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Connection formulas for Askey–Wilson polynomials and related expansions

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Dedicated to Krishna Alladi on the occasion of his 70 th birthday

Abstract
We derive and investigate expansions involving Askey–Wilson polynomials. We study these expansions and examine some limits to the continuous dual q-Hahn, Al-Salam–Chihara, continuous big q-Hermite and continuous q-Hermite polynomials and their q^{-1}-analogues. The Poisson kernel for the infinite discrete orthogonality relation for the q^{-1}-Al-Salam–Chihara polynomials is derived which in a special case reduces to the Gupta–Masson biorthogonal rational ${}_4\phi_3$-functions. This Poisson kernel implies new infinite series connection relations for the Askey–Wilson polynomials involving these rational ${}_4\phi_3$-functions. We also consider various interesting limits.

Keywords Basic hypergeometric series · Askey–Wilson polynomials

Mathematics Subject Classification 33D15 · 33D50

1 Introduction

The Askey–Wilson polynomials [1] form a family of orthogonal polynomials that are on the top-level of the Askey-scheme of q-hypergeometric orthogonal polynomials, implying that they contain many well-known families of orthogonal polynomials as

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special cases or limit cases. This makes them a central object in the study of special functions. This paper contributes to this study by considering several summation formulas involving Askey–Wilson polynomials and some of their limit cases.

Askey–Wilson polynomials are defined explicitly as

$$p_n(x;a,b,c,d|q) = a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right),$$
$$x = \frac{1}{2}(z+z^{-1}),$$

where we use the usual notations for q-shifted factorials and q-hypergeometric series. Clearly, the polynomials p_n depend, besides the 'base' parameter q, on the four parameters a, b, c, d. A fundamental result for Askey–Wilson polynomials is the connection formula

$$p_n(x;a,b,c,d|q) = \sum_{k=0}^{n} c_{k,n}\, p_k(x;e,f,g,h|q),$$

relating the family of polynomials depending on a, b, c, d to the polynomials depending on different parameters e, f, g, h. Askey and Wilson computed [1] the connection coefficients $c_{k,n}$ in case $d = h$ as balanced ${}_5\phi_4$-series. Ismail and Zhang [11, Theorem 3.2] obtained a formula for connection coefficients in the general case as a double sum. In this paper we obtain an explicit expression as balanced ${}_4\phi_3$-functions for the coefficients in the slightly adjusted connection problem

$$\frac{p_n(x;a,b,c,d|q)}{(cz,cz^{-1};q)_\infty} = \sum_{k=0}^{\infty} C_{k,n}\, \frac{p_k(x;a,b,e,f|q)}{(ez,ez^{-1};q)_\infty}. \tag{1}$$

To compute the coefficients $C_{k,n}$ we first evaluate a Poisson kernel for Al-Salam–Chihara polynomials in base q^{-1}. Then using a summation formula which relates q^{-1}-Al-Salam–Chihara polynomials to Askey–Wilson polynomials it follows that a special case of the Poisson kernel gives the connection coefficients. Furthermore, from orthogonality relations for q^{-1}-Al-Salam–Chihara polynomials it follows that the coefficients $C_{k,n}$ can be considered as biorthogonal functions in k (or n). This leads to another proof of biorthogonality relations for ${}_4\phi_3$-functions obtained by Gupta and Masson [8].

The motivation for looking at these summations comes from their appearance in the representation theory of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, e.g., see [6, 7]. In [6] the Askey–Wilson polynomials, including the denominator factors in (1), appear as overlap coefficients between bases $\{v_n\}$ and $\{\tilde{v}_n\}$ of the representation space. These two bases diagonalize two different operators in $\mathcal{U}_q(\mathfrak{sl}_2)$ and can be written in terms of Al-Salam–Chihara polynomials in base q and q^{-1}. This goes back to Rosengren's work [14] on generalized group elements in $\mathcal{U}_q(\mathfrak{sl}_2)$. The connection formula (1) can be interpreted in this setting as a result of computing overlap coefficients in two different ways: directly using the bases $\{v_n\}$ and $\{\tilde{v}_n\}$, or in two steps by first computing overlap coefficients between $\{v_n\}$ and a natural 'in between' basis $\{\tilde{v}'_n\}$ and then between $\{\tilde{v}'_n\}$

and $\{\tilde{v}_n\}$. Computing overlap coefficients between $\{\tilde{v}'_n\}$ and $\{\tilde{v}_n\}$ requires an evaluation of the Poisson kernel mentioned above. We stress that the approach in the current paper does not rely anywhere on knowledge of the algebraic interpretations.

The paper is organized as follows. In this paper we consider expansions using Askey–Wilson polynomials. In Sect. 2 we introduce various mathematical concepts which are required in order to read the paper. In Sect. 3 we consider certain interesting limits and specializations of the nonstandard generating function for Askey–Wilson polynomials which was originally published in [5, Theorem 2.1]. In Sect. 4 we consider a Poisson kernel for the q^{-1}-Al-Salam–Chihara polynomials and its limit to a Poisson kernel for the continuous big q^{-1}-Hermite polynomials. In Sect. 5, we utilize a bilinear sum over a product of Al-Salam–Chihara and q^{-1}-Al-Salam–Chihara polynomials originally presented in [6] along with an orthogonality relation for q^{-1}-Al-Salam–Chihara polynomials in order to obtain in Lemma 5.1, a bilinear sum of the Al-Salam–Chihara polynomials in terms of a product of Askey–Wilson polynomials and q^{-1}-Al-Salam–Chihara polynomials. We also compute some interesting limits of Lemma 5.1. In Sect. 6 we utilize Lemma 5.1 in order to derive two connection relations for the Askey–Wilson polynomials in Theorem 6.1 whose coefficients are given in terms of ${}_4\phi_3$-functions. We also consider interesting limits of these connection relations. In Sect. 7 we focus on the relation between the connection coefficients and Gupta and Masson's biorthogonal rational ${}_4\phi_3$-functions. Among other things, we obtain in Corollary 7.2 an integral representation for the rational ${}_4\phi_3$-functions which generalizes the well-known Askey–Wilson orthogonality relations.

2 Preliminaries

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, and we use the set $\mathbb{C}$ which represents the complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$, $\mathbb{C}^\ddagger := \mathbb{C} \setminus (\{0\} \cup \{z \in \mathbb{C}^* : |z| = 1\})$.

Definition 2.1 Throughout this paper we adopt the following conventions for succinctly writing elements of lists. To indicate sequential positive and negative elements, we write

$$\pm a := \{a, -a\}.$$

We also adopt an analogous multiset notation

$$z^{\pm} := z^{\pm 1} := \{z, z^{-1}\}.$$

Consider $q \in \mathbb{C}^\dagger$. Define for $n \in \mathbb{N}_0$ the sets $\Omega_q^n := \{q^{-k} : k \in \mathbb{N}_0,\ 0 \le k \le n-1\}$, $\Omega_q := \Omega_q^\infty = \{q^{-k} : k \in \mathbb{N}_0\}$. In order to obtain our derived identities, we rely on properties of the q-Pochhammer symbol (a.k.a. the q-shifted factorial). For any $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a \in \mathbb{C}$, the q-shifted factorial is defined as

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \in \mathbb{N}_0, \quad \text{and } (a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

We will also use, for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0 \cup \{\infty\}$, the common notational product convention

$$(a_1, ..., a_k; q)_n := (a_1; q)_n \cdots (a_k; q)_n.$$

For $n \in \mathbb{C}$, one has [4, (I.5)]

$$(q^n a; q)_\infty = \frac{(a; q)_\infty}{(a; q)_n}. \tag{2}$$

Basic hypergeometric series, which we will often use, are defined for $q \in \mathbb{C}^\dagger$, $z \in \mathbb{C}^*$, such that $s, r \in \mathbb{N}_0$, $b_j \notin \Omega_q$, $j = 1, ..., s$, as [12, (1.10.1)]

$${}_r\phi_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(q, b_1, ..., b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k. \tag{3}$$

where for $r = s+1$ we assume $|z| < 1$ for convergence, if the series does not terminate. (For $r \leq s$ the series converges for any z, due to the quadratic powers of q appearing in the terms of the series.) We refer to a basic hypergeometric series as balanced (or Saalschützian) if $r = s + 1$ and $qa_1 \cdots a_r = b_1 \cdots b_s$. In the sequel, we will use the following notation ${}_{r+1}\phi_s^m$, $m \in \mathbb{Z}$ (originally due to van de Bult and Rains [15, p. 4]), for basic hypergeometric series with zero parameter entries. Consider $p \in \mathbb{N}_0$. Then define

$${}_{r+1}\phi_s^{-p}\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := {}_{r+p+1}\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1}, \overbrace{0, \ldots, 0}^{p} \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right), \tag{4}$$

$${}_{r+1}\phi_s^{p}\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := {}_{r+1}\phi_{s+p}\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_s, \underbrace{0, \ldots, 0}_{p} \end{matrix}; q, z\right), \tag{5}$$

where $b_1, \ldots, b_s \notin \Omega_q \cup \{0\}$, and ${}_{r+1}\phi_s^0 := {}_{r+1}\phi_s$. The nonterminating basic hypergeometric series ${}_{r+1}\phi_s^m(\mathbf{a}; \mathbf{b}; q, z)$, $\mathbf{a} := \{a_1, \ldots, a_{r+1}\}$, $\mathbf{b} := \{b_1, \ldots, b_s\}$, is well-defined for $s - r + m \geq 0$. In particular ${}_{r+1}\phi_s^m$ is an entire function of z for $s-r+m > 0$, convergent for $|z| < 1$ for $s-r+m = 0$ and divergent if $s-r+m < 0$. Note that we will move interchangeably between the van de Bult & Rains notation and the alternative notation with vanishing numerator and denominator parameters which are used on the right-hand sides of (4) and (5).

