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RESEARCH PAPER

# APPLICATIONS OF HILFER-PRABHAKAR OPERATOR TO OPTION PRICING FINANCIAL MODEL

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### Abstract

In this paper, we focus on option pricing models based on time-fractional diffusion with generalized Hilfer-Prabhakar derivative. It is demonstrated how the option is priced for fractional cases of European vanilla option pricing models. Series representations of the pricing formulas and the risk-neutral parameter under the time-fractional diffusion are also derived.

*MSC 2010*: 26A33; 34A08; 91B25; 91G20

*Key Words and Phrases*: Hilfer-Prabhakar derivatives; Mittag-Leffler functions; European pricing model; Cauchy problem; heat equation; fractional diffusion

## 1. Introduction

Fractional calculus has celebrated great success in recent decades. It is a powerful tool, which has been recently employed to model complex biological systems with non-linear behavior and long-term memory. First attempts on fractional calculus were made by Leibniz, Liouville, Riemann, or Riesz. As it turns out, the fractional calculus does not provide a unique definition of the fractional derivative operator. Therefore one can find definitions according to Riemann and Liouville, Riesz and Feller, Caputo, Gründwald and Letnikov, Hilfer, Prabhakar and many more, see, e.g., Refs. [29, 31, 27, 11, 28]. As a result, different definitions of fractional derivative also have different properties in general; linearity is the only property


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that is always satisfied. The main application of the fractional operators is to define a class of integrodifferential equations, called fractional diffusion equations, where the ordinary first temporal and second spatial derivative is replaced by various fractional derivatives. The main physical purpose for adopting and investigating diffusion equations of fractional order is to describe phenomena of anomalous diffusion usually met in transport processes through complex or disordered systems, including fractal media. In this paper, we will consider the fractional diffusion model involving fractional operator in time. Fractional derivatives represent an interpolation between various well-known classes of ordinary or partial differential equations. This enables to model processes with large jumps (fractional spatial derivatives) or with memory (fractional temporal derivative). Let us mention the famous space-time fractional diffusion equation introduced by Mainardi, Pagnini, and Luchko [23]. This equation is equipped with a time derivative of the Caputo type and space derivative of the Riesz-Feller type.

We generalize this fractional diffusion equation by considering a generalization of the Caputo derivative called Hilfer-Prabhakar derivative. It combines the Hilfer derivative, which is a generalization of both Caputo and Riemann-Liouville derivative with Prabhakar integral. The Hilfer-Prabhakar derivative is then used as the temporal derivative operator as the generalization of heat and wave equations. With the help of the Fourier-Laplace transform, we derive two representations of the fundamental solution of this equation in terms of an infinite series. Both of the representations can be used in different situations and are complementary. Finally, we apply these results to option pricing of the European vanilla options and derive formulas for option prices under the diffusion with Hilfer-Prabhakar derivatives.

## 2. Definitions of fractional derivatives

Before introducing the regularized and non-regularized Hilfer-Prabhakar differential operators, we briefly review the definitions of the most commonly used operators of classical fractional calculus. In particular, the classical Riemann-Liouville derivative and the Caputo derivative of fractional order are introduced. For more information about these derivatives, we refer the reader to classical references [15, 16, 19, 26].

Definition 2.1. (Riemann-Liouville integral). Let $f \in L^1_{\text{loc}}[a,b]$, where $-\infty \leq a < t < b \leq \infty$, be a locally integrable real-values function. The Riemann-Liouville integral is defined as

$$(I^{\alpha}_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(u)}{(t-u)^{1-\alpha}}\, du = (f * \Phi_\alpha)(t), \quad \alpha > 0, \tag{2.1}$$

where $\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

Using this integral we can next define the Riemann-Liouville derivative.

Definition 2.2. (Riemann-Liouville derivative). Let $f \in L^1[a,b]$, where $-\infty \le a < t < b \le \infty$, and $f * \Phi_{m-\alpha} \in W^{m,1}[a,b], m = \lceil \alpha \rceil$ with $W^{m,1}[a,b]$ is the Sobolev space defined by

$$W^{m,1}[a,b] = \left\{ f \in L^1[a,b] : \frac{d^m}{dt^m} f \in L^1[a,b] \right\}$$

The Riemann-Liouville derivative of order $\alpha > 0$ is defined as

$$(D^\alpha_{a^+} f)(t) = \frac{d^m}{dt^m} I^{m-\alpha}_{a^+} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t \frac{f(u)}{(t-u)^{m-1-\alpha}}\, du. \quad (2.2)$$

For $n \in \mathbf{N}$, we denote by $\mathrm{AC}^n[a,b]$ the space of real-valued functions with $(n-1)$ continuous derivatives on $[a,b]$ such that $f^{n-1}(t) \in AC[a,b]$.

We now continue with the Caputo derivative in which definition the differentiation proceeds integration.

