} \text{ for } t = 1,\dots,2L-1.
\end{aligned}$$

Due to the L_q bound of Proposition 10, we have for any $q \ge 2$ and $t = L + \lceil nb_n\rceil, \dots, n$,

$$\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_q}^2 = \mathcal{O}\Big(\log(n)^{2\lceil\eta\rceil} b_n^{2\eta} + \tfrac{1}{nb_n} + \log(n)^2 n^{-2(1\wedge\eta)}\Big) \ \le\ \mathcal{O}\Big(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\Big).$$

Furthermore, for $t = 2L, \dots, L + \lceil nb_n\rceil$, the local polynomial estimator effectively uses the smaller bandwidth $\frac{t-L}{n} \ge \frac{L}{n}$, see Remark 5. Hence

$$\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_q}^2 = \mathcal{O}\Big(\log(n)^{2\lceil\eta\rceil}(\tfrac{t-L_n}{n})^{2\eta} + \tfrac{1}{t-L_n} + \log(n)^2 n^{-2(1\wedge\eta)}\Big).$$

Moreover, $\|\psi_{t-L,n} - \psi_{t,n}\|^2 = \mathcal{O}((L_n/n)^{2\eta})$, and hence $\| \max_{t=2L,\dots,n} |\hat{\psi}_{t,n} - \psi_{t,n}| \|_{L_q} = \mathcal{O}(n^{\frac{1}{q}-\epsilon}) + \mathcal{O}((L_n/n)^{\eta})$, for some $\epsilon > 0$, which tends to zero for q large enough. Using the assumptions on b_n, we find

$$\begin{aligned}\frac{1}{n}\sum_{t=2L}^{n} \|\hat{\psi}_{t,n} - \psi_{t,n}\|_{L_q}^2 &\le C\Big(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\Big) + \frac{C}{n}\sum_{t=2L}^{L+\lceil nb_n\rceil}\left[\log(n)^{2\lceil\eta\rceil}\left(\tfrac{t-L}{n}\right)^{2\eta} + \frac{1}{t-L}\right] \\ &\le C\Big(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\Big) + Cb_n^{1+2\eta}\log(n)^{2\lceil\eta\rceil} + C\frac{\log(n)}{n} \\ &= \mathcal{O}\Big(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\Big).\end{aligned}$$

Thanks to the uniform convergence, we may restrict our attention to the event

$$A_n = \Big\{\frac{1}{2}\psi_{t,n} \le \hat{\psi}_{t,n} \le 2\psi_{t,n}, \qquad \forall\, t = 1,\dots,n\Big\},$$

as $P(A_n) \to 1$ as $n \to \infty$. For this event, a Taylor expansion of m around $\hat{\psi}_{t,n}$ yields

$$\begin{aligned}&\widehat{\mathcal{H}}(u) - \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} m(\psi_{t,n}) \\ &= \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor}\big\{m(\hat{\psi}_{t,n}) + Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \hat{\psi}_{t,n})\big\} - \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} m(\psi_{t,n}) + \mathcal{O}_P\Big(\frac{L_n}{n}\Big) \\ &= \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \psi_{t,n}) + \mathcal{O}_P\Big(\frac{1}{n}\sum_{t=1}^{n}\|\hat{\psi}_{t,n} - \psi_{t,n}\|^2\Big) + \mathcal{O}_P\Big(\frac{L_n}{n}\Big) \qquad \text{(B.7)} \\ &= \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \psi_{t,n}) + \mathcal{O}_P\Big(\log(n)^{2\lceil\eta\rceil} n^{-\frac{1}{2}-r}\Big) + \mathcal{O}_P\Big(\frac{L_n}{n}\Big), \qquad \text{(B.8)}\end{aligned}$$

using in (B.7) that the second derivative $D^2 m(\psi)$ is bounded in the specified neighborhood of $\psi_{t,n}$, and the special definition $\hat{\psi}_{t,n} = \psi_{t,n}$ for $t = 1,\dots,2L-1$. By our assumption on L_n, the remainder terms in (B.8) are of order $o_P(1/\sqrt{n})$. Moreover, the approximation (B.7) holds uniformly in $u \in [0,1]$.