One has the following nonterminating transformation between a ${}_2\phi_2$ and a ${}_2\phi_1$ cf. [4, (III.4)]

$${}_2\phi_2\left(\begin{matrix} a, b \\ c, \frac{abz}{c} \end{matrix}; q, z\right) = \frac{(\frac{bz}{c}; q)_\infty}{(\frac{abz}{c}; q)_\infty} {}_2\phi_1\left(\begin{matrix} a, \frac{c}{b} \\ c \end{matrix}; q, \frac{bz}{c}\right). \tag{6}$$

Another important nonterminating transformation that we will use is [13, (17.9.13)]

$$
{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right)=\frac{(\frac{e}{b},\frac{e}{c};q)_\infty}{(e,\frac{e}{bc};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{d}{a},b,c\\ d,\frac{qbc}{e}\end{matrix};q,q\right)
+\frac{(\frac{d}{a},b,c,\frac{de}{bc};q)_\infty}{(d,e,\frac{bc}{e},\frac{de}{abc};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{e}{b},\frac{e}{c},\frac{de}{abc}\\ \frac{de}{bc},\frac{qe}{bc}\end{matrix};q,q\right). \tag{7}
$$

Another important nonterminating transformation formula that we will use is Bailey's transformation of a nonterminating very-well-poised ${}_8W_7$ to a sum of two nonterminating balanced ${}_4\phi_3$'s [13, (17.9.16)]

$$
{}_8W_7\left(a;b,c,d,e,f;q,\frac{q^2a^2}{bcdef}\right)=\frac{(qa,\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef};q)_\infty}{(\frac{qa}{def},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{qa}{bc},d,e,f\\ \frac{qa}{b},\frac{qa}{c},\frac{def}{a}\end{matrix};q,q\right)
+\frac{(qa,\frac{q^2a^2}{bdef},\frac{q^2a^2}{cdef},\frac{qa}{bc},d,e,f;q)_\infty}{(\frac{q^2a^2}{bcdef},\frac{def}{qa},\frac{qa}{b},\frac{qa}{c},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f};q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\frac{q^2a^2}{bcdef},\frac{qa}{de},\frac{qa}{df},\frac{qa}{ef}\\ \frac{q^2a^2}{bdef},\frac{q^2a^2}{cdef},\frac{q^2a}{def}\end{matrix};q,q\right), \tag{8}
$$

where $\frac{qa}{b},\frac{qa}{c},\frac{qa}{d},\frac{qa}{e},\frac{qa}{f},\frac{qa}{def},\frac{def}{qa},\frac{q^2a^2}{bcdef}\notin\Omega_q$. Let $p\in\mathbb{Z}$. Consider the nonterminating very-well-poised basic hypergeometric series with vanishing denominator parameters ${}_{r+1}W_r^p$ [15, p. 4]

$$
{}_{r+1}W_r^p(a;a_4,\ldots,a_{r+1};q,z):={}_{r+1}\phi_r^p\left(\begin{matrix}a,\pm q\sqrt{a},a_4,\ldots,a_{r+1}\\ \pm\sqrt{a},\frac{qa}{a_4},\ldots,\frac{qa}{a_{r+1}}\end{matrix};q,z\right), \tag{9}
$$

where $\sqrt{a},\frac{qa}{a_4},\ldots,\frac{qa}{a_{r+1}}\notin\Omega_q$. Recognize that ${}_{r+1}W_r^0={}_{r+1}W_r$. Also note that in the case where $p<0$, then the nonterminating basic hypergeometric series is divergent, so we will not consider this case. As pointed out in [15, p. 4], usually one cannot obtain the nonterminating very-well-poised series with vanishing denominator parameters as a limit from very-well-poised series. However in cases we will treat below, this limit will be well-defined. This fact leads to some interesting nonterminating transformation formulas for these functions.

Theorem 2.2 *Let $q\in\mathbb{C}^\dagger$, $a,b,c,d,e\in\mathbb{C}^*$. Then*

$$
{}_7W_6^1\left(a;b,c,d,e;q,\frac{q^2a^2}{bcde}\right)=\frac{(qa,\frac{qa}{de};q)_\infty}{(\frac{qa}{d},\frac{qa}{e};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{qa}{bc},d,e\\ \frac{qa}{b},\frac{qa}{c}\end{matrix};q,\frac{qa}{de}\right), \tag{10}
$$

$$
\begin{aligned}
{}_6W_5^2\left(a;b,c,d;q,\frac{q^2a^2}{bcd}\right)&=\frac{(qa;q)_\infty}{(\frac{qa}{d};q)_\infty}\,{}_2\phi_2\left(\begin{matrix}\frac{qa}{bc},d\\ \frac{qa}{b},\frac{qa}{c}\end{matrix};q,\frac{qa}{d}\right)\\
&=\frac{(qa,\frac{qa}{bd};q)_\infty}{(\frac{qa}{b},\frac{qa}{d};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}b,d\\ \frac{qa}{c}\end{matrix};q,\frac{qa}{bd}\right),
\end{aligned} \tag{11}
$$

where the identities hold wherever the displayed series converge and the expressions are defined, and elsewhere by meromorphic continuation.

Proof For (10), start with Bailey's transformation of a nonterminating very-well-poised ${}_8W_7$ to a sum of two nonterminating balanced ${}_4\phi_3$'s (8) and take the limit $b \to \infty$ (or $c \to \infty$). Then replace $f \mapsto b$ (or $f \mapsto c$), and apply the three-term nonterminating transformation for a ${}_3\phi_2$ (7). For (11), start with (10), taking the limit as $e \to \infty$ (or $d \to \infty$) produces the ${}_2\phi_2$ representation. Taking the limit as $c \to \infty$ (or $b \to \infty$) and then replacing variables accordingly so that b, c, d remain, produces the result. □

Two useful terminating summations that we will use are the terminating q-binomial theorem, namely [12, (1.11.2)]

$$ {}_1\phi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}; q, z\right) = (q^{-n}z; q)_n, \tag{12} $$

and the q-Chu–Vandermonde sum [13, (17.6.3)]

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ b \end{matrix}; q, q\right) = \frac{a^n(\frac{b}{a}; q)_n}{(b; q)_n}. \tag{13} $$

2.1 The Askey–Wilson polynomials and their symmetric subfamilies

Below we present definitions for orthogonal polynomials we are using here. All basic hypergeometric orthogonal polynomials which we use are adopted from the definitions in Koekoek et al. [12]. The Askey–Wilson polynomials [9, Chap. 15] (see also [10, Chap. 8]) form the most general family of orthogonal polynomials in x that are also basic hypergeometric series. They contain, in addition to the variable x and the base q, four extra parameters a, b, c, d. The n-th degree Askey–Wilson polynomial is denoted by $p_n(x; a, b, c, d|q)$. These polynomials are symmetric in the four parameters a, b, c, d and have the following terminating balanced ${}_4\phi_3$ basic hypergeometric series representation [3, (15)] (with obvious additional ones corresponding to the symmetries in the four parameters). Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^{\ddagger}$, $x = \frac{1}{2}(z + z^{-1})$, $z, a, b, c, d \in \mathbb{C}^*$. Then the Askey–Wilson polynomials can be defined as follows [3, Theorem 7]

$$ p_n(x; a, b, c, d|q) = a^{-n}(ab, ac, ad; q)_n \, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az^{\pm} \\ ab, ac, ad \end{matrix}; q, q\right). \tag{14} $$

Using Sears' transformations [4, (III.15), (III.16)], several other ${}_4\phi_3$-expressions for the Askey–Wilson polynomials can be obtained.

Taking limits as parameters tend to 0 gives several subfamilies of the Askey–Wilson polynomials. The continuous dual q-Hahn polynomials can be given by

$$ p_n(x; a, b, c|q) := a^{-n}(ab, ac; q)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, az^{\pm} \\ ab, ac \end{matrix}; q, q\right). \tag{15} $$

The Al-Salam–Chihara polynomials are given by

$$Q_n(x;a,b|q) := a^{-n}(ab;q)_n \, {}_3\phi_1^1\left(\begin{matrix} q^{-n}, az^{\pm} \\ ab \end{matrix}; q, q\right) \tag{16}$$

$$= z^n(bz^{-1};q)_n \, {}_2\phi_1\left(\begin{matrix} q^{-n}, az \\ \frac{q^{1-n}}{b}z \end{matrix}; q, \frac{q}{bz}\right). \tag{17}$$

The continuous big q-Hermite polynomials are given by

$$H_n(x;a|q) := a^{-n} \, {}_3\phi_0^2\left(\begin{matrix} q^{-n}, az^{\pm} \\ - \end{matrix}; q, q\right) \tag{18}$$

$$= z^n \, {}_2\phi_0\left(\begin{matrix} q^{-n}, az \\ - \end{matrix}; q, \frac{q^n}{z^2}\right). \tag{19}$$

Replacing q by q^{-1} in the Askey–Wilson polynomials does not essentially change the polynomials. However, for the above given subfamilies of polynomials replacing q by q^{-1} leads to a different class of orthogonal polynomials. We also list the corresponding q^{-1}-subfamilies of polynomials.

The continuous dual q^{-1}-Hahn polynomials can be obtained from the Askey–Wilson polynomials as follows

$$p_n(x;a,b,c|q^{-1}) = q^{-3\binom{n}{2}}(-abc)^n \lim_{d\to 0} d^n \, p_n(x;\tfrac{1}{a},\tfrac{1}{b},\tfrac{1}{c},\tfrac{1}{d}|q). \tag{20}$$

Then the continuous dual q^{-1}-Hahn polynomials are given by

$$p_n(x;a,b,c|q^{-1}) = q^{-2\binom{n}{2}}(abc)^n \left(\frac{1}{ab},\frac{1}{ac};q\right)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{z^{\pm}}{a} \\ \frac{1}{ab}, \frac{1}{ac} \end{matrix}; q, \frac{q^n}{bc}\right). \tag{21}$$

The q^{-1}-Al-Salam–Chihara polynomials are given by

$$Q_n(x;a,b|q^{-1}) = q^{-\binom{n}{2}}(-b)^n \left(\frac{1}{ab};q\right)_n \, {}_3\phi_1\left(\begin{matrix} q^{-n}, \frac{z^{\pm}}{a} \\ \frac{1}{ab} \end{matrix}; q, \frac{q^n a}{b}\right). \tag{22}$$