Definition 2.3. (Caputo derivative). Let $\alpha > 0$, $m = \lceil \alpha \rceil$, and $f \in AC^m[a,b]$. The Caputo derivative of order $\alpha > 0$ is the defined as

$$({}^C\!D^\alpha_{a^+} f)(t) = (I^{m-\alpha}_{a^+} \frac{d^m}{dt^m} f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \frac{d^m}{ds^m} f(s)\, ds. \quad (2.3)$$

In the space of functions belonging to $AC^m[a,b]$ the following relation between the Riemann-Liouville and the Caputo derivative holds.

Theorem 2.1. *For $f \in AC^m[a,b]$, $m = \lceil \alpha \rceil$, $\alpha \in \mathbb{R}^+$, $m \in \mathbb{N}$ the Riemann-Liouville derivative of order $\alpha$ exists almost everywhere and can be written as*

$$(D^\alpha_{a^+} f)(t) = ({}^C\!D^\alpha_{a^+} f)(t) + \sum_{k=0}^{m-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+). \quad (2.4)$$

2.1. **Hilfer derivatives.** Hilfer introduced a generalized fractional operator in [15, 16] which combines the Caputo and Riemann-Liouville derivatives presented in the following definition.

Definition 2.4. (Hilfer Derivative) Let $\mu \in (0,1), \nu \in [0,1]$ and $f \in L^1[a,b]$, $-\infty \le a < t < b \le \infty$, $f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The Hilfer derivative is defined as

$$(D^{\mu,\nu}_{a^+}f)(t) = \left( I^{(1-\mu)\nu}_{a^+} \frac{d}{dt}(I^{(1-\nu)(1-\mu)}_{a^+} f) \right)(t). \tag{2.5}$$

We remark that Eq.(2.5) reduces to the Riemann-Liouville derivative for $\nu = 0$ and to the Caputo derivative for $\nu = 1$. Furthermore, from now on we will set $a = 0$ without loss of generality, for convenience. An important issue is which initial conditions should be imposed when considering Cauchy problems with involving Hilfer derivatives. The expression of the Laplace transform of Eq.(2.5) [15, 16]

$$\mathcal{L}\left[(D^{\mu,\nu}_{0^+}f)(t)\right](s) = s^{\mu}\mathcal{L}[f(t)](s) - s^{\nu(\mu-1)}\left(I^{(1-\nu)(1-\mu)}_{0^+} f\right)(0^+), \tag{2.6}$$

shows that the initial conditions should be of the form $\left(I^{(1-\nu)(1-\mu)}_{0^+} f\right)(0^+)$, which do not have a clear physical meaning except when $\nu = 1$, that is, in the Caputo case. However, we can regularize the Hilfer derivative by restricting ourselves to the space of absolutely continuous function $AC^1[0, b]$ and apply Theorem 2.1. We then obtain for $\mu \in (0, 1)$

$$\begin{aligned}(D^{\mu,\nu}_{0^+}f)(t) &= \left(I^{(1-\nu)\mu}_{0^+} \tfrac{d}{dt}(I^{(1-\nu)(1-\mu)}_{0^+} f)\right)(t) = \left(I^{\nu(1-\mu)}_{0^+} I^{(1-\nu)(1-\mu)}_{0^+} \tfrac{d}{dt} f\right)(t) \\ &\quad + I^{\nu(1-\mu)}_{0^+} \tfrac{t^{\nu\mu-\mu-\nu} f(0^+)}{\Gamma(1-\nu-\mu+\nu\mu)} = I^{1-\mu}_{0^+} \tfrac{d}{dt} f(t) + \tfrac{t^{-\mu} f(0^+)}{\Gamma(1-\mu)} \\ &= ({}^{C}\mathcal{D}^{\mu}_{0^+} f)(t) + \tfrac{t^{-\mu} f(0^+)}{\Gamma(1-\mu)}. \end{aligned} \tag{2.7}$$

Hence the Hilfer derivative reduces to the Riemann-Liouville as defined in (2.2) in the space $AC[0, b]$. We can regularize the Hilfer derivative by subtracting the divergent term, which is commonly referred to as the regularized Hilfer derivative.

Definition 2.5. (Regularized Hilfer Derivative). Under the same conditions as in definition of the Hilfer derivative we define the regularized Hilfer derivative $(\bar{D}^{\mu,\nu}_{0^+}f)(t)$ as

$$(\bar{D}^{\mu,\nu}_{0^+}f)(t) = (D^{\mu,\nu}_{0^+}f)(t) - \frac{t^{-\mu} f(0^+)}{\Gamma(1-\mu)}. \tag{2.8}$$

We immediately see from Eq. (2.8) that the regularized Hilfer derivative is equal to the Caputo derivative and does hence not depend on $\nu$.