Next, note that the Hölder regularity of H_v with exponent η yields

$$\frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} m(\psi_{t,n}) = \frac{1}{n}\sum_{t=1}^{\lfloor un\rfloor} H_{t/n} = \int_0^u H_v\, dv + \mathcal{O}(n^{-\eta}),$$

uniformly in $u \in [0,1]$. Since $\eta > \frac{1}{2}$ by assumption, we have shown that

$$\sqrt{n}\Big[\widehat{\mathcal{H}}(u) - \mathcal{H}(u)\Big] = \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor un\rfloor} Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \psi_{t,n}) + o_P(1).$$

By Lemma 4, $|\mathbb{E}(Z_{t,n}) - \psi_{t,n}| = \mathcal{O}(\log(n)n^{-\eta}) = o(n^{-\frac{1}{2}})$, and we obtain

$$\sqrt{n}\Big[\widehat{\mathcal{H}}(u) - \mathcal{H}(u)\Big] = \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor un\rfloor} Dm(\hat{\psi}_{t,n})\cdot(Z_{t,n} - \mathbb{E}Z_{t,n}) + o_P(1). \qquad \text{(B.9)}$$

Step (ii): We now apply the multiplier FCLT of Theorem 5 to the leading term in (B.9), with

$$\hat{g}_{t,n} = Dm(\hat{\psi}_{t,n})\mathbb{1}_{A_n}, \qquad g_{t,n} = Dm(\psi_{t,n}), \qquad g_u = Dm(\psi_u), \qquad Y_{t,n} = Z_{t,n} - \mathbb{E}Z_{t,n}.$$

For this definition of $g_{t,n}$ and $\hat{g}_{t,n}$, (A.4) holds, Φ_n is bounded, and $\Psi_n = \mathcal{O}(n^{1-\eta})$. Moreover, Proposition 10 shows that

$$\begin{aligned}\Lambda_n^2 &\le C\sum_{t=2L}^{n}\Big(\|\hat{\psi}_{t,n} - \psi_{t-L,n}\|_{L_2}^2 + \|\psi_{t-L,n} - \psi_{t,n}\|^2\Big) \\ &\le C\Big(\log(n)^2 n^{\frac{1}{2}-r} + L_n^{2\eta} n^{1-2\eta}\Big) + C\sum_{t=2L_n}^{L_n+\lceil nb_n\rceil}\Big(\log(n)^2(\tfrac{t-L_n}{n})^{2\eta} + \tfrac{1}{t-L_n}\Big) \\ &\le C\Big(\log(n)^2 n^{\frac{1}{2}-r} + L_n n^{1-2\eta} + \log(n)^2\tfrac{(nb_n)^{2\eta+1}}{n^{2\eta}} + \log(nb_n)\Big) \\ &= \mathcal{O}\Big(\log(n)^2 n^{\frac{1}{2}-r} + L_n n^{1-2\eta}\Big)\end{aligned}$$

It remains to check the conditions (A.1), (A.2), (A.3) for $Y_{t,n}$. To write $Y_{t,n}$ in the form $G_{t,n}(\epsilon_t)$ as in Appendix A, note that $Y_{t,n}$ is a functional of the driving Brownian motion B_s, see Definition (2). Since all Polish spaces are Borel-isomorphic, there exists a Borel-isomorphism $\varphi : (0,1) \to C[0,1]$, and iid random variables $\epsilon_t \sim U(0,1)$ such that $\varphi(\epsilon_t) = [\tilde{B}_u]_{u\in[0,1]} = \sqrt{n}[B_{\frac{u+t-1}{n}} - B_{\frac{t-1}{n}}]_{u\in[0,1]} = \varphi(\epsilon_t)$,

for a standard Brownian motion $\tilde{B}$. With this notation, we may write

$$\frac{n^{H_{i/n}}}{\sigma_{i/n}}\chi_{i,n} = \frac{n^{H_{i/n}}}{\sigma_{i/n}}\int_{-\infty}^{i/n} g_{i,n}(s)\,dB_s =: \widetilde{G}_{i,n}(\boldsymbol{\epsilon}_i),$$
$$g_{i,n}(s) = \sigma_s\left[(\tfrac{i}{n}-s)_+^{H_s-\frac{1}{2}} - 2(\tfrac{i}{n}-\tfrac{1}{n}-s)_+^{H_s-\frac{1}{2}} + (\tfrac{i}{n}-\tfrac{2}{n}-s)_+^{H_s-\frac{1}{2}}\right].$$

In particular, we regard $\chi_{i,n}$ as a measurable function of $\varphi(\epsilon_t) = \sqrt{n}[B_{\frac{u+t-1}{n}} - B_{\frac{t-1}{n}}]_{u\in[0,1]}$ for $t = i, i-1, \ldots$, which is in turn a measurable function of $\boldsymbol{\epsilon}_i = (\epsilon_i, \epsilon_{i-1}, \ldots)$. This is summarized by the notation $\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_i)$. This also allows us to plug-in a shifted sequence $\boldsymbol{\epsilon}_j$ for $j \neq i$, and this effectively shifts the time of the Brownian motion, but leaves the kernel $g_{i,n}$ untouched. That is,