We will mostly be interested in these polynomials in case $x = \frac{1}{2}(z+z^{-1})$ with $z = q^{-m}a$, $m \in \mathbb{N}_0$. In this case we denote the polynomials by $Q_n[q^{-m}a;a,b|q^{-1}]$, and we have

$$Q_n[q^{-m}a;a,b|q^{-1}] = q^{-\binom{n}{2}}q^{-\binom{m}{2}}\left(-\frac{a}{qb}\right)^m (-b)^n \frac{(\frac{1}{ab};q)_n(\frac{qb}{a};q)_m}{(\frac{1}{ab};q)_m}$$

$$\times \, {}_2\phi_1\left(\begin{matrix} q^{-m}, \frac{q^m}{a^2} \\ \frac{qb}{a} \end{matrix}; q, q^{n+1}\right). \tag{23}$$

Note that the ${}_2\phi_1$-function is a polynomial in q^n of degree m, which can be identified as a little q-Jacobi polynomial. The q^{-1}-Al-Salam–Chihara polynomials have the following generating function

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q;q)_n} Q_n(x;a,b|q^{-1}) = \frac{(-tz^{\pm};q)_\infty}{(-ta,-tb;q)_\infty}. \tag{24}$$

If one chooses $z = q^{-m}a$, then one has

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q;q)_n} Q_n[q^{-m}a;a,b|q^{-1}] = q^{-\binom{m}{2}} \frac{(-\frac{t}{a};q)_\infty}{(-bt;q)_\infty} \frac{(-\frac{q}{at};q)_m}{(-\frac{t}{a};q)_m} \left(\frac{at}{q}\right)^m. \tag{25}$$

This generating function will be useful below. The continuous big q^{-1}-Hermite polynomials are given by

$$H_n(x;a|q^{-1}) = a^{-n}\, {}_3\phi_0\left(\begin{matrix} q^{-n}, \frac{z^{\pm}}{a} \\ - \end{matrix}; q, q^n a^2\right). \tag{26}$$

Similar as for the q^{-1}-Al-Salam–Chihara polynomials we use the notation $H_n[aq^{-m};a|q^{-1}]$ for $H_n(x;a|q^{-1})$ with $x = \frac{1}{2}(aq^{-m} + a^{-1}q^m)$.

2.2 Connection coefficients for Askey–Wilson polynomials

The Askey–Wilson polynomials have the following connection relation due to Ismail and Zhang [11, Theorem 3.2] (see also [9, Theorem 16.4.2]),

$$\begin{aligned} p_n(x;a,b,c,d|q) &= (q,ad,bd,cd;q)_n \\ &\times \sum_{k=0}^{n} \frac{p_k(x;e,f,g,h|q)(q^k d)^{k-n}(q^{n-1}abcd;q)_k}{(q;q)_{n-k}(q,ad,bd,cd,q^{k-1}efgh;q)_k} \\ &\times \sum_{l=0}^{n-k} \left(\frac{qd}{h}\right)^l \frac{(q^{k-n},q^{n+k-1}abcd,q^k eh,q^k fh,q^k gh;q)_l}{(q,q^{2k}efgh,q^k ad,q^k bd,q^k cd;q)_l} \\ &\times {}_4\phi_3\left(\begin{matrix} q^{-(n-k-l)}, q^{n+k+l-1}abcd, q^{k+l}dh, \frac{d}{h} \\ q^{k+l}ad, q^{k+l}bd, q^{k+l}cd \end{matrix}; q, q\right). \end{aligned} \tag{27}$$

This connection coefficient simplifies to a ${}_5\phi_4$ in the case $d = h$, and this result was first given in the Askey–Wilson memoir [1, §6]. In the one free parameter case $(f,g,h) = (b,c,d)$ one obtains a ${}_3\phi_2$ which can be summed using q-Pfaff–Saalschütz

[13, (17.7.5)]. The result is as follows

$$p_n(x;a,b,c,d|q) = \frac{(q,ab,ac,bc;q)_n}{(abce;q)_n}\sum_{k=0}^{n} p_k(x;a,b,c,e|q) \times \frac{e^{n-k}(\frac{d}{e};q)_{n-k}(\frac{abce}{q},q^{n-1}abcd;q)_k(abce;q)_{2k}}{(q;q)_{n-k}(q,ab,ac,bc,q^n abce;q)_k(\frac{abce}{q};q)_{2k}}. \quad (28)$$

3 Nonstandard generating function for Askey–Wilson polynomials

In [5, Theorem 2.1], Groenevelt derived the following nonstandard generating function for Askey–Wilson polynomials, which can be considered as a generalization of the connection relation for Askey–Wilson polynomials (28) in case $(e, f, g) = (a, b, c)$, see [5, Remark 2.2],

$$\sum_{n=0}^{\infty} \frac{t^n\,(abcd;q)_{2n}(\frac{abcd}{q},\frac{r}{t},\frac{abc}{r};q)_n\,p_n(x;a,b,c,d|q)}{(\frac{abcd}{q};q)_{2n}(q,ab,ac,bc,\frac{abcdt}{r},dr;q)_n} = \frac{(abcd,dt,\frac{abctz}{r},rz;q)_\infty}{(\frac{abcdt}{r},dr,abcz,tz;q)_\infty}\,{}_8W_7\left(\frac{abcz}{q};az,bz,cz,\frac{r}{t},\frac{abc}{r};q,\frac{t}{z}\right), \quad (29)$$

where $|t| < \min\{|z|, |z|^{-1}\}$.

There are some interesting limits of this nonstandard generating function. First we consider the limits which give nonstandard generating functions over Askey–Wilson polynomials. First there is the limit as $t \to 0$.

Corollary 3.1 *Let $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $a, b, c, d, t \in \mathbb{C}^*$, $|t| < |c|$. Then*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}t^n(abcd;q)_{2n}(\frac{abcd}{q},-\frac{abc}{t};q)_n}{(\frac{abcd}{q};q)_{2n}(q,ab,ac,bc,-dt;q)_n}\,p_n(x;a,b,c,d|q) = \frac{(abcd,-\frac{t}{c};q)_\infty}{(ab,-dt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}-\frac{abc}{t},cz^{\pm}\\ ac,bc\end{matrix};q,-\frac{t}{c}\right). \quad (30)$$

Proof Start with (29) and take the limit as $t \to 0$. Then replace $r \mapsto -r$. Finally taking advantage of the nonterminating transformation (10) and replacing $r \mapsto t$ completes the proof. □

Another interesting limit to Askey–Wilson polynomials is the limit as $t \to 0$ in the above nonstandard generating function. This produces the following result.

Corollary 3.2 *Let* $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}$, $a, b, c, d \in \mathbb{C}^*$, $|c| < \min\{|z|, |z|^{-1}\}$. *Then*

$$\sum_{n=0}^{\infty} \frac{q^{2\binom{n}{2}}(abc)^n(abcd;q)_{2n}(\frac{abcd}{q};q)_n}{(\frac{abcd}{q};q)_{2n}(q,ab,ac,bc;q)_n} p_n(x;a,b,c,d|q) = \frac{(abcd,\frac{c}{z};q)_\infty}{(ac,bc;q)_\infty} {}_2\phi_1\left(\begin{matrix} az, bz \\ ab \end{matrix}; q, \frac{c}{z}\right). \tag{31}$$

Proof Start with Corollary 3.1 and take the limit as $t \to 0$. This produces a ${}_2\varphi_2$ on the right-hand side which can be converted to a ${}_2\varphi_1$ using the nonterminating transformation (6). □

There are also some interesting limits of the nonstandard generating function for Askey–Wilson polynomials which generate nonstandard generating functions for continuous dual q-Hahn polynomials. For instance, the $a \to 0$ limit gives the following result.

Corollary 3.3 *Let* $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}$, $t, r, a, b, c \in \mathbb{C}^*$, $|t| < \min\{|z|, |z|^{-1}\}$. *Then*

$$\sum_{n=0}^{\infty} \frac{t^n\,(\frac{r}{t};q)_n\, p_n(x;a,b,c|q)}{(q,bc,ar;q)_n} = \frac{(at,rz;q)_\infty}{(ar,tz;q)_\infty} {}_3\phi_2\left(\begin{matrix} bz, cz, \frac{r}{t} \\ bc, rz \end{matrix}; q, \frac{t}{z}\right). \tag{32}$$

Proof Start with (29) and take the limit as $a \to 0$ followed by replacing $d \mapsto a$. □

There is also the limit as $t \to 0$ of the above result.

Corollary 3.4 *Let* $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}$, $t, a, b, c \in \mathbb{C}^*$, $|t| < |b|$. *Then*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}t^n\, p_n(x;a,b,c|q)}{(q,bc,-at;q)_n} = \frac{(-tz;q)_\infty}{(-at;q)_\infty} {}_2\phi_1\left(\begin{matrix} bz^{\pm} \\ bc \end{matrix}; q, -\frac{t}{b}\right). \tag{33}$$

Proof Start with Corollary 3.3, take the limit as $t \to 0$, and then replace $r \mapsto -t$. This completes the proof. □

One also has a nice $d \to 0$ limit for the nonstandard Askey–Wilson generating function.

Corollary 3.5 *Let* $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}^*$, $t, r, a, b, c \in \mathbb{C}^*$, $|t| < \min\{|z|, |z|^{-1}\}$. *Then*

$$\sum_{n=0}^{\infty} \frac{t^n(\frac{r}{t},\frac{abc}{r};q)_n\, p_n(x;a,b,c|q)}{(q,ab,ac,bc;q)_n} = \frac{(\frac{abctz}{r},rz;q)_\infty}{(abcz,tz;q)_\infty} \times {}_8W_7\left(\frac{abcz}{q}; az, bz, cz, \frac{r}{t}, \frac{abc}{r}; q, \frac{t}{z}\right). \tag{34}$$

Proof Start with (29) and take the limit as $d \to 0$. □

The $d \to 0$ limit followed by the $t \to 0$ limit is also interesting.

Corollary 3.6 *Let* $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $t, a, b, c \in \mathbb{C}^*$, $|t| < |c|$. *Then*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n (-\frac{abc}{t}; q)_n \, p_n(x; a, b, c|q)}{(q, ab, ac, bc; q)_n} = \frac{(-\frac{t}{c}; q)_\infty}{(ab; q)_\infty} \, {}_3\phi_2\left(\begin{matrix} -\frac{abc}{t}, cz^{\pm} \\ ac, bc \end{matrix} ; q, -\frac{t}{c} \right). \tag{35}$$

Proof Start with Corollary 3.5 and then take the $t \to 0$ limit followed by replacing $r \mapsto -t$. □

4 Poisson kernel for q^{-1}-Al-Salam–Chihara polynomials

One can derive a bilinear sum of a product of two q^{-1}-Al-Salam–Chihara polynomials which is a Poisson kernel. See also [7, Lemma 2.1] where the following result is derived.