2.2. **Prabhakar derivative.** We can further generalize the Hilfer derivative by replacing the integral kernel $1/t^{\mu-1}$ occurring in the Riemann-Liouville integrals by the function $e^{\gamma}_{\rho,\mu,\omega}(t)$ with

$$e^{\gamma}_{\rho,\mu,\omega}(t) = t^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega t^{\rho}), \quad t \in \mathbb{R},\ \rho, \mu, \omega, \gamma \in \mathbb{C},\ \mathcal{R}(\rho), \mathcal{R}(\mu) > 0,$$

where $E^{\gamma}_{\rho,\mu}(\omega t^{\rho})$ is the generalized Mittag-Leffler (Prabhakar) function which is defined as

$$E^{\gamma}_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} t^k.$$

By definition, it follows that

$$E^{1}_{\rho,\mu}(t) = E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k+\mu)},$$

and

$$E^{1}_{\rho,1}(t) = E_{\rho}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k+1)}.$$

The Laplace transform of $e^{\gamma}_{\rho,\mu,\omega}(t)$ is given by

$$\mathcal{L}[e^{\gamma}_{\rho,\mu,\omega}(t)](s) = s^{-\mu}(1-\omega s^{-\rho})^{-\gamma} \quad (\left|\omega s^{-\rho}\right| < 1).$$

The following definition was put forward in a paper by Prabhakar [25] and later investigated in [29, 31]. Using the integral kernel $e^{\gamma}_{\rho,\mu,\omega}(t)$ enables us to define so-called Prabhakar integral.

Definition 2.6. (Prabhakar integral). Let $f \in L^1[0,b]$, $0 < t < b \leq \infty$. The Prabhakar integral can be written as

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+} f)(t) = \int_0^t (t-y)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-y)^{\rho}] f(y) dy = f * e^{\gamma}_{\rho,\mu,\omega}(t), \quad (2.9)$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C},\ \mathcal{R}(\rho), \mathcal{R}(\mu) > 0$.

The inverse operation, the Prabhakar derivative, can now be introduced in the following form.

Definition 2.7. (Prabhakar derivative). Let $f \in L^1[0,b], 0 < x < b \leq \infty$ and $f * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in W^{m,1}[0,b]$, $m = \lceil \mu \rceil$. The Prabhakar derivative is defined as

$$(\mathbf{D}^{\gamma}_{\rho,\mu,\omega,0^+} f)(x) = \frac{d^m}{dx^m} (\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,0^+} f)(x), \quad (2.10)$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C},\ \mathcal{R}(\rho), \mathcal{R}(\mu) > 0$.

2.3. **Generalized Hilfer-Prabhakar derivative.** We can next define the generalized Hilfer-Prabhakar derivative (GHP).

Definition 2.8. Let $0 < \nu \leq 1,\ n-1 < \mu \leq n,\ n \in \mathbb{N},\ \omega, \gamma \in \mathbb{R},\ \rho > 0$ and let $f \in L^1[0,b], f * e^{-\gamma}_{\rho,\mu,\omega}(\cdot) \in AC^1[0,b]$. The generalized Hilfer-Prabhakar derivative is defined by

$$(D^{\gamma,\mu,\nu}_{\rho,\omega,0^+}f)(t) = \left(\mathbf{E}^{-\gamma\nu}_{\rho,\nu(\mathbf{n}-\mu),\omega,\mathbf{0}^+}\frac{\mathbf{d^n}}{\mathbf{dt^n}}\left(\mathbf{E}^{-\gamma(\mathbf{1}-\nu)}_{\rho,(\mathbf{1}-\nu)(\mathbf{n}-\mu),\omega,\mathbf{0}^+}\mathbf{f}\right)\right)(t), \qquad (2.11)$$

where $(\mathbf{E}^{0}_{\rho,0,\omega,0^+}f)(t) = f$.

The special case $n = 1$ of this definition, was introduced and considered by Garra et al. [11]. We observe that (2.11) reduces to the generalized Hilfer derivative for $\gamma = 0$, defined by Hilfer et al. [17]. We will introduce an extended Laplace transform formula of the GHP derivative, which has not yet appeared in the literature.

LEMMA 2.1. *(Laplace transform of the GHP derivative). The Laplace transform of the GHP derivative (2.11) with $0 < \nu \leq 1$ and $n-1 < \mu \leq n$ and $n \in \mathbb{N}$ is given by*

$$\mathcal{L}\left((D^{\gamma,\mu,\nu}_{\rho,\omega,0^+}f)(t)\right)(s) = s^{\mu}(1-\omega s^{-\rho})^{\gamma}\mathcal{L}\left(f(t)\right)(s) - s^{-\nu(n-\mu)}(1-\omega s^{-\rho})^{\gamma\nu} \times \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}f\right)(t)\Big|_{t=0^+}. \qquad (2.12)$$