$$\begin{aligned}
\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_j) &= \frac{n^{H_{i/n}}}{\sigma_{i/n}}\int_{-\infty}^{i/n} g_{i,n}(s)\,dB_{s+\frac{j-i}{n}} \\
&= \frac{n^{H_{i/n}}}{\sigma_{i/n}}\sum_{k=0}^{\infty}\int_{(i-k-1)/n}^{(i-k)/n} g_{i,n}(s)\,dB_{s+\frac{j-i}{n}} \qquad = \frac{n^{H_{i/n}}}{\sigma_{i/n}}\sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} g_{i,n}(\tfrac{s}{n})\,\tfrac{1}{\sqrt{n}}\,d\tilde{B}_{s+(j-i)} \\
&= \sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \tilde{g}_{i,n}(s)\,d\tilde{B}_{s+(j-i)} \\
&= \sum_{k=0}^{\infty}\zeta_{i,k,n}(\boldsymbol{\epsilon}_{j-k}), \qquad \zeta_{i,k,n} : (0,1)\to\mathbb{R},\ \zeta_{i,k,n}(\boldsymbol{\epsilon}_{j-k}) \in L_2(P), \\
\tilde{g}_{i,n}(s) &= \frac{\sigma_{\frac{s}{n}}}{\sigma_{\frac{i}{n}}} n^{H_{\frac{i}{n}}-H_{\frac{s}{n}}}\,\bar{g}(s-i, H_{\frac{s}{n}}), \\
\bar{g}_i(s,H) &= (-s)_+^{H-\frac{1}{2}} - 2(-1-s)_+^{H-\frac{1}{2}} + (-2-s)_+^{H-\frac{1}{2}}.
\end{aligned}$$

The observation that the Bernoulli shift of the ϵ_t is equivalent to shifting the driving Brownian motion will be used to upper bound the physical dependence measure of $\widetilde{G}_{i,n}$ in the sequel.

Ad (A.2): The $\zeta_{i,k,n}(\epsilon_j)$ are centered Gaussian random variables with variance

$$\begin{aligned}
\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 &= \int_{i-k-1}^{i-k} |\tilde{g}_{i,n}(s)|^2\,ds \\
&\le Cn^{2(H_{\frac{i}{n}}-H_{\frac{i-k}{n}})+Cn^{-\eta}}(k+1)^{2H_{\frac{i-k}{n}}-5+Cn^{-\eta}} \\
&\le Cn^{2(H_{\frac{i}{n}}-H_{\frac{i-k}{n}})}(k+1)^{2H_{\frac{i-k}{n}}-5+\delta},
\end{aligned}$$

for $\delta > 0$ small enough, and n large enough. If $H_{\frac{i}{n}} \le H_{\frac{i-k}{n}}$, then $\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 \le C(k+1)^{2\overline{H}-5+\delta} \le C(k+1)^{-3}$ for δ small enough, because $\overline{H} < 1$. If instead $H_{\frac{i}{n}} > H_{\frac{i-k}{n}}$, we further distinguish two cases to obtain

$$\begin{aligned}
\|\zeta_{i,k,n}(\epsilon_j)\|_{L_2}^2 &\le \begin{cases} C(k+1)^{\frac{2}{1-\delta}(H_{\frac{i}{n}}-H_{\frac{i-k}{n}})}(k+1)^{2H_{\frac{i-k}{n}}-5+\delta}, & k \ge n^{1-\delta}, \\ Cn^{Cn^{-\delta\eta}}(k+1)^{2H_{\frac{i-k}{n}}-5+\delta}, & k < n^{1-\delta}, \end{cases} \\
&\le \begin{cases} C(k+1)^{\frac{2}{1-\delta}H_{\frac{i}{n}}-\frac{2\delta}{1-\delta}H_{\frac{i-k}{n}}-5+\delta}, & k \ge n^{1-\delta}, \\ C(k+1)^{2\overline{H}-5+\delta}, & k < n^{1-\delta}, \end{cases} \\
&\le \begin{cases} C(k+1)^{\frac{2}{1-\delta}\overline{H}-5+\delta}, & k \ge n^{1-\delta}, \\ C(k+1)^{2\overline{H}-5+\delta}, & k < n^{1-\delta}, \end{cases} \\
&\le C(k+1)^{-3}, \qquad \text{(B.10)}
\end{aligned}$$

for δ small enough. Since $\zeta_{i,k,n}(\epsilon_j)$ is Gaussian, the latter L_2 bound also holds in L_q. Thus, we have shown that for all $q \ge 2$, and for some $C = C_q$,

$$\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0) - \widetilde{G}_{i,n}(\tilde{\boldsymbol{\epsilon}}_{0,h})\|_{L_q} \le C\|\zeta_{i,h,n}(\epsilon_j)\|_{L_2} \le C(h+1)^{-\frac{3}{2}},$$
$$\|\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_0)\|_{L_q} \le C.$$