Theorem 4.1 *Let* $m, m' \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b, c, d \in \mathbb{C}^*$, $t \in \mathbb{C}$ *such that* $|bdt| < 1$. *Then*

$$\begin{aligned} P_{m,m'}(t; a, b, c, d) &:= \sum_{n=0}^{\infty} \frac{q^{2\binom{n}{2}} t^n}{(q, \frac{1}{cd}; q)_n} Q_n[q^{-m} a; a, b|q^{-1}] Q_n[q^{-m'} c; c, d|q^{-1}] \\ &= q^{-\binom{m}{2}} q^{-\binom{m'}{2}} \left(-\frac{c}{qd}\right)^{m'} \left(-\frac{adt}{q}\right)^{m} \frac{(\frac{dt}{a}; q)_\infty}{(bdt; q)_\infty} \frac{(\frac{q}{adt}; q)_m}{(\frac{dt}{a}; q)_m} \frac{(\frac{qd}{c}; q)_{m'}}{(\frac{1}{cd}; q)_{m'}} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} q^{-m'}, \frac{q^{m'}}{c^2}, adt, bdt \\ q^{-m} adt, q^m \frac{dt}{a}, \frac{qd}{c} \end{matrix} ; q, q \right). \end{aligned} \tag{36}$$

Proof Start with the bilinear sum for $P_{m,m'}(t; a, b, c, d) := P_{m,m'}(t)$. From (23) it follows that

$$Q_n[q^{-m} a; a, b|q^{-1}] = \mathcal{O}(b^n q^{-\binom{n}{2}})$$

for $n \to \infty$, so that the sum converges absolutely for $|bdt| < 1$. Then replace one of the q^{-1}-Al-Salam–Chihara polynomials with its ${}_2\phi_1$ representation using (23). Replace the ${}_2\phi_1$ with its infinite series representation and reverse the order of the sums. This produces

$$P_{m,m'}(t) = q^{-\binom{m'}{2}} \left(-\frac{c}{qd}\right)^{m'} \frac{(\frac{qd}{c};q)_{m'}}{(\frac{1}{cd};q)_{m'}} \sum_{k=0}^{\infty} q^k \frac{(q^{-m'}, \frac{q^{m'}}{c^2};q)_k}{(q, \frac{qd}{c};q)_k} \times \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-q^k dt)^n}{(q;q)_n} Q_n[q^{-m}a;a,b|q^{-1}], \tag{37}$$

in which the sum over n is a rescaled form of the generating function (25). Inserting this result in (37) completes the proof. □

Remark 4.2 Note that the ${}_4\phi_3$ in Theorem 4.1 becomes balanced if $ab = cd$. In this case there is an obvious symmetry: $P_{m,m'}(t;a,b,c,d) = P_{m',m}(t;c,d,a,b)$.

If you scale the variable t and then take the limit as $b, d \to 0$, one can obtain a corresponding bilinear sum for continuous big q^{-1}-Hermite polynomials.

Corollary 4.3 *Let $m, m' \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b \in \mathbb{C}^*$, $t \in \mathbb{C}$. Then*

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} t^n}{(q;q)_n} H_n[q^{-m}a;a|q^{-1}] H_n[q^{-m'}b;b|q^{-1}] = q^{-\binom{m}{2}} q^{-2\binom{m'}{2}} \left(\frac{b^2}{q}\right)^{m'} \left(\frac{at}{qb}\right)^m (-\tfrac{t}{ab};q)_\infty \frac{(-\frac{qb}{at};q)_m}{(-\frac{t}{ab};q)_m} \times {}_3\phi_2\left(\begin{matrix} q^{-m'}, \frac{q^{m'}}{b^2}, -\frac{at}{b} \\ -q^{-m}\frac{at}{b}, -q^m\frac{t}{ab} \end{matrix}; q, q\right). \tag{38}$$

Proof Evaluate $P_{m,m'}(-\frac{t}{cd};a,b,c,d)$ using Theorem 4.1. Then take the limit as $b, d \to 0$. Finally replacing $c \mapsto b$ completes the proof. □

5 Expansion of Al-Salam–Chihara polynomials as a bilinear sum over Askey–Wilson polynomials and q^{-1}-Al-Salam–Chihara polynomials

An expansion of the Askey–Wilson polynomial in terms of Al-Salam–Chihara and q^{-1}-Al-Salam–Chihara polynomials, was given by Groenevelt [6, Lemma 4.6], namely

$$p_n(x;a,b,c,d|q) = q^{\binom{n}{2}}(-d)^n \frac{(cz^{\pm};q)_\infty (ab,ac,bc;q)_n}{(ac,bc;q)_\infty} \times \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(-1)^k}{(q,ab;q)_k} \left(\frac{qabc}{d}\right)^{\frac{k}{2}} Q_k\left[\frac{q^{\frac{1}{2}-n}}{\sqrt{abcd}}; \frac{\sqrt{q}}{\sqrt{abcd}}, \frac{\sqrt{cd}}{\sqrt{qab}} \middle| q^{-1}\right] Q_k(x;a,b|q). \tag{39}$$

Furthermore, for $0 < q < 1$, $ab > 1$ and $qb < a$ the q^{-1}-Al-Salam–Chihara

polynomials satisfy the following orthogonality relation

$$\sum_{m=0}^{\infty} Q_k[q^{-m}a;a,b|q^{-1}]Q_{k'}[q^{-m}a;a,b|q^{-1}]W_m(a,b;q) = N_k(a,b;q)\delta_{k,k'}, \tag{40}$$

with

$$W_m(a,b;q) = \frac{(\frac{qb}{a};q)_\infty}{(qa^{-2};q)_\infty} q^{m^2} \frac{1-q^{2m}a^{-2}}{1-a^{-2}} \frac{(a^{-2},\frac{1}{ab};q)_m}{(q,\frac{qb}{a};q)_m} \left(\frac{b}{a}\right)^m, \tag{41}$$

$$N_k(a,b;q) = q^{-2\binom{k}{2}} \left(\frac{ab}{q}\right)^k \left(q,\frac{1}{ab};q\right)_k. \tag{42}$$

This orthogonality relation is essentially the dual orthogonality relation for the little q-Jacobi polynomials. Using the above expansion and applying the orthogonality relation, we can obtain an expansion for the Al-Salam–Chihara polynomials as a bilinear sum over a product of q^{-1}-Al-Salam–Chihara polynomials and Askey–Wilson polynomials.

Lemma 5.1 *Let $n \in \mathbb{N}_0$, $0 < q < 1$, $a,b,c,d \in \mathbb{C}^*$ such that $0 < ab, cd < 1$, and $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}$, then*

$$\begin{aligned}\sum_{k=0}^{\infty} q^{\binom{k}{2}}(-c)^k \frac{(\frac{abcd}{q},\pm\sqrt{qabcd};q)_k}{(q,\pm\sqrt{\frac{abcd}{q}},ac,bc,cd;q)_k} Q_n\left[\frac{q^{1/2-k}}{\sqrt{abcd}};\sqrt{\frac{q}{abcd}},\sqrt{\frac{cd}{qab}}\middle|q^{-1}\right] \\ \times p_k(x;a,b,c,d|q) \\ = q^{-\binom{n}{2}} \frac{(abcd,cz^{\pm};q)_\infty}{(ac,bc,cd;q)_\infty} \left(-\sqrt{\frac{c}{qabd}}\right)^n Q_n(x;a,b|q).\end{aligned} \tag{43}$$

Proof Start with (39) and replace

$$\left(\sqrt{\frac{q}{abcd}},\sqrt{\frac{cd}{qab}},\sqrt{\frac{bd}{qac}},\sqrt{\frac{bc}{qad}}\right) \mapsto (a,b,c,d). \tag{44}$$

Then multiply both sides of the equation by the weight function in (41) multiplied by $Q_{k'}[q^{-n}a;a,b|q^{-1}]$, $k' \in \mathbb{N}_0$, and sum over all $n \in \mathbb{N}_0$ utilizing the orthogonality relation. This selects out the $k = k'$ term in the infinite sum over k. Then replacing $k \leftrightarrow n$, and performing the inverse to the map (44) throughout, completes the proof. □

There are some interesting special cases which can be derived from this expansion. First for continuous dual q-Hahn polynomials

$$p_m(x;a,b,c|q) = q^{\binom{m}{2}}(-c)^m(ab;q)_m \sum_{n=0}^{m} \left(\frac{q}{c}\right)^n \frac{(q^{-m};q)_n}{(q,ab;q)_n} Q_n(x;a,b|q) \tag{45}$$

$$= q^{\binom{m}{2}}(-c)^m \frac{(bz^{\pm};q)_\infty (ab;q)_m}{(ab;q)_\infty} \sum_{n=0}^{\infty} \frac{(qb/c)^{\frac{n}{2}}}{(q;q)_n} H_n\left[\sqrt{\frac{q}{bc}}; \frac{q^{\frac{1}{2}-m}}{\sqrt{bc}}\middle| q\right] H_n(x;a|q), \tag{46}$$

where the first and second expansions follow respectively by starting with (39) and taking $c \to 0$ and $a \to 0$ then relabeling the parameters. The continuous big q-Hermite polynomial in the second expansion comes from expanding the q^{-1}-Al-Salam–Chihara polynomial in terms of its ${}_2\phi_1$ representation, taking the necessary limit and re-expressing it as a continuous big q-Hermite polynomial. Then for the Al-Salam–Chihara polynomials one has

$$Q_m(x;a,b|q) = q^{\binom{m}{2}}(-b)^m \sum_{n=0}^{m} \left(\frac{q}{b}\right)^n \frac{(q^{-m};q)_n}{(q;q)_n} H_n(x;a|q) \tag{47}$$

$$= q^{\binom{m}{2}}(-b)^m (az^{\pm};q)_\infty \sum_{n=0}^{\infty} \frac{(qa/b)^{\frac{n}{2}}}{(q;q)_n} H_n\left[\sqrt{\frac{q}{ab}}; \frac{q^{\frac{1}{2}-m}}{\sqrt{ab}}\middle| q\right] H_n(x|q), \tag{48}$$

which follow respectively by setting $b \to 0$, $c \to 0$ in (45), (46) and then relabeling the parameters. For the continuous big q-Hermite polynomials, one has

$$H_m(x;a|q) = q^{\binom{m}{2}}(-a)^m \sum_{n=0}^{m} \left(\frac{q}{a}\right)^n \frac{(q^{-m};q)_n}{(q;q)_n} H_n(x|q), \tag{49}$$

where $H_n(x|q)$ is the continuous q-Hermite polynomial. This expansion follows by starting with the first expansion for the Al-Salam–Chihara polynomials, taking the limit as $a \to 0$ and relabeling the parameters.