P r o o f. Using the convolution property for Laplace transform, we obtain

$$\mathcal{L}\left((D^{\gamma,\mu,\nu}_{\rho,\omega,0^+}f)(t)\right)(s) = \mathcal{L}\left[e^{-\gamma\nu}_{\rho,\nu(n-\mu),\omega}(t)\right](s) \times \mathcal{L}\left[\frac{d^n}{dt^n}\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}f\right)(t)\right](s)$$

$$= s^{-\nu(n-\mu)}(1-\omega s^{-\rho})^{\gamma\nu}\left\{s^n\mathcal{L}\left[e^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega}(t)\right](s) \times\mathcal{L}\left(f(t)\right)(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}f\right)(t)\,|_{t=0^+}\right\}$$

$$= s^{-\nu(n-\mu)}(1-\omega s^{-\rho})^{\gamma\nu}s^n s^{-(1-\nu)(n-\mu)}(1-\omega s^{-\rho})^{\gamma(1-\nu)}\mathcal{L}\left(f(t)\right)(s) - s^{-\nu(n-\mu)}(1-\omega s^{-\rho})^{\gamma\nu}\sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}f\right)(t)\,|_{t=0^+}$$

$$= s^{\mu}(1-\omega s^{-\rho})^{\gamma}\mathcal{L}\left(f(t)\right)(s) - s^{\nu(n-\mu)}(1-\omega s^{-\rho})^{\gamma\nu} \times \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left(\mathbf{E}^{-\gamma(1-\nu)}_{\rho,(1-\nu)(n-\mu),\omega,0^+}f\right)(t)\,|_{t=0^+}.$$

□

In order to consider Cauchy problems involving initial conditions depending only on the function and its integer order derivatives, we use the *regularized version* of the Hilfer-Prabhakar (RHP) derivative that we next define.

DEFINITION 2.9. (Regularized Hilfer Prabhakar derivative). For $0 < \nu \leq 1$, $n-1 < \mu \leq n$ and $n \in \mathbb{N}$ the RHP of $f \in AC^n[0,b]$ is given as

$$({}^C D^{\gamma,\mu}_{\rho,\omega,0^+} f)(t) = \left( \mathbf{E}^{-\gamma\nu}_{\rho,\nu(n-\mu),\omega,0^+} \mathbf{E}^{-\gamma(1-\nu)}_{\rho,\nu(n-\mu),\omega,0^+} \frac{d^n}{dt^n} f \right)(t) = \left( \mathbf{E}^{-\gamma}_{\rho,\nu(n-\mu),\omega,0^+} \frac{d^n}{dt^n} f \right)(t). \tag{2.13}$$

We proceed by calculating the Laplace transform of RHP, which we will need in the next section.

LEMMA 2.2. *The Laplace transform of the RHP derivative (2.13) with $0 < \nu \leq 1$ and $n-1 < \mu \leq n$ and $n \in \mathbb{N}$ is given by*

$$\mathcal{L}\left( ({}^C\!D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} f)(t) \right)(s) = s^{\mu}(1-\omega s^{-\rho})^{\gamma} \mathcal{L}\left(f(t)\right)(s) - s^{\mu-n}(1-\omega s^{-\rho})^{\gamma} \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0^+). \tag{2.14}$$

P r o o f. Using the convolution property of Laplace transform, we obtain

$$\mathcal{L}\left( ({}^C\!D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} f)(t) \right)(s) = \mathcal{L}\left[ e^{-\gamma}_{\rho,n-\mu,\omega}(t) \right](s) \mathcal{L}\left[ f^{(n)}(t) \right](s)$$
$$= s^{\mu-n}(1-\omega s^{-\rho})^{\gamma} \left[ s^n \mathcal{L}[f(t)](s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0^+) \right]$$
$$= s^{\mu}(1-\omega s^{-\rho})^{\gamma} \mathcal{L}\left(f(t)\right)(s) - s^{\mu-n}(1-\omega s^{-\rho})^{\gamma} \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0^+).$$

□

## 3. Fractional-diffusion equation with Hilfer-Prakhbar derivative

The theory developed in the previous section will next be applied to a few problems that are of interest in finance and mathematical physics. Let us study a generalization of the ordinary fractional wave equation, by generalizing the time derivative to a Hilfer-Prabhakar derivative. We then find the following theorem.

THEOREM 3.1. *The solution to the Cauchy problem*

$$\begin{cases} (\mathcal{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u)(x,t) = K\frac{\partial^2}{\partial x^2}u(x,t), \ t>0, \ x\in\mathbb{R}, \\ \left[\mathbf{E}^{-\gamma(1-\nu)}_{\rho(1-\nu)(2-\mu),\omega,0^+}u(x,t)\right]_{t=0^+} = g(x) \\ \left[\frac{\partial}{\partial t}\mathbf{E}^{-\gamma(1-\nu)}_{\rho(1-\nu)(2-\mu),\omega,0^+}u(x,t)\right]_{t=0^+} = 0 \\ \lim_{x\to\pm\infty} u(x,t) = 0, \end{cases} \tag{3.1}$$

*where* $\mu\in(1,2), \nu\in[0,1], \omega\in\mathbb{R}, \ K,\rho>0, \ \gamma\geq 0$ *is given by*