Now note that $Y_{t,n}$ is a function of the $\widetilde{G}_{i,n}(\boldsymbol{\epsilon}_i)$, in particular

$$Y_{i,n} = G_{t,n}(\boldsymbol{\epsilon}_j) = \begin{pmatrix} \widetilde{G}_{t,n}(\boldsymbol{\epsilon}_j)^2 \\ (\widetilde{G}_{t,n}(\boldsymbol{\epsilon}_j) + 2\widetilde{G}_{t-1,n}(\boldsymbol{\epsilon}_{j-1}) + \widetilde{G}_{t-2,n}(\boldsymbol{\epsilon}_{j-2}))^2 \end{pmatrix}, \qquad \text{(B.11)}$$

and it is straightforward to derive

$$\|G_{t,n}(\epsilon_0) - G_{t,n}(\bar{\epsilon}_{0,h})\|_{L_q} \le C(h+1)^{-\frac{3}{2}}. \tag{B.12}$$

This establishes (A.2) with $\Theta_n = \mathcal{O}(1)$ and exponent $\beta = \frac{3}{2}$.
Ad (A.1): Observe that

$$\begin{aligned}
&\left\|\widetilde{G}_{i,n}(\epsilon_0) - \widetilde{G}_{i-1,n}(\epsilon_0)\right\|_{L_2}^2 \\
&= \left\|\sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \tilde{g}_{i,n}(s)\, d\tilde{B}_{s-i} - \sum_{k=0}^{\infty}\int_{(i-1)-k-1}^{(i-1)-k} \tilde{g}_{i-1,n}(s)\, d\tilde{B}_{s-(i-1)}\right\|_{L_2}^2 \\
&= \left\|\sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \left[\tilde{g}_{i,n}(s) - \tilde{g}_{i-1,n}(s-1)\right] d\tilde{B}_{s-i}\right\|_{L_2}^2 \\
&= \sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} |\tilde{g}_{i,n}(s) - \tilde{g}_{i-1,n}(s-1)|^2\, ds \\
&\le \sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \left|\frac{\sigma_{\frac{s}{n}}}{\sigma_{\frac{i}{n}}} n^{H_{\frac{i}{n}} - H_{\frac{s}{n}}} - \frac{\sigma_{\frac{s}{n}}}{\sigma_{\frac{i-1}{n}}} n^{H_{\frac{i-1}{n}} - H_{\frac{s}{n}}}\right|^2 |\bar{g}\left(s-i, H_{\frac{s}{n}}\right)|^2\, ds \\
&\quad + \sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \left|\frac{\sigma_{\frac{s}{n}}}{\sigma_{\frac{i-1}{n}}} n^{H_{\frac{i-1}{n}} - H_{\frac{s}{n}}}\right|^2 |\bar{g}\left(s-i, H_{\frac{s}{n}}\right) - \bar{g}\left(s-i, H_{\frac{s-1}{n}}\right)|^2\, ds \\
&\le \sum_{k=0}^{\infty}\left|1 - \frac{\sigma_{\frac{i}{n}}}{\sigma_{\frac{i-1}{n}}} n^{H_{\frac{i-1}{n}} - H_{\frac{i}{n}}}\right| \int_{i-k-1}^{i-k} |\tilde{g}_{i,n}(s)|^2\, ds \\
&\quad + \sum_{k=0}^{\infty}\int_{i-k-1}^{i-k} \left|n^{H_{\frac{i-1}{n}} - H_{\frac{s}{n}}}\right|^2 |\bar{g}\left(s-i, H_{\frac{s}{n}}\right) - \bar{g}\left(s-i, H_{\frac{s-1}{n}}\right)|^2\, ds \\
&\le C\log(n)\, n^{-2\eta} \sum_{k=0}^{\infty}(k+1)^{-3} \\
&\quad + C\sum_{k=0}^{\infty} \log(n) n^{-\eta} n^{2H_{\frac{i-1}{n}} - 2H_{\frac{i-k}{n}}} \int_{i-k-1}^{i-k} |\bar{g}\left(s-i, H_{\frac{s}{n}}\right) - \bar{g}\left(s-i, H_{\frac{s-1}{n}}\right)|^2\, ds.
\end{aligned} \tag{B.13}$$

To bound the latter integral, we exploit the fact that

$$\begin{aligned}
&\frac{d}{dH}\bar{g}(s,H) = \mathbb{1}_{s<0}\log(-s)(-s)^{H-\frac{1}{2}} - 2\mathbb{1}_{s<-1}\log(-s-1)(-s-1)^{H-\frac{1}{2}} + \mathbb{1}_{s<2}\log(-s-2)(-s-2)^{H-\frac{1}{2}}, \\
&\rightsquigarrow \quad \left|\frac{d}{dH}\bar{g}(s,H)\right| \le C(1+|\log s|)\min\left(|s|^{H-\frac{1}{2}}, |s|^{H-\frac{5}{2}}\right).
\end{aligned}$$