6 A general expansion for Askey–Wilson polynomials

We are able to derive a bilinear sum over Askey–Wilson polynomials which generalizes the expansion given in (39).

Theorem 6.1 *Let* $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $z, a, b, c, d, e, f \in \mathbb{C}^*$, *such that* $\max\{|cz|, |c/z|\} < 1$. *Then*

$$p_n(x;a,b,c,d|q) = \frac{(ae,be,ef,abcf,cz^{\pm};q)_\infty}{(ac,bc,cf,abef,ez^{\pm};q)_\infty} (ac,bc,cd;q)_n \times \sum_{k=0}^{\infty} c^{k-n} \frac{(abef;q)_{2k} (\frac{e}{c}, \frac{abef}{q};q)_k}{(\frac{abef}{q};q)_{2k} (q,ae,be,ef,abcf;q)_k} p_k(x;a,b,e,f|q) \times {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, cf, \frac{qc}{e} \\ cd, q^k abcf, q^{1-k}\frac{c}{e} \end{matrix}; q, q\right) \tag{50}$$

$$= \frac{(ae, be, ef, abcf, cz^{\pm}; q)_\infty}{(ac, bc, cf, abef, ez^{\pm}; q)_\infty} \frac{(ab, ac, bc, \frac{d}{f}; q)_n}{(abcf; q)_n}$$
$$\times \sum_{k=0}^{\infty} f^{n-k} \frac{(abef; q)_{2k} (\frac{abef}{q}; q)_k}{(\frac{abef}{q}; q)_{2k} (q, ab, ae, be; q)_k} p_k(x; a, b, e, f|q)$$
$$\times {}_4\phi_3 \left(\begin{matrix} q^{-k}, q^{k-1}abef, cf, \frac{qf}{d} \\ ef, q^n abcf, q^{1-n}\frac{f}{d} \end{matrix}; q, q \right). \tag{51}$$

Proof Start with Lemma 5.1. Replace $c \mapsto e$ and $d \mapsto f$ and divide both sides by the prefactor of the Al-Salam–Chihara polynomial on the right-hand side. Multiply both sides by

$$q^{\binom{m}{2}} (-d)^m \frac{(cz^{\pm}; q)_\infty (ab, ac, bc; q)_m}{(ac, bc; q)_\infty} \frac{q^{\binom{n}{2}} (-1)^n}{(q, ab; q)_n} \left(\frac{qabc}{d} \right)^{\frac{n}{2}}$$
$$\times Q_n \left[\frac{q^{\frac{1}{2}-m}}{\sqrt{abcd}}; \frac{\sqrt{q}}{\sqrt{abcd}}, \frac{\sqrt{cd}}{\sqrt{qab}} \middle| q^{-1} \right],$$

and then sum both sides of the resulting expression over $n \geq 0$. Applying (39) on the right-hand side gives the Askey–Wilson polynomial $p_m(x; a, b, c, d|q)$. The left-hand side is a double sum over n and k. Interchanging the n and k sums and identifying the sum over n using Theorem 4.1 with

$$\{m', m, a, b, c, d, t\} \mapsto \left\{ m, k, \sqrt{\frac{q}{abef}}, \sqrt{\frac{ef}{qab}}, \sqrt{\frac{q}{abcd}}, \sqrt{\frac{cd}{qab}}, qab\sqrt{\frac{cf}{de}} \right\}$$

and replacing $m \mapsto n$ in the resulting expression produces (50). Alternatively, by utilizing the apparent symmetry in Remark 4.2, interchanging the n and k sums and identifying the sum over n using Theorem 4.1 with

$$\{m', m, c, d, a, b, t\} \mapsto \left\{ k, m, \sqrt{\frac{q}{abef}}, \sqrt{\frac{ef}{qab}}, \sqrt{\frac{q}{abcd}}, \sqrt{\frac{cd}{qab}}, qab\sqrt{\frac{cf}{de}} \right\}$$

and replacing $m \mapsto n$ in the resulting expression produces (51). In order to determine the range of parameters over which the infinite series (50), (51) respectively are convergent, write the right-hand sides respectively as

$$p_m(x; a, b, c, d|q) = \frac{(abcf, ae, be, ef, cz^{\pm}; q)_\infty}{(abef, ac, bc, fc, ez^{\pm}; q)_\infty}$$
$$\times (ac, bc, cd; q)_m \sum_{k=0}^{\infty} G_{k,m}(z; a, b, c, d, e, f|q), \tag{52}$$

$$p_m(x;a,b,c,d|q) = \frac{(ae,be,ef,abcf,cz^{\pm};q)_\infty}{(ac,bc,cf,abef,ez^{\pm};q)_\infty}\frac{(ab,ac,bc,\frac{d}{f};q)_m}{(abcf;q)_m} \times \sum_{k=0}^{\infty} H_{k,m}(z;a,b,c,d,e,f|q). \tag{53}$$

It is then straightforward to show using Ismail's asymptotic expansion for large degree of Askey–Wilson polynomials [13, (18.29.1)] that as $k \to \infty$ one has respectively

$$G_{k,m}(z;a,b,c,d,e,f|q) \sim \frac{(\frac{e}{c},abef;q)_\infty c^{-m}}{(q,ae,be,ef,abcf;q)_\infty} \times \left((cz)^k \frac{(\frac{a}{z},\frac{b}{z},\frac{e}{z},\frac{f}{z};q)_\infty}{(z^{-2};q)_\infty} + \left(\frac{c}{z}\right)^k \frac{(az,bz,ez,fz;q)_\infty}{(z^2;q)_\infty}\right), \tag{54}$$

$$H_{k,m}(z;a,b,c,d,e,f|q) \sim \frac{(\frac{e}{c},abef;q)_\infty (cd,abcf;q)_m c^{-m}}{(q,ae,be,ef,abcf;q)_\infty (ab,\frac{d}{f};q)_m} \times \left((cz)^k \frac{(\frac{a}{z},\frac{b}{z},\frac{e}{z},\frac{f}{z};q)_\infty}{(z^{-2};q)_\infty} + \left(\frac{c}{z}\right)^k \frac{(az,bz,ez,fz;q)_\infty}{(z^2;q)_\infty}\right), \tag{55}$$

which provides the restrictions on the parameters c and z in order for the infinite sum to converge. Furthermore, using the identity (2) and

$$\frac{(abcf;q)_\infty(\frac{e}{c};q)_k(cd;q)_n}{(abcf;q)_k}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,cf,\frac{qc}{e}\\ cd,q^kabcf,q^{1-k}\frac{c}{e}\end{matrix};q,q\right) = \sum_{j=0}^{n}\frac{(q^{-n},q^{n-1}abcd,cf;q)_j(cdq^j;q)_{n-j}(abcfq^{k+j};q)_\infty q^j}{(q;q)_j} \times \begin{cases}\left(-\frac{eq^k}{c}\right)^j q^{-\binom{j+1}{2}}(\frac{e}{c};q)_{k-j}(\frac{qc}{e};q)_j, & j\le k,\\ \left(-\frac{eq^k}{c}\right)^k q^{-\binom{k+1}{2}}(\frac{c}{e}q^{j-k+1};q)_k, & j>k,\end{cases} \tag{56}$$

it follows that, away from the remaining denominator zeros, the summand is analytic in $z,a,b,c,d,e,f \in \mathbb{C}^*$ and $q \in \mathbb{C}^\dagger$. The series converges locally uniformly on this domain provided $\max\{|cz|,|c/z|\} < 1$, and therefore its sum is analytic there. By meromorphic continuation, the identity extends beyond the parameter restrictions imposed in Lemma 5.1, subject to this convergence condition. At exceptional parameter values, the identity is understood by taking limits whenever the singularities are removable. This completes the proof. □

Remark 6.2 The above expansion has an evident symmetry in a,b.

Remark 6.3 It is interesting to compare the above infinite expansion for the Askey–Wilson polynomials with the connection coefficients for Askey–Wilson polynomials

which are obtained by setting $h = d$ and then replacing $a \leftrightarrow d$ followed by setting $g = b$ in (27), namely

$$p_n(x;a,b,c,d|q) = (q,ab,ac,ad;q)_n \sum_{k=0}^{n} p_k(x;a,b,e,f|q) \times \frac{\left(q^k a\right)^{k-n}(q^{n-1}abcd;q)_k}{(q;q)_{n-k}(q,ab,ac,ad,q^{k-1}abef;q)_k} {}_4\phi_3\left(\begin{matrix} q^{k-n}, q^k ae, q^k af, q^{n+k-1}abcd \\ q^k ac, q^k ad, q^{2k}abef \end{matrix}; q, q\right). \tag{57}$$

Both expansions involve the Askey–Wilson polynomial $p_k(x;a,b,e,f|q)$ multiplied by terminating ${}_4\phi_3$'s, except (57) truncates because of the prefactor $(q;q)_{n-k}$ term in the denominator.

One interesting limit for the above expansion is the limit $c \to e$. In this case, the expansion becomes the connection coefficients for the Askey–Wilson polynomials with one free parameter, which is well known [1, (6.5)] to be a simple product (28). Examining this limit gives the following interesting result.