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{K^{n+1}}e^{-\gamma(n+2)}_{\rho,\nu(2-\mu)-\mu n-1,\omega}(t)\int_{-\infty}^{\infty}\frac{\hat{g}(k)e^{-i\kappa x}}{\kappa^{2n+2}}d\kappa. \tag{3.2}$$

P r o o f. We let $\tilde{u}(x,s)$ denote the Laplace transform (with respect to time) of $u(x,t)$, that is $\tilde{u}(x,s) = \mathcal{L}[u(x,t)]$. The Fourier transform (with respect to $x$) is designated by $\hat{u}(\kappa,t) = \mathcal{F}[u(x,t)]$. We calculate the solution of Eq. (3.1) by taking the Laplace-Fourier transform of Eq. (3.1) and using Lemma 1 (Eq. (2.12). This yields:

$$s^\mu(1-\omega s^{-\rho})^\gamma\hat{\tilde{u}}(\kappa,s) - s^{-\nu(2-\mu)}(1-\omega s^{-\rho})^{2\gamma}s\hat{g}(\kappa) = -K\kappa^2\hat{\tilde{u}}(\kappa,s). \tag{3.3}$$

From Eq. (3.3) we derive the following power series representation for $\hat{\tilde{u}}(\kappa,s)$:

$$\begin{aligned} \hat{\tilde{u}}(\kappa,s) &= \frac{s^{-\nu(2-\mu)+1}(1-\omega s^{-\rho})^{2\gamma}}{s^\mu(1-\omega s^{-\rho})^\gamma + K\kappa^2}\hat{g}(\kappa) = \frac{1}{K\kappa^2}\frac{s^{-\nu(2-\mu)+1}(1-\omega s^{-\rho})^{2\nu}}{1+\frac{s^\mu(1-\omega s^{-\rho})^\gamma}{K\kappa^2}}\hat{g}(\kappa) \\ &= \frac{s^{-\nu(2-\mu)+1}(1-\omega s^{-\rho})^{2\nu}}{K\kappa^2}\sum_{n=0}^{\infty}\left(-\frac{1}{K\kappa^2}\right)^n s^{\mu n}(1-\omega s^{-\rho})^{\gamma n}\hat{g}(\kappa) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{K^{n+1}}s^{\mu n-\nu(2-\mu)+1}(1-\omega s^{-\rho})^{\gamma(n+2)}\frac{\hat{g}(\kappa)}{\kappa^{2n+2}}. \end{aligned} \tag{3.4}$$

Eq. (3.4) is valid when the sum converges, that is when

$$\left|\frac{s^\mu(1-\omega s^{-\rho})^\gamma}{K\kappa^2}\right| < 1.$$

We conclude the proof by applying the inverse Fourier transform to Eq. (3.4) to obtain

$$u(x,t) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{K^{n+1}}e^{-\gamma(n+2)}_{\rho,\nu(2-\mu)-\mu n-1,\omega}(t)\int_{-\infty}^{\infty}\frac{\hat{g}(k)e^{-i\kappa x}}{\kappa^{2n+2}}d\kappa.$$

□

EXAMPLE 1. For the initial condition $g(x) = \delta(x)$, where $\delta$ is the Dirac delta function, expression (3.2) reduces to

$$u(x,t) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{|x|^{2n+1}}{(2n+1)!K^{n+1}} e^{-\gamma(n+2)}_{\rho,\nu(2-\mu)-\mu n-1,\omega}(t).$$

In order to consider Cauchy problems involving initial conditions depending only on the function and its integer order derivatives, we calculate the solution to problem similar to the first problem, but this time with a RHP derivative. The solution can again be expressed in power series and is presented in the following theorem.

THEOREM 3.2. *The solution to the Cauchy problem*

$$\begin{cases} ({}^C\mathcal{D}^{\gamma,\mu,\nu}_{\rho,\omega,0^+}u)(x,t) = K\frac{\partial^2}{\partial x^2}u(x,t),\ t>0,\ x\in\mathbb{R}, \\ u(x,0^+) = g(x) \\ \frac{\partial}{\partial t}u(x,t)\big|_{t=0^+} = h(x) \\ \lim_{x\to\pm\infty} u(x,t) = 0, \end{cases} \tag{3.5}$$

*where* $\mu \in (1,2), \nu \in [0,1], \omega \in \mathbb{R},\ K,\rho > 0,\ \gamma \geq 0$ *is given by*

$$u(x,t) = \frac{1}{2\pi}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{K^{n+1}} e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+2,\omega}(t)\int_{-\infty}^{\infty}\frac{\hat{h}(k)e^{-i\kappa x}}{\kappa^{2n+2}}d\kappa\right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{n+1}} e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(t)\int_{-\infty}^{\infty}\frac{\hat{g}(k)e^{-i\kappa x}}{\kappa^{2n+2}}d\kappa\right]. \tag{3.6}$$

P r o o f. We first take the Laplace transform of (3.5) and use Lemma 2.2, (2.14). This yields