Hence, the mean value theorem yields some $\tilde{H}_s$ between $H_{\frac{s}{n}}$ and $H_{\frac{s-1}{n}}$ such that, for $k \ge 1$,

$$\begin{aligned}
&\int_{i-k-1}^{i-k} |\bar{g}\left(s-i, H_{\frac{s}{n}}\right) - \bar{g}\left(s-i, H_{\frac{s-1}{n}}\right)|^2\, ds \\
&\le \int_{i-k-1}^{i-k} |H_{\frac{s}{n}} - H_{\frac{s-1}{n}}|^2 \left|\frac{d}{dH}\bar{g}_{0,n}(s-i,H)|_{H=\tilde{H}_s}\right|^2 ds \\
&\le Cn^{-2\eta}\int_{i-k-1}^{i-k} \left[1+|\log(s-i)|\right]\left[\mathbb{1}_{|s-i|<1}|s-i|^{2\tilde{H}_s-1} + \mathbb{1}_{|s-i|\ge 1}|s-i|^{2\tilde{H}_s-5}\right] ds \\
&\le Cn^{-2\eta}\int_{i-k-1}^{i-k} \left[1+|\log(s-i)|\right]|s-i|^{2\tilde{H}_s-5}\, ds \\
&\le Cn^{-2\eta}(k+1)^{2H_{\frac{i-k}{n}}-3+Ck^{-\eta}}\log(k) \\
&\le Cn^{-2\eta}(k+1)^{2H_{\frac{i-k}{n}}-3+\delta},
\end{aligned}$$

for $\delta > 0$ small. This term may be further bounded as in (B.10) to find

$$\log(n) n^{-\eta} n^{2H_{\frac{i-1}{n}} - 2H_{\frac{i-k}{n}}} \int_{i-k-1}^{i-k} |\bar{g}\left(s-i, H_{\frac{s}{n}}\right) - \bar{g}\left(s-i, H_{\frac{s-1}{n}}\right)|^2\, ds \le C(k+1)^{-3}.$$

Using this in (B.13), and exploiting the Gaussianity, we obtain for all $q \ge 2$

$$\left\|\widetilde{G}_{i,n}(\epsilon_0) - \widetilde{G}_{i-1,n}(\epsilon_0)\right\|_{L_q}^2 \le Cn^{-2\eta}\sum_{k=0}^{\infty}(k+1)^{-3} \le Cn^{-2\eta}.$$

Since $\|\tilde{G}_{i,n}\|_{L_q}$ is bounded, it is straightforward to conclude from this and (B.11) that

$$\left\|G_{i,n}(\epsilon_0) - G_{i-1,n}(\epsilon_0)\right\|_{L_2}^2 \leq Cn^{-2\eta}, \tag{B.14}$$

and hence (A.1) holds with $\Theta_n = \mathcal{O}(1)$ and $\Gamma_n = n^{1-\eta}$.
Ad (A.3): For $u \in [0,1]$, define the limiting kernels

$$\widetilde{G}_u(\epsilon_j) = \int_{-\infty}^{0} \bar{g}(s, H_u)\, d\tilde{B}_{s+j}, \tag{B.15}$$

and, in view of (B.11),

$$G_u(\epsilon_j) = \begin{pmatrix} \widetilde{G}_u(\epsilon_j)^2 \\ \left(\widetilde{G}_u(\epsilon_j) + 2\widetilde{G}_u(\epsilon_{j-1}) + \widetilde{G}_u(\epsilon_{j-2})\right)^2. \end{pmatrix} \tag{B.16}$$