Corollary 6.4 *Let $n,k \in \mathbb{N}_0$ such that $n \geq k$, $q \in \mathbb{C}^{\ddagger}$, $a,b,c \in \mathbb{C}$. Then*

$$\lim_{x\to 1}(x;q)_k \, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}ab, c, \frac{q}{x} \\ q^k ac, b, \frac{q^{1-k}}{x} \end{matrix}; q, q\right) = c^{n-k}\frac{(q,a;q)_n(\frac{b}{c};q)_{n-k}(c,ac,q^{n-1}ab;q)_k}{(b,ac;q)_n(q;q)_{n-k}(a,q^n ac;q)_k}. \tag{58}$$

Proof Take the limit as $e \to c$ in Theorem 6.1 and the contribution from $k > n$ vanishes. Then compare the limit with the connection coefficient with one free parameter (28). Then after straightforward substitutions, the result follows. □

If one starts with (51) and sets $e \mapsto c$ followed by $f \mapsto e$, one obtains the following result.

Corollary 6.5 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^{\dagger}$, $z,a,b,c,d,e \in \mathbb{C}^*$, such that* $\max\{|ez|,|e/z|\} < 1$. *Then one has the following connection relation for Askey–Wilson polynomials*

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,bc,\frac{d}{e};q)_n}{(abce;q)_n} \times \sum_{k=0}^{\infty} \frac{e^{n-k}(abce;q)_{2k}(\frac{abce}{q};q)_k}{(\frac{abce}{q};q)_{2k}(q,ab,ac,bc;q)_k} p_k(x;a,b,c,e|q) \, {}_3\phi_2\left(\begin{matrix} q^{-k}, q^{k-1}abce, \frac{qe}{d} \\ q^n abce, \frac{q^{1-n}e}{d} \end{matrix}; q, q\right). \tag{59}$$

Then if one starts with (59) and takes the limit as $d \to 0$ followed by $e \mapsto d$, one obtains the following expansion of the continuous dual q-Hahn polynomials in terms of the Askey–Wilson polynomials.

Corollary 6.6 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b, c, d \in \mathbb{C}^*$. Then*

$$p_n(x; a, b, c|q) = \frac{(ab, ac, bc; q)_n \, d^n}{(abcd; q)_n} \times \sum_{k=0}^{n} \frac{q^{-\binom{k}{2}} \left(-\frac{q^n}{d}\right)^k (abcd; q)_{2k} (q^{-n}, \frac{abcd}{q}; q)_k}{(\frac{abcd}{q}; q)_{2k} (q, ab, ac, bc, q^n abcd; q)_k} p_k(x; a, b, c, d|q). \quad (60)$$

Proof Start with (59), take the limit as $d \to 0$, replace $e \mapsto d$. In this limit the ${}_3\phi_2$ becomes a ${}_2\phi_1$ which can then be evaluated using the q-Gauss sum [13, (17.6.1)] which after simplification completes the proof. □

Note that the limits $a \to 0$ and $b \to 0$ in (59) are equivalent and produce the following terminating connection relation for continuous dual q-Hahn polynomials.

Corollary 6.7 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b, c, d \in \mathbb{C}^*$. Then*

$$p_n(x; a, b, c|q) = d^n (bc, \tfrac{a}{d}; q)_n \sum_{k=0}^{n} \left(\frac{q}{a}\right)^k \frac{(q^{-n}; q)_k}{(q, bc, \frac{q^{1-n}d}{a}; q)_k} p_k(x; b, c, d|q). \quad (61)$$

Proof Take the limit as $a \to 0$ in (59) and then relabeling the parameters completes the proof. □

There are some other interesting limits of Theorem 6.1. First consider some expansions of the Askey–Wilson polynomial whose expansion coefficients are the continuous dual q-Hahn polynomials.

Corollary 6.8 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $a, b, c, d, e \in \mathbb{C}^*$. Then*

$$p_n(x; a, b, c, d|q) = \frac{(ae, be, cz^{\pm}; q)_\infty}{(ac, bc, ez^{\pm}; q)_\infty} (ac, bc, cd; q)_n \times \sum_{k=0}^{\infty} \frac{c^{k-n} (\frac{e}{c}; q)_k}{(q, ae, be; q)_k} p_k(x; a, b, e|q) \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n-1}abcd, \frac{qc}{e} \\ cd, q^{1-k}\frac{c}{e} \end{matrix}; q, q\right) \quad (62)$$

$$= \frac{(abce, cz^{\pm}; q)_\infty}{(ac, bc, ec; q)_\infty} (ac, bc, cd; q)_n \times \sum_{k=0}^{\infty} \frac{c^{k-n}}{(q, abce; q)_k} p_k(x; a, b, e|q) \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ce \\ cd, q^k abce \end{matrix}; q, q^{k+1}\right) \quad (63)$$

$$= \frac{(abcd, cz^{\pm}; q)_\infty}{(ac, bc, cd; q)_\infty} \frac{(ac, bc, cd; q)_n}{(abcd; q)_{2n}} \sum_{k=0}^{\infty} \frac{c^k}{(q, q^{2n}abcd; q)_k} p_{n+k}(x; a, b, d|q) \tag{64}$$

$$= \frac{(\frac{qa}{d}, \frac{qb}{d}, cz^{\pm}; q)_\infty}{(ac, bc, \frac{q}{d}z^{\pm}; q)_\infty} q^{\binom{n}{2}} (-d)^n (ab, ac, bc; q)_n$$
$$\times \sum_{k=0}^{\infty} \frac{c^k (q^n ab, \frac{q^{1-n}}{cd}; q)_k}{(q, ab, \frac{qa}{d}, \frac{qb}{d}; q)_k} p_k(x; a, b, \tfrac{q}{d}|q). \tag{65}$$

Proof The first result is obtained by taking the limit as $f \to 0$ in Theorem 6.1. The second is obtained by taking the limit as $e \to 0$ in Theorem 6.1 and then replacing $f \mapsto e$. The third result is obtained by setting $e = d$ in the second result, in which case the ${}_3\phi_2$ becomes a ${}_2\phi_1$ that can be evaluated using the q-Chu–Vandermonde sum [13, (17.6.2)]. The resulting expression only survives for $k \geq n$. Then shifting the summation index gives (64). The fourth result is obtained by setting $e = q/d$ in the first result. One can then simplify the resulting expression using the q-Chu–Vandermonde sum (13) which provides (65). □

Remark 6.9 If one considers expansions of the Askey–Wilson polynomial whose expansion coefficients are Al-Salam–Chihara polynomials, it is straightforward to show that if one takes the limit $e, f \to 0$ in Theorem 6.1, then one can see that it is a generalization of (39).

We also have an expansion of a continuous dual q-Hahn polynomial in terms of Askey–Wilson polynomials.

Corollary 6.10 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $a, b, c, d, e \in \mathbb{C}^*$. Then*

$$p_n(x; a, b, c|q) = \frac{(abce, ad, bd, ed, cz^{\pm}; q)_\infty}{(abde, ac, bc, ec, dz^{\pm}; q)_\infty} (ac, bc; q)_n$$
$$\times \sum_{k=0}^{\infty} \frac{c^{k-n}(abde; q)_{2k} (\frac{abde}{q}, \frac{d}{c}; q)_k}{(\frac{abde}{q}; q)_{2k} (q, ad, bd, ed, abce; q)_k} p_k(x; a, b, d, e|q)$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{-n}, ce, \frac{qc}{d} \\ q^k abce, q^{1-k}\frac{c}{d} \end{matrix}; q, q \right). \tag{66}$$

Proof This result follows by starting with Theorem 6.1 and taking the limit as $d \to 0$, followed by replacing $(e, f) \mapsto (d, e)$. □

Now we consider expansions of continuous dual q-Hahn polynomials in terms of continuous dual q-Hahn polynomials.

Corollary 6.11 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $a, b, c, d, e \in \mathbb{C}^*$. Then*

$$p_n(x; a, b, c|q) = \frac{(ae, ed, cz^{\pm}; q)_\infty}{(ac, cd, ez^{\pm}; q)_\infty} (ac, bc; q)_n$$

$$\times \sum_{k=0}^{\infty} \frac{c^{k-n}\,(\frac{e}{c};q)_k}{(q,ae,de;q)_k}\, p_k(x;a,d,e|q)\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{qc}{e}, cd \\ bc, q^{1-k}\frac{c}{e} \end{matrix}; q, q\right) \tag{67}$$

$$= \frac{(ad,bd,cz^{\pm};q)_\infty}{(ac,bc,dz^{\pm};q)_\infty} \frac{(ac,bc;q)_n}{(ad,bd;q)_n} \sum_{k=0}^{\infty} \frac{c^k(\frac{d}{c};q)_k\, p_{n+k}(x;a,b,d|q)}{(q,q^n ad,q^n bd;q)_k} \tag{68}$$

$$= \frac{(abcd,cz^{\pm};q)_\infty}{(ac,bc,cd;q)_\infty}(ac,bc;q)_n$$
$$\times \sum_{k=0}^{\infty} \frac{c^{k-n}\, p_k(x;a,b,d|q)}{(q,abcd;q)_k}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, cd \\ q^k abcd \end{matrix}; q, q^{k+1}\right). \tag{69}$$

Proof The first result is obtained by taking the limit as $a \to 0$ in Theorem 6.1 followed by replacing $d \mapsto a$, and $a \leftrightarrow b$, $f \mapsto d$. The second result follows by taking the limit as $d \to 0$ in (62) followed by replacing $e \mapsto d$, which evaluates to a ${}_2\phi_1$ which can be evaluated using the q-Chu–Vandermonde sum (13). Notice that the sum only survives for $k \geq n$. After shifting the summation index one obtains (68). The third result follows by setting $d \to 0$ in (63) and replacing $e \mapsto d$. □

If we start with (65), replace $(a,b,c) \mapsto (\frac{1}{a},\frac{1}{b},\frac{1}{c})$ using (20) and take the $d \to 0$ limit, then we obtain the following expansion of a continuous dual q^{-1}-Hahn polynomial in terms of Al-Salam–Chihara polynomials.