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\hat{\tilde{u}}(\kappa,s) - s^{\mu-2}(1-\omega s^{-\rho})^{\gamma}\left[\hat{h}(\kappa) + s\hat{g}(\kappa)\right] = -K\kappa^2\hat{\tilde{u}}(\kappa,s), \tag{3.7}$$

from which we can express $\hat{\tilde{u}}(\kappa,s)$ in term of the Fourier transforms of the functions $g$ and $h$

$$\hat{\tilde{u}}(\kappa,s) = \frac{s^{\mu-2}(1-\omega s^{-\rho})^{\gamma}\hat{h}(\kappa) + s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}\hat{g}(\kappa)}{s^{\mu}(1-\omega s^{-\rho})^{\gamma} + K\kappa^2}. \tag{3.8}$$

Under the condition

$$\left|\frac{s^{\mu}(1-\omega s^{-\rho})^{\gamma}}{K\kappa^2}\right| < 1,$$

we can expand the denominator of Eq. (3.8), which gives

$$\hat{\tilde{u}}(\kappa,s) = \frac{1}{K\kappa^2}\left[\sum_{n=0}^{\infty}\left(-\frac{1}{K\kappa^2}\right)^n s^{\mu(n+1)-2}(1-\omega s^{-\rho})^{\gamma(n+1)}\hat{h}(\kappa)\right.$$
$$\left. + \sum_{n=0}^{\infty}\left(-\frac{1}{K\kappa^2}\right)^n s^{\mu(n+1)-1}(1-\omega s^{-\rho})^{\gamma(n+1)}\hat{g}(\kappa)\right]. \tag{3.9}$$

Taking the inverse Laplace transform results in

$$\hat{u}(\kappa,t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+2,\omega}(t)\frac{\hat{h}(\kappa)}{\kappa^{2n+2}}$$
$$+ \sum_{n=0}^{\infty}\frac{(-1)^n}{K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(t)\frac{\hat{g}(\kappa)}{\kappa^{2n+2}}. \tag{3.10}$$

By next applying the inverse Fourier transform on $\hat{u}(\kappa,t)$ we obtain result Eq. (3.6). □

EXAMPLE 2. Similarly to the previous case, when $g(x) = \delta(x)$ and $h(x) = 0$, we obtain

$$u(x,t) = \sum_{n=0}^{\infty}\frac{|x|^{2n+1}}{(2n+1)!K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(t). \tag{3.11}$$

Alternatively, we can derive another representation for this example. Let us rewrite the fundamental solution in the Fourier-Laplace representation (Eq. 3.8) as

$$\hat{\tilde{u}}(\kappa,s) = \frac{s^{-1}}{1+K\kappa^2\left(\frac{s^{\rho\gamma-\mu}}{(s^\rho-\omega)^\gamma}\right)}. \tag{3.12}$$

Let us now apply the series expansion, in contrast to the previous section, directly to $-K\kappa^2\left(\frac{s^{\rho\gamma-\mu}}{(s^\rho-\omega)^\gamma}\right)$ The reason for that is that while the former representation is good for direct calculations with the fundamental solution, the latter is better for integration over initial conditions. Moreover, the two representations are complimentary.

Thus we can write

$$\hat{u}(\kappa,t) = \sum_{n=0}^{\infty}(-K)^n\kappa^{2n}e^{\gamma n}_{\rho,\mu n+1,\omega}(t). \tag{3.13}$$

By taking the inverse Fourier transform, we obtain the expansion in terms of the integration kernel $\Phi_\alpha(x)$ function, as defined in Eq. (2.1) so we have

$$u(x,t) = \sum_{n=0}^{\infty}\Phi_{-2n}(x)(-K)^n e^{\gamma n}_{\rho,\mu n+1,\omega}(t). \tag{3.14}$$

We note that for $n \in \mathbb{N}$ we have according to [12]

$$\Phi_{-n} = \lim_{\alpha \to n} \Phi_{-\alpha}(x) = \delta^{(n)}(x). \tag{3.15}$$

Of course, both representations have a different range of validity. In the next section we will show that we can combine both representations to obtain the correct option pricing formula.

## 4. Applications to option pricing model

Applications of fractional calculus in option pricing started in connection with stable distributions, which can be naturally described by fractional diffusion equation with Riesz-Feller derivative. The first generalization of Black-Scholes model [5, 6] to the model with stable distributions was the Finite moment log-stable option pricing model introduced by Carr and Wu [7]. Later several other models were introduced that included time-fractional [13, 21, 10] derivatives, space-fractional derivatives, [8] space-time fractional derivatives [20, 1, 2, 3, 4] derivatives of fractional order [22], etc. It has been shown that the orders of fractional derivatives play the role of risk redistribution parameters in price and time [20].