Then

$$\begin{aligned}
&\|\widetilde{G}_u(\epsilon_0) - \widetilde{G}_{\lfloor un\rfloor,n}(\epsilon_0)\|_{L_q} \\
&\leq C\|\widetilde{G}_u(\epsilon_0) - \widetilde{G}_{\lfloor un\rfloor,n}(\epsilon_0)\|_{L_2} \\
&\leq \int_{-\infty}^{\lfloor un\rfloor} |\bar{g}(s - \lfloor un\rfloor, H_u) - \tilde{g}_{\lfloor un\rfloor,n}(s)|^2\, ds \\
&= \int_{-\infty}^{0} \left| \bar{g}(s, H_u) - \frac{\sigma_{\frac{\lfloor un\rfloor+s}{n}}}{\sigma_{\frac{\lfloor un\rfloor}{n}}} n^{H_{\frac{\lfloor un\rfloor}{n}} - H_{\frac{\lfloor un\rfloor+s}{n}}} \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\leq 2\int_{-\infty}^{0} \left| \bar{g}(s, H_u) - \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\quad + 2\int_{-\infty}^{0} \left| 1 - \frac{\sigma_{\frac{\lfloor un\rfloor+s}{n}}}{\sigma_{\frac{\lfloor un\rfloor}{n}}} n^{H_{\frac{\lfloor un\rfloor}{n}} - H_{\frac{\lfloor un\rfloor+s}{n}}} \right|^2 \left| \bar{g}(s, H_{\frac{\lfloor un\rfloor+s}{n}}) \right|^2 ds \\
&\leq Cn^{-2\eta} \int_{-1}^{0} ds + C\int_{-\infty}^{-1} (\tfrac{|s|}{n})^{2\eta} \sup_{H\in[\underline{H},\overline{H}]} \left| \tfrac{d}{dH} \bar{g}(s, H) \right|^2 ds \\
&\quad + C\log(n) \int_{-n^{1-\delta}}^{-1} (\tfrac{|s|}{n})^{2\eta} \sup_{H\in[\underline{H},\overline{H}]} \left| \bar{g}(s, H) \right|^2 ds \\
&\quad + C\log(n) \int_{-\infty}^{-n^{1-\delta}} n^{2\overline{H}} \sup_{H\in[\underline{H},\overline{H}]} \left| \bar{g}(s, H) \right|^2 ds \\
&\leq Cn^{-2\eta} + C\log(n)n^{-2\eta} \int_{-\infty}^{-1} |s|^{2\overline{H}+2\eta-5}\, ds + C\log(n) \int_{-\infty}^{-n^{1-\delta}} |s|^{2\overline{H}-5}\, ds \\
&\leq C\log(n)n^{-2\eta} + C\log(n)n^{-2(1-\delta)} \qquad \leq C\log(n)n^{-2\eta}
\end{aligned}$$

for any small $\delta > 0$. Here, we used that $|\bar{g}(s, H)| \leq C|s|^{H-\frac{5}{2}}$ and $\frac{d}{dH}|\bar{g}(s, H)| \leq C(1 + \log|s|)|s|^{H-\frac{5}{2}}$ for $|s| \geq 1$. That is, we find that for any $q \geq 2$

$$\|\widetilde{G}_{\lfloor un\rfloor,n}(\epsilon_0) - \widetilde{G}_u(\epsilon_0)\|_{L_q} \leq Cn^{-\eta}. \tag{B.17}$$

Again, this directly yields that

$$\|G_{\lfloor un\rfloor,n}(\epsilon_0) - G_u(\epsilon_0)\|_{L_q} \leq Cn^{-\eta}, \tag{B.18}$$

and hence $\int_0^1 \|G_{\lfloor un\rfloor,n}(\epsilon_0) - G_u(\epsilon_0)\|_{L_2}\, du \to 0$ as $n \to \infty$. This establishes (A.3).
Rate constraints: In the previous steps, we have verified the conditions of Theorem 5 for any $q \geq 2$, for $\beta = \frac{3}{2}$, and with $\Gamma_n \asymp n^{1-\eta}$, $\Lambda_n^2 \asymp \log(n)^2 n^{\frac{1}{2}-r} + L_n^{2\eta} n^{1-2\eta}$, $\Psi_n \asymp n^{1-\eta}$, $\Phi_n = \mathcal{O}(1)$, $\Theta_n = \mathcal{O}(1)$. Upon choosing q large enough, we have $\xi(q,\beta) = \frac{\beta-1}{4\beta-2} = \frac{1}{8}$, hence

$$(\Gamma_n + \Psi_n)^{\frac{\beta-1}{2\beta}} \sqrt{\log(n)} n^{-\xi(q,\beta)} = n^{\frac{1-\eta}{6}} \sqrt{\log(n)} n^{-\frac{1}{8}},$$

which tends to zero if $\eta > \frac{1}{2}$. Moreover, upon choosing q large enough, we have

$$\begin{aligned}
&n^{-\frac{1}{2}}\Lambda_n^2 + \Lambda_n L_n^{1-\beta} + n^{\frac{1}{q}-\frac{1}{2}} L_n \to 0 \\
&\quad \Longleftarrow n^{\frac{1}{2}-r} \log(n)^2 \ll L_n \ll n^{\frac{1}{2}-\delta} \quad \text{for some } \delta > 0, \text{ and } L_n \ll n^{1-\frac{1}{4\eta}}.
\end{aligned}$$

Thus, we have shown that Theorem 5 is applicable under the conditions formulated in Theorem 3.