Corollary 6.12 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}^*$, $a,b,c \in \mathbb{C}^*$. Then*

$$p_n(x;a,b,c|q^{-1}) = q^{-2\binom{n}{2}} \frac{(\frac{1}{c}z^{\pm};q)_\infty}{(\frac{1}{ac},\frac{1}{bc};q)_\infty}(abc)^n \left(\tfrac{1}{ab},\tfrac{1}{ac},\tfrac{1}{bc};q\right)_n$$
$$\times \sum_{k=0}^{\infty} \left(\frac{1}{c}\right)^k \frac{(\frac{q^n}{ab};q)_k}{(q,\frac{1}{ab};q)_k} Q_k(x;\tfrac{1}{a},\tfrac{1}{b}|q). \tag{70}$$

Proof First start with (65), replace $(a,b,c) \mapsto (\frac{1}{a},\frac{1}{b},\frac{1}{c})$, multiply both sides of the resulting equation by $q^{-3\binom{n}{2}}(-abcd)^n$, use (20) and take the $d \to 0$ limit. Then using the terminating q-binomial theorem (12) completes the proof. □

Now we consider expansions of Al-Salam–Chihara polynomials in terms of Al-Salam–Chihara polynomials.

Corollary 6.13 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z+z^{-1}) \in \mathbb{C}$, $a,b,c,d \in \mathbb{C}^*$. Then*

$$Q_n(x;a,b|q) = \frac{(cd,az^{\pm};q)_\infty}{(ad,cz^{\pm};q)_\infty}(ab;q)_n \sum_{k=0}^{\infty} \frac{a^{k-n}(\frac{c}{a};q)_k}{(q,cd;q)_k} Q_k(x;c,d|q)$$
$$\times\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{qa}{c}, ad \\ q^{1-k}\frac{a}{c}, ab \end{matrix}; q, q\right) \tag{71}$$

$$= \frac{(bc, az^{\pm}; q)_\infty}{(ab, cz^{\pm}; q)_\infty} \frac{(ab; q)_n}{(bc; q)_n} \sum_{k=0}^{\infty} \frac{a^k (\frac{c}{a}; q)_k}{(q, q^n bc; q)_k} Q_{n+k}(x; b, c|q) \quad (72)$$

$$= \frac{(bz^{\pm}; q)_\infty}{(ab, bc; q)_\infty} (ab; q)_n \sum_{k=0}^{\infty} \frac{b^{k-n}}{(q; q)_k} Q_k(x; a, c|q)$$

$$\times {}_2\phi_1\left(\begin{matrix} q^{-n}, bc \\ 0 \end{matrix}; q, q^{k+1}\right). \quad (73)$$

Proof The first result is obtained by starting with (67) and taking the limit as $a \to 0$ followed by $c \mapsto a$, $e \mapsto c$. The second result is obtained by starting with (68) and taking the limit as $a \to 0$ followed by $c \mapsto a$, $d \mapsto c$. The third result is obtained by starting with (69) and taking the limit as $a \to 0$, $c \mapsto a$, $d \mapsto c$, $a \leftrightarrow b$. □

One can also obtain a connection formula between continuous big q-Hermite polynomials and Al-Salam–Chihara polynomials.

Corollary 6.14 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $a, b, c \in \mathbb{C}^*$. Then*

$$H_n(x; a|q) = \frac{(bc, az^{\pm}; q)_\infty}{(ab, cz^{\pm}; q)_\infty} \sum_{k=0}^{\infty} \frac{a^{k-n} (\frac{c}{a}; q)_k}{(q, bc; q)_k} Q_k(x; b, c|q)\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{qa}{c}, ab \\ q^{1-k}\frac{a}{c}, 0 \end{matrix}; q, q\right). \quad (74)$$

Proof Let $b \to 0$ in (71) and then replacing $d \mapsto b$ completes the proof. □

One can also obtain a connection formula between the continuous big q-Hermite polynomials and the Al-Salam–Chihara polynomials.

Corollary 6.15 *Let $n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $x = \frac{1}{2}(z + z^{-1}) \in \mathbb{C}$, $a, b \in \mathbb{C}^*$. Then*

$$H_n(x; a|q) = \frac{(ab; q)_\infty}{(bz^{\pm}; q)_\infty} \frac{1}{(ab; q)_n} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-b)^k}{(q, q^n ab; q)_k} Q_{n+k}(x; a, b|q). \quad (75)$$

Proof This result can be obtained by taking the limit $a \to 0$ in (72) and then replacing $(b, c) \mapsto (a, b)$. □

7 The Gupta–Masson rational ${}_4\phi_3$-functions

From orthogonality relations for the q^{-1}-Al-Salam–Chihara polynomials it follows that the Poisson kernel in Theorem 4.1 in the case $ab = cd$ satisfies biorthogonality relations. This leads to another proof of the biorthogonality relation for rational ${}_4\phi_3$-functions in [8, Corollary 4.4], see also [7] where this is proved using unitary maps. Let us briefly give a direct proof here.

Assume $0 < q < 1$, $ab = cd$, $ab > 1$, $qb < a$, $qd < c$, so that the weight functions $W_m(a, b; q)$ and $W_m(c, d; q)$ for the q^{-1}-Al-Salam–Chihara polynomials,

see (41), are positive. Using the shorthand notation

$$P^t_{m,m'} = P_{m,m'}\left(\frac{qt}{ab}; a, b, c, d\right), \tag{76}$$

the Poisson kernel from Theorem 4.1 can be written as

$$P^t_{m,m'} = \sum_{n=0}^{\infty} h^t_n Q_n[aq^{-m}; a, b|q^{-1}] Q_n[cq^{-m'}; c, d|q^{-1}],$$

with

$$h^t_n = \frac{q^{2\binom{n}{2}}}{(q, \frac{1}{ab}; q)_n}\left(\frac{qt}{ab}\right)^n.$$

Let $|\frac{qb}{c}| < |t| < |\frac{c}{qb}|$, so that the sums for $P^t_{m,l}$ and $P^{t^{-1}}_{m,l}$ both converge absolutely. Now we have

$$\begin{aligned}\sum_{l=0}^{\infty} W_l(c,d) P^t_{m,l} P^{t^{-1}}_{m',l} &= \sum_{n,n'=0}^{\infty} h^t_n h^{t^{-1}}_{n'} Q_n[aq^{-m}; a, b|q^{-1}] Q_{n'}[aq^{-m'}; a, b|q^{-1}] \\ &\quad \times \sum_{l=0}^{\infty} W_l(c,d) Q_n[cq^{-l}; c, d|q^{-1}] Q_{n'}[cq^{-l}; c, d|q^{-1}] \\ &= \sum_{n=0}^{\infty} h^t_n h^{t^{-1}}_n Q_n[aq^{-m}; a, b|q^{-1}] Q_n[aq^{-m'}; a, b|q^{-1}] N_n(c, d; q),\end{aligned}$$

where we have used the orthogonality relations (40). From the definitions of $N_n(a, b; q)$ and h^t_n it follows that $h^t_n h^{t^{-1}}_n N_n(c, d; q) = (N_n(a, b; q))^{-1}$, so that applying the orthogonality relations dual to (40) gives biorthogonality relations for the Poisson kernel $P_{m,m'}$,

$$\sum_{l=0}^{\infty} W_l(c,d) P^t_{m,l} P^{t^{-1}}_{m',l} = \frac{\delta_{m,m'}}{W_m(a,b)}. \tag{77}$$

Using the symmetry from Remark 4.2 it follows that the dual relations also hold,

$$\sum_{m=0}^{\infty} W_m(a,b) P^t_{m,l} P^{t^{-1}}_{m,l'} = \frac{\delta_{l,l'}}{W_l(c,d)}. \tag{78}$$

Explicitly, this gives the following biorthogonality relations. Under the parameter conditions given above, these are with respect to a positive weight function. The conditions can be removed using analytic continuation, leading to biorthogonality with respect to a complex-valued weight.

Theorem 7.1 *Let $0 < q < 1$, $t, a, b, c, d \in \mathbb{C}^*$, such that $ab = cd$. Then*

$$\sum_{l=0}^{\infty} \frac{(1-\frac{q^{2l}}{c^2})}{(1-\frac{1}{c^2})} \frac{(\frac{qd}{c}, \frac{1}{c^2}; q)_l}{(q, \frac{1}{ab}; q)_l} \left(\frac{c}{qd}\right)^l {}_4\phi_3\left(\begin{matrix} q^{-l}, \frac{q^l}{c^2}, \frac{qat}{c}, \frac{qbt}{c} \\ \frac{q^{1-m}at}{c}, \frac{q^{m+1}t}{ac}, \frac{qd}{c} \end{matrix}; q, q\right)$$
$$\times {}_4\phi_3\left(\begin{matrix} q^{-l}, \frac{q^l}{c^2}, \frac{qa}{ct}, \frac{qb}{ct} \\ \frac{q^{1-m'}a}{ct}, \frac{q^{m'+1}}{act}, \frac{qd}{c} \end{matrix}; q, q\right)$$
$$= \frac{(\frac{q}{a^2}, \frac{q}{c^2}, \frac{qbt^{\pm}}{c}; q)_\infty}{(\frac{qb}{a}, \frac{qd}{c}, \frac{qt^{\pm}}{ac}; q)_\infty} \left(\frac{c}{qd}\right)^m \frac{(1-\frac{1}{a^2})}{(1-\frac{q^{2m}}{a^2})} \frac{(q, \frac{qb}{a}, \frac{qt^{\pm}}{ac}; q)_m}{(\frac{1}{a^2}, \frac{1}{ab}, \frac{ct^{\pm}}{a}; q)_m} \delta_{m,m'}, \tag{79}$$

$$\sum_{m=0}^{\infty} \frac{(1-\frac{q^{2m}}{a^2})}{(1-\frac{1}{a^2})} \frac{(\frac{1}{a^2}, \frac{1}{ab}, \frac{ct^{\pm}}{a}; q)_m}{(q, \frac{qb}{a}, \frac{qt^{\pm}}{ac}; q)_m} \left(\frac{qd}{c}\right)^m {}_4\phi_3\left(\begin{matrix} q^{-l}, \frac{q^l}{c^2}, \frac{qat}{c}, \frac{qbt}{c} \\ \frac{q^{1-m}at}{c}, \frac{q^{m+1}t}{ac}, \frac{qd}{c} \end{matrix}; q, q\right)$$
$$\times {}_4\phi_3\left(\begin{matrix} q^{-l'}, \frac{q^{l'}}{c^2}, \frac{qa}{ct}, \frac{qb}{ct} \\ \frac{q^{1-m}a}{ct}, \frac{q^{m+1}}{act}, \frac{qd}{c} \end{matrix}; q, q\right)$$
$$= \frac{(\frac{q}{a^2}, \frac{q}{c^2}, \frac{qbt^{\pm}}{c}; q)_\infty}{(\frac{qb}{a}, \frac{qd}{c}, \frac{qt^{\pm}}{ac}; q)_\infty} \left(\frac{qd}{c}\right)^l \frac{(1-\frac{1}{c^2})}{(1-\frac{q^{2l}}{c^2})} \frac{(q, \frac{1}{cd}; q)_l}{(\frac{1}{c^2}, \frac{qd}{c}; q)_l} \delta_{l,l'}, \tag{80}$$

where $m, m' \in \mathbb{N}_0$, $|qb| < |a|$ in (79) and $l, l' \in \mathbb{N}_0$, $|qd| < |c|$ in (80).