Let us now consider an option $\mathcal{O}$, which is a function of the underlying asset price $S_t$, strike price $\mathcal{K}$, maturity time $\tau = T - t$ (and then other market parameters and parameters of the underlying distribution) which price is given by integration over the initial condition

$$\mathcal{O}(S_t, \mathcal{K}, \tau) = \int \mathrm{d}S_t \mathcal{O}(S_T, \mathcal{K}) u(S_T, 0 | S_t, \tau). \tag{4.1}$$

This can be rewritten as

$$\mathcal{O}(S_t, \mathcal{K}, \tau, r, q) = \int \mathrm{d}x F(S_t e^{(r+q)\tau + x}, \mathcal{K}) u(x, \tau), \tag{4.2}$$

where $r$ is the interest rate and $q$ is the risk-neutral factor. The reason for introduction of the risk neutral factor is transform from the log-returns, which are described by the diffusion equations to prices. We obtain it as the Radon-Nikodyn derivate of the equivalent martingale measure with respect to the original measure. The risk-neutral parameter can be expressed as:

$$q = -\log \mathbb{E}^{\mathbb{P}}[e^{S_{t-1}}]. \tag{4.3}$$

The expectation in definition (4.3) over the probability measure $\mathbb{P}$ is called Esscher's transform [9], which can be calculated in terms of its probability densities $u(x, \tau)$, that is [1]

$$q = -\log \int \exp(x) u(x, \tau = 1) dx. \tag{4.4}$$

We consider the fundamental solution in the following form:

$$
\begin{aligned}
u(x,t) &= c^{+}u^{+}(x,t)\xi_{x\geq 0}(x) + c^{-}u^{-}(x,t)\xi_{x<0}(x) \\
&= c^{+}\xi_{x\geq 0}(x)\sum_{n=0}^{\infty}\Phi_{-2n}(x)(-K)^{n}e^{\gamma n}_{\rho,\mu n+1,\omega}(t) \\
&+ c^{-}\xi_{x<0}(x)\sum_{n=0}^{\infty}\frac{|x|^{2n+1}}{(2n+1)!K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(t).
\end{aligned} \tag{4.5}
$$

Let us consider the European option, where $F(S_T,\mathcal{K}) = \max\{S_T-\mathcal{K},0\}$.

Let us divide the calculation to two parts, so $\mathcal{O} = c^{+}\mathcal{O}^{+} + c^{-}\mathcal{O}^{-}$, which we obtain from integration over positive or negative $x$. From the first part, we get

$$
\mathcal{O}^{+}(S_t,\mathcal{K},\tau,r,q,\mu,\gamma,\rho,\omega,K) = \sum_{n=0}^{\infty}\frac{\partial^{2n}F(S_T,\mathcal{K})}{\partial x^{2n}}|_{x=0}(-K)^{n}e^{\gamma n}_{\rho,\mu n+1,\omega}(\tau), \tag{4.6}
$$

$$
\begin{aligned}
\mathcal{O}^{+}(\mathcal{F},\mathcal{K},\tau,\mu,\gamma,\rho,\omega,K) &= \max\{\mathcal{F}-\mathcal{K},0\} \\
&+ \phi_{\mathcal{K}}(\mathcal{F})\sum_{n=1}^{\infty}(-K)^{n}e^{\gamma n}_{\rho,\mu n+1,\omega}(\tau),
\end{aligned} \tag{4.7}
$$

where $\mathcal{F} = S_t e^{(r+q)\tau}$, $\phi_{\mathcal{K}}(x) = x$ if $x > \mathcal{K}$ and $\phi_{\mathcal{K}}(x) = 0$ if $x < \mathcal{K}$.

For the second part, we get

$$
\begin{aligned}
\mathcal{O}^{-}(S_t,\mathcal{K},\tau,r,q,\mu,\gamma,\rho,\omega,K) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(\tau) \\
&\times \int_{-(r+q)\tau+\log(\mathcal{K}/S_t)}^{0}\mathrm{d}x|x|^{2n+1}\left(e^{(r+q)\tau+x}-K\right)
\end{aligned} \tag{4.8}
$$

which can be expressed as

$$
\begin{aligned}
\mathcal{O}^{-}(\mathcal{F},\mathcal{K},\tau,\mu,\gamma,\rho,\omega,K) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!K^{n+1}}e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(\tau) \\
&\times\left((\mathcal{F}/S_t)\gamma(2(n+1),\log(\mathcal{F}/\mathcal{K})) - \frac{\mathcal{K}\ (log(\mathcal{F}/\mathcal{K}))^{2(n+1)}}{2(n+1)}\right),
\end{aligned} \tag{4.9}
$$

where

$$
\gamma(s,x) = \int_0^x t^{s-1}\exp(-t)\mathrm{d}t
$$

denotes the incomplete Gamma function. Note that the constants $c^{\pm}$ have to be chosen in order to fit the data. Since $\mathcal{O}^{+}$ is linear in $S$ and $\mathcal{O}^{-}$ might not converge for certain values, the constants should be chosen carefully from the data to describe the prices accurately.