Determining the asymptotic variance: The asymptotic local variance is given by

$$\Sigma_u = \sum_{h=-\infty}^{\infty} g_u \operatorname{Cov}(G_u(\epsilon_0), G_u(\epsilon_h)) g_u^T,$$
$$g_u = \frac{1}{2\Gamma_{H_u}(0)} \begin{pmatrix} -1 \\ 2^{-2H_u} \end{pmatrix}.$$

The autocovariances may be computed as in Lemma 5, hence $\operatorname{Cov}(G_u(\epsilon_0), G_u(\epsilon_h)) = \Sigma_{H_u}(h)$ as therein. Thus,

$$\begin{aligned}
\Sigma_u &= \tau^2(H_u), \\
\tau^2(H) &= \frac{1}{4\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} (-1, 2^{-2H}) \Sigma_H(h) \begin{pmatrix} -1 \\ 2^{-2H} \end{pmatrix} \\
&= \frac{1}{2\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} \Big\{ \Gamma_H(h)^2 + 2^{-4H} (2\Gamma_H(h) + \Gamma_H(h-1) + \Gamma_H(h+1))^2 \\
&\qquad - 2^{-2H} \left[(\Gamma_H(h) + \Gamma_H(h-1))^2 + (\Gamma_H(h) + \Gamma_H(h-1))^2 \right] \Big\} \\
&= \frac{1}{2\Gamma_H(0)^2} \sum_{h=-\infty}^{\infty} \Big\{ \Gamma_H(h)^2 + 2^{-4H} (2\Gamma_H(h) + \Gamma_H(h-1) + \Gamma_H(h+1))^2 \\
&\qquad - 2^{-2H+1} (\Gamma_H(h) + \Gamma_H(h-1))^2 \Big\}.
\end{aligned} \tag{B.19}$$

Note that this is the same asymptotic variance as for the local nonparametric estimator given in Theorem 1.

□

Proof of Theorem 4. By Theorem 1, and as established in the proof of Theorem 3, we have for any $q \geq 2$

$$\max_{t=2L_n,\dots,n} |\hat{H}_n(\tfrac{t}{n}) - H_{\frac{t}{n}}| \quad \xrightarrow{\mathbb{P}} \quad 0.$$

Hence,

$$\sup_{u\in[0,1]} \left| \frac{1}{n} \sum_{t=2L}^{\lfloor un \rfloor} \tau^2(H(\tfrac{t}{n})) - \frac{1}{n} \sum_{t=2L}^{\lfloor un \rfloor} \tau^2(\hat{H}_n(\tfrac{t}{n})) \right| \xrightarrow{\mathbb{P}} 0.$$

Moreover, continuity of $v \mapsto H_v$, and $L_n/n \to 0$, yields

$$\sup_{u\in[0,1]} \left| \int_0^u \tau^2(H_v)\, dv - \frac{1}{n} \sum_{t=2L_n}^{\lfloor un \rfloor} \tau^2(H(\tfrac{t}{n})) \right| \to 0.$$

□

Proof of Corollary 1. Since $H_v \in [\underline{H}, \overline{H})$, and since Δ_v is bounded, we find we may set $\underline{H}^* = \underline{H}/2 > 0$ and $\overline{H}^* = \frac{1+\overline{H}}{2} < 1$ such that for sufficiently large n, it holds $H_{v,n} = H_v + \Delta_v/\sqrt{n} \in [\underline{H}^*, \overline{H}^*]$ for all v. Moreover, the error bound in Proposition 10 still holds with H_v replaced by $H_{v,n}$. This is the case because the implicit constants in the bias bound of Lemma 4 only depends on the upper bounds and Hölder constants of (σ_s, H_s), which are uniformly bounded for $v \mapsto H_{v,n}$. The same holds the implicit constants in the variance bound of Lemma 5, which therefore transfers to the implicit constants in Proposition 10. Thus, as in (B.9), we may derive

$$\sqrt{n} \left[\hat{\mathcal{H}}(u) - \mathcal{H}(u) - \tfrac{1}{\sqrt{n}} \int_0^u \Delta_v\, dv \right] = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor un \rfloor} Dm(\hat{\psi}_{t,n}) \cdot (Z_{t,n} - \mathbb{E} Z_{t,n}) + o_P(1), \tag{B.20}$$

and we may follow the same arguments as in Theorem 3 to show that the latter term converges to $\int_0^u \tau(H_v)\, dW_v$, by verifying conditions (A.1), (A.2), (A.4).

The only difference compared to the proof of Theorem 3 is in establishing (A.3) starting at (B.15). As the Hurst function now depends on n, we define the pre-limiting kernel

$$\widetilde{G}_u^n(\epsilon_j) = \int_{-\infty}^0 \bar{g}(s, H_{u,n})\, d\tilde{B}_{s+j}.$$

As in (B.17), we find that

$$\|\widetilde{G}_{\lfloor un \rfloor,n}(\epsilon_0) - \widetilde{G}_u^n(\epsilon_0)\|_{L_q} \leq C n^{-\eta}.$$

To establish condition (A.3), it thus suffices to prove that, uniformly in $u \in [0,1]$,

$$0 \overset{!}{=} \lim_{n\to\infty} \left\| \int_{-\infty}^0 \bar{g}(s, H_u) - \bar{g}(s, H_{u,n})\, dB_s \right\|_{L_2}^2$$

$$= \lim_{n\to\infty} \int_{-\infty}^{0} \left|\overline{g}(s, H_u) - \overline{g}(s, H_{u,n})\right|^2 ds,$$

$$\overline{g}(s, H) = (-s)_+^{H-\frac{1}{2}} - 2(-1-s)_+^{H-\frac{1}{2}} + (-2-s)_+^{H-\frac{1}{2}}.$$