Proof These biorthogonality relations are obtained by using (77), (78) and substituting the definitions of the weight function W and the Poisson kernel $P_{m,m'}$ (76). The conditions on convergence are obtained by performing an asymptotic analysis as the sum indices l, m become very large respectively. For (79), (80) write the summands on the left-hand sides as $A_{l,m,m'}(t; a, b, c, d; q)$, $B_{m,l,l'}(t; a, b, c, d; q)$ respectively. Then it is straightforward to show using Sears' balanced transformation [13, (17.9.14)] that one has the following asymptotic approximations, namely

$$A_{l,m,m'}(t; a, b, c, d; q) \sim a_{m,m'}(t; a, b, c, d; q) \left(\frac{qb}{a}\right)^l, \text{ as } l \to \infty, \tag{81}$$

$$B_{m,l,l'}(t; a, b, c, d; q) \sim b_{l,l'}(t; a, b, c, d; q) \left(\frac{qd}{c}\right)^m \text{ as } m \to \infty, \tag{82}$$

where $a_{m,m'}$ is independent of l and $b_{l,l'}$ is independent of m. This completes the proof. □

Using Theorem 6.1 and orthogonality relations for Askey–Wilson polynomials, we can express the biorthogonal rational ${}_4\phi_3$-functions in terms of Askey–Wilson polynomials. The well-known orthogonality relation for the Askey–Wilson polynomials

is given by [12, (14.1.2)]

$$\int_{-1}^{1} p_n(x;a,b,c,d|q)p_{n'}(x;a,b,c,d|q)w(x;a,b,c,d|q)\,\frac{dx}{\sqrt{1-x^2}} = \frac{2\pi(q^{n-1}abcd;q)_n(q^{2n}abcd;q)_\infty}{(q^{n+1},q^nab,q^nac,q^nad,q^nbc,q^nbd,q^ncd;q)_\infty}\delta_{n,n'}, \tag{83}$$

where $0 < q < 1$, the parameters a, b, c, d are real or appear in pairs of complex conjugates and $|a|, |b|, |c|, |d| < 1$, and

$$w(x;a,b,c,d|q) := \frac{(z^{\pm 2};q)_\infty}{(az^{\pm},bz^{\pm},cz^{\pm},dz^{\pm};q)_\infty}, \quad x = \frac{1}{2}(z+z^{-1}).$$

Using (83) with Theorem 6.1 we obtain the following expressions for the rational ${}_4\phi_3$-function, where we have chosen to write the Askey–Wilson polynomials in their 'natural' parameters, so that the parameters a, b, c, d below are different from the parameters in the biorthogonality relations above.

Corollary 7.2 *Let $n, k \in \mathbb{N}_0$, $0 < q < 1$, $a, b, c, d, e, f \in \mathbb{C}^*$ such that $|a|, |b|, |c|, |f| < 1$. Then*

$${}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, cf, \frac{qc}{e} \\ cd, q^kabcf, q^{1-k}\frac{c}{e}\end{matrix}; q, q\right) = c^{n-k}\frac{(q,cf,q^nac,q^nbc,q^kab,q^kaf,q^kbf;q)_\infty}{2\pi(cd;q)_n(\frac{e}{c};q)_k(q^kabcf;q)_\infty}\mathcal{I}_{n,k}(a,b,c,d,e,f|q), \tag{84}$$

$${}_4\phi_3\left(\begin{matrix} q^{-k}, q^{k-1}abef, cf, \frac{qf}{d} \\ ef, q^nabcf, q^{1-n}\frac{f}{d}\end{matrix}; q, q\right) = f^{k-n}\frac{(q,cf,q^nab,q^nac,q^nbc,q^kaf,q^kbf;q)_\infty}{2\pi(\frac{d}{f};q)_n(ef;q)_k(q^nabcf;q)_\infty}\mathcal{I}_{n,k}(a,b,c,d,e,f|q), \tag{85}$$

where

$$\mathcal{I}_{n,k}(a,b,c,d,e,f|q) := \int_{-1}^{1} p_n(x;a,b,c,d|q)p_k(x;a,b,e,f|q) \times w(x;a,b,c,f|q)\,\frac{dx}{\sqrt{1-x^2}}. \tag{86}$$

Proof In order to write the Gupta–Masson rational ${}_4\phi_3$-functions in terms of an integral over Askey–Wilson polynomials, assume $a, b, c, e, f \in (-1, 1)$. Then multiply the first identity in Theorem 6.1 by

$$p_k(x; a, b, e, f|q)\frac{w(x; a, b, c, f|q)}{\sqrt{1-x^2}} = p_k(x; a, b, e, f|q)\frac{(ez^{\pm}; q)_\infty w(x; a, b, e, f|q)}{(cz^{\pm}; q)_\infty\sqrt{1-x^2}},$$

integrate over $x \in (-1, 1)$ and use the Askey–Wilson orthogonality relation (83). This produces (84). The proof of (85) is similar, using (51) instead. The restrictions on the parameters can be removed afterwards by analytic continuation. □

Remark 7.3 Note that Corollary 7.2 can be considered as a generalization of the Askey–Wilson orthogonality relation (83). Clearly, for $(e, f) \mapsto (c, d)$ the integral $\mathcal{I}_{n,k}$ reduces to the integral in (83). In this case, the ${}_4\phi_3$ rational functions become ${}_3\phi_2$'s which can be evaluated using the q-Pfaff–Saalschütz sum [13, (17.7.4)].

For (84), one has the following specializations and limits. For $f = d$ the left-hand side becomes a q-Hahn type rational function, see e.g., [2]. Furthermore, letting $f \to 0$ or $d \to 0$, we find an identity expressing a q-Hahn type rational function in terms of an integral of a product of a continuous dual q-Hahn polynomial and an Askey–Wilson polynomial. Letting $e, f \to 0$ the ${}_4\phi_3$ on the left-hand side becomes a ${}_2\phi_1$-function, which is essentially a q^{-1}-Al-Salam–Chihara polynomial, leading to an identity which is equivalent to (39).

Finally, let us remark that by writing an Askey–Wilson polynomial as a ${}_4\phi_3$-series using (14) and using the symmetry $p_n(x; a, b, c, d|q) = p_n(x; c, d, a, b|q)$ it follows that the ${}_4\phi_3$ rational function can be written in terms of an Askey–Wilson polynomial as

$$\begin{aligned}&{}_4\phi_3\left(\begin{matrix}q^{-n}, q^{n-1}abcd, cf, \frac{qc}{e}\\ cd, q^m abcf, q^{1-m}\frac{c}{e}\end{matrix}; q, q\right)\\ &= \frac{\left(c\sqrt{\frac{qf}{e}}\right)^n}{(cd, q^m abcf, q^{1-m}\frac{c}{e}; q)_n}\, p_n\left[\sqrt{\tfrac{ef}{q}}; q^{m-\frac{1}{2}}ab\sqrt{ef}, \tfrac{q^{\frac{1}{2}-m}}{\sqrt{ef}}, c\sqrt{\tfrac{qf}{e}}, d\sqrt{\tfrac{e}{qf}}\,|q\right].\end{aligned} \tag{87}$$

Using this identity one can rewrite identities for Askey–Wilson polynomials as identities for the ${}_4\phi_3$ rational functions. As an example, the connection relation (28) with one free parameter can be turned into a connection formula for the ${}_4\phi_3$ rational functions.

Corollary 7.4 *Let $m, n \in \mathbb{N}_0$, $0 < q < 1$, $a, b, c, d, e, f, g \in \mathbb{C}^*$, then*

$${}_4\phi_3\left(\begin{matrix}q^{-n}, q^{n-1}abcd, cf, \frac{qc}{e}\\ cd, q^m abcf, q^{1-m}\frac{c}{e}\end{matrix}; q, q\right)$$

$$= \frac{(q,ab;q)_n}{(cd,\frac{qabcfg}{e};q)_n}\sum_{k=0}^{n}\frac{(\frac{de}{qfg};q)_{n-k}(\frac{abcfg}{e},\frac{qcfg}{e},q^{n-1}abcd;q)_k(\frac{qabcfg}{e};q)_{2k}}{(q;q)_{n-k}(q,ab,\frac{q^{n+1}abcfg}{e};q)_k(\frac{abcfg}{e};q)_{2k}} \tag{88}$$

$$\times\left(\frac{qcfg}{e}\right)^{n-k}{}_4\phi_3\left(\begin{matrix}q^{-k},\frac{q^k abcfg}{e},cf,\frac{qc}{e}\\ \frac{qcfg}{e},q^m abcf,q^{1-m}\frac{c}{e}\end{matrix};q,q\right).$$

Proof Using (28) with $x=\frac{1}{2}\left(\sqrt{\frac{ef}{q}}+\sqrt{\frac{q}{ef}}\right)$ and parameters

$$(a,b,c,d,e)\mapsto\left(q^{m-\frac{1}{2}}ab\sqrt{ef},\frac{q^{\frac{1}{2}-m}}{\sqrt{ef}},c\sqrt{\frac{qf}{e}},d\sqrt{\frac{e}{qf}},g\sqrt{\frac{qf}{e}}\right),$$

identity (87) and a similar identity for $p_k\left[\sqrt{\frac{ef}{q}};q^{m-\frac{1}{2}}ab\sqrt{ef},\frac{q^{\frac{1}{2}-m}}{\sqrt{ef}},c\sqrt{\frac{qf}{e}},g\sqrt{\frac{qf}{e}}|q\right]$ completes the proof. □

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Data availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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