The risk-neutral factor can be calculated in the similar way. We express it in the following way:

$$\begin{aligned} q &= -\log\left(\int_{-\infty}^{0} \mathrm{d}x \exp(x) c^- u^-(x) + \int_0^{\infty} \mathrm{d}x \exp(x) c^+ u^+(x)\right) \\ &= -\log(c^- q^- + c^+ q^+), \end{aligned} \tag{4.10}$$

where

$$q^- = \sum_{n=0}^{\infty} \frac{1}{K^{n+1}} e^{-\gamma(n+1)}_{\rho,-\mu(n+1)+1,\omega}(1) \tag{4.11}$$

and

$$q^+ = \sum_{n=0}^{\infty} (-K)^n e^{\gamma n}_{\rho,\mu n+1,\omega}(1). \tag{4.12}$$

Example 3. Let us calculate the value of $q^+$ for a special configuration of parameters $\mu = 2\rho$, $\gamma = 1$ and $\omega^2 > 4K$. Then we get rewrite $q^+$ as

$$q^+ = \sum_{n=0}^{\infty} (-K)^n E^n_{\rho,2\rho n+1}(\omega) = 1 - K \sum_{n=0}^{\infty} (-K)^n E^{n+1}_{\rho,2\rho(n+1)+1}(\omega). \tag{4.13}$$

Here we can apply the straightforward formula

$$\sum_{n=0}^{\infty} (-xy)^n E^{n+1}_{\alpha,2\alpha n+\beta}(x+y) = \frac{x E_{\alpha,\beta}(x) - y E_{\alpha,\beta}(y)}{x-y} \quad (x \neq y), \tag{4.14}$$

where we identify $\alpha = \rho$, $\beta = 2\alpha + 1$, $xy = K$ and $x + y = \omega$. Thus, we get the expressions

$$x = \frac{\omega + \sqrt{\omega^2 - 4K}}{2} \tag{4.15}$$

$$y = \frac{\omega - \sqrt{\omega^2 - 4K}}{2} \tag{4.16}$$

and express $q^+$ as

$$\begin{aligned} q^+ = 1 - \frac{K}{\sqrt{\omega^2 - 4K}} &\left[\frac{\omega + \sqrt{\omega^2 - 4K}}{2} E_{\rho,2\rho+1}\left(\frac{\omega + \sqrt{\omega^2 - 4K}}{2}\right)\right. \\ &\left. - \frac{\omega - \sqrt{\omega^2 - 4K}}{2} E_{\rho,2\rho+1}\left(\frac{\omega - \sqrt{\omega^2 - 4K}}{2}\right)\right]. \end{aligned} \tag{4.17}$$

Example 4. Similarly, we can calculate the value of $q^-$ for the similar conditions, i.e., $\mu = 2\rho$, $\gamma = 1$. Going back to the Laplace image, we can write

$$\mathcal{L}[q^-](s) = \frac{1}{s} \sum_{n=0}^{\infty} \frac{1}{K^{n+1}} s^{2\rho(n+1)} (1 - \omega s^{-\rho})^{n+1}$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{s^{2\rho}-\omega s^{\rho}}{K}\right)^{n+1} = \frac{s^{2\rho-1}-\omega s^{\rho-1}}{\omega s^{\rho}-s^{2\rho}+K}\,. \tag{4.18}$$

The roots of the denominator can be calculated as

$$u = \frac{\omega+\sqrt{\omega^2+4K}}{2}, \tag{4.19}$$

$$v = \frac{\omega-\sqrt{\omega^2+4K}}{2}, \tag{4.20}$$

so we can write

$$\mathcal{L}[q^-](s) = \frac{1}{s} - \frac{uv}{u-v}\left(\frac{s^{-1}}{s^{\rho}-u} - \frac{s^{-1}}{s^{\rho}-v}\right). \tag{4.21}$$

With the inverse Laplace transform, we finally get

$$q^- = \frac{uE_\rho(v)-vE_\rho(u)}{u-v}$$
$$= \frac{\frac{\omega+\sqrt{\omega^2+4K}}{2}E_\rho\left(\frac{\omega-\sqrt{\omega^2+4K}}{2}\right) - \frac{\omega-\sqrt{\omega^2+4K}}{2}E_\rho\left(\frac{\omega+\sqrt{\omega^2+4K}}{2}\right)}{\sqrt{\omega^2+4K}}\,. \tag{4.22}$$

## 5. **Conclusions**

In this paper, we have calculated fundamental solutions of diffusion equations with Hilfer-Prabhakar time-fractional derivative and applied them to the problem of option pricing. We have shown two representations of the fundamental solutions, one expressed in a series of $x^n$, one in a series of $\delta^{(n)}(x)$. Each representation has its range of validity. Therefore, it is convenient to describe the resulting fundamental solution as a general linear combination of both representations. Since fractional calculus also appears in other parts of finance [18, 14, 24, 30] it will also be natural to consider the Hilfer-Prabhakar derivative in other financial applications. The parameters of Hilfer-Prabhakar derivative enable one to model a large scale of different aspects of financial processes, including scaling, large jumps, or different types of memory effects.

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