Using the bounds

$$\left|\frac{d}{dH}\overline{g}(s, H)\right| \le C|\log(|s|)| \cdot \left[|s|^{H-\frac{1}{2}} + |s+1|^{H-\frac{1}{2}} + |s+2|^{H-\frac{1}{2}}\right], \qquad |s| \le 3,$$

$$\left|\frac{d}{dH}\overline{g}(s, H)\right| \le C(1+\log|s|)|s|^{H-\frac{5}{2}}, \qquad |s| \ge 3,$$

we find that $\frac{d}{dH}\overline{g}(s, H)$ admits an integrable majorizing function $\overline{g}^*(s) = \overline{g}^*(s, \overline{H}, \underline{H})$

$$\int_{-\infty}^{0} \left|\overline{g}(s, H_u) - \overline{g}(s, H_{u,n})\right|^2 ds \le \frac{C \sup_v |\Delta_v|}{\sqrt{n}} \int_{-\infty}^{0} \overline{g}^*(s)\, ds \le \frac{C \sup_v |\Delta_v|}{\sqrt{n}},$$

which clearly tends to zero.

The consistency of the variance estimator claimed in Theorem 4 remains valid because $\widehat{H}_n(u)$ is still consistent, as all inequalities on therein only use upper bounds on H_v and its Hölder norm. □

Proof of Proposition 2.

Convergence under the null is a direct consequence of Theorems 3 and 4, i.e. $q_n(\alpha) \to q(\alpha)$ by Theorem 4, where $q(\alpha)$ is the $1-\alpha$ quantile of $Y = \sup_u |W(\int_0^u \tau^2(H_v)\, dv) - u \cdot W(\int_0^1 \tau^2(H_v)\, dv)|$, and $\sqrt{n}\widehat{T}_{\mathrm{CUSUM}}(H) \Rightarrow Y$.

Consistency against a fixed alternative holds because Theorem 3 shows that $\sup_{u\in[0,1]} |\widehat{\mathcal{H}}(u) - u\widehat{\mathcal{H}}(u)| \to \sup_{u\in[0,1]} |\mathcal{H}(u) - u\mathcal{H}(u)| \neq 0$, and thus $\sqrt{n}T_{\mathrm{CUSUM}}(\widehat{H}) \to \infty$, while $q_n(\alpha) \to q_\alpha < \infty$.

Under the local alternative model, we still have $q_n(\alpha) \to q(\alpha)$, which is a quantile of Y. On the other hand, by Corollary 1,

$$\sqrt{n}T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) \Rightarrow Y_c := \sup_u \left|W^*(u) - uW^*(1) + c\int_0^u \Delta_v\, dv - u\int_0^1 \Delta_v\, dv\right|$$

Hence,

$$\lim_{n\to\infty} P\left(\sqrt{n}T_{\mathrm{CUSUM}}(\widehat{\mathcal{H}}) > q_n(\alpha)\right) = P\left(Y_c > q(\alpha)\right).$$

Since $Y_c \to \infty$ in probability as $c \to \infty$, the latter expression tends to 1 as $c \to \infty$.

□

Proof of Proposition 3. The first claim holds because $q_n(\alpha) \to q(\alpha)$ by Theorem 4, where $q(\alpha)$ is the $1-\alpha$ quantile of $Z = \sup_u |W(\int_0^u \tau^2(H_v)\, dv)|$, and $\sqrt{n}\widehat{T}(\mathcal{G}_0) \le \sqrt{n}\widehat{T}(H) \Rightarrow Z$.

We proceed to prove the second claim. If $\mathcal{G}_0$ is closed, then $\widetilde{\mathcal{G}}_0 = \{u \mapsto \int_0^u \tilde{H}(v)\, dv \mid \tilde{H} \in \mathcal{G}_0\}$ is also closed, and $\int H(v)\, dv \notin \widetilde{\mathcal{G}}_0$. Hence, $\sup_{u\in[0,1]} |\int_0^u \tilde{H}(v)\, dv - \int_0^u \tilde{H}(v)\, dv| > 0$ for all $\tilde{H} \in \mathcal{G}_0$. As $\sqrt{n}(\widehat{\mathcal{H}} - \mathcal{H}) = o_P(1)$, this implies that $\sqrt{n}\widehat{T}(\mathcal{G}_0) \xrightarrow{\mathbb{P}} \infty$. On the other hand, Theorem 4 yields convergence of $q_n(\alpha)$ to a finite number, establishing consistency of the test. □

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