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Generalization of the Weibull probabilistic compatible model to assess fatigue data into three domains: LCF, HCF and VHCF

A. Fernández Canteli [a,*], E. Castillo [b,c], S. Blasón [d], J.A.F.O. Correia [e,f], A.M.P. de Jesus [e]

[a] Dept. of Construction and Manufacturing Engineering, University of Oviedo, Campus de Viesques, 33203 Gijón, Spain
[b] Royal Academy of Engineering, Don Pedro 10, 28005 Madrid, Spain
[c] Royal Academy of Sciences, Valverde 24, 8004 Madrid, Spain
[d] Dept. of Component Safety, Bundesanstalt für Materialforschung und -prüfung (BAM), Unter den Eichen 87, 12205 Berlin, Germany
[e] INEGI & CONSTRUCT, Faculty of Engineering, University of Porto, Rua Dr. Roberto Frias, 4200-465 Porto, Portugal
[f] Faculty of Civil Engineering and Geosciences, Delft University of Technology, 2600 GA Delft, Netherlands

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ABSTRACT

In this work, three classes of fatigue models are reviewed according to the fatigue regimes commonly considered in the current components design. Particular attention is devoted to the so-called Class III fatigue models, covering the three fatigue regimes, namely, LCF, HCF and VHCF. The applicability and limitations of the proposed analytical sigmoidal solutions are discussed from the viewpoint of practical design. The compatible Weibull S-N model by Castillo and Canteli is revisited and improved by considering a new reference parameter GP $= \mathrm{E}\cdot\sigma_M \cdot(d\varepsilon/d\sigma)|_M$ as the driving force alternative to the conventional stress range. In this way, the requirement, $\sigma_M \leq \sigma_u$, according to the real experimental conditions, is fulfilled and the parametric limit number of cycles, N_0, recovers its meaning. The probabilistic definition of the model on the HCF and VHCF regimes is maintained and extended to the LCF regime. The strain gradients may be calculated from the monotonic or cyclic stress–strain curve of the material although a direct derivation from the hysteresis loop is recommended. Some Class III fatigue models from the literature and another one improved by the authors are applied to the assessment of one experimental campaign under different stress ratios conditions and the results compared accordingly. Finally, the new probabilistic GP-N field is evaluated. The results confirm the practical confluence of the stress- and the strain-based approaches into a single and advantageous unified methodology.

1. Introduction

The validity of a fatigue model can be referred to the fatigue regimes implied in the current component design, determined by the loading type characteristics and the material used. Accordingly, a plausible classification of fatigue models may obey the following categories as suggested by Strzelecki [1]: Class I models, describing a bounded fatigue life range only, which among others, disregards the inclusion of a fatigue limit; Class II models, including the relationships on the middle- and long-life region and the existence of a fatigue limit, and Class III models, including all the three domains, low-cycle fatigue (LCF), high-cycle fatigue (HCF) and very-high-cycle fatigue (VHCF).

Further enrichment within each category can be envisaged by including additional subcategories to account for relevant characteristics of the model such as its probabilistic or stochastic character and how the fatigue limit is defined. The fatigue limit may be derived either as a constant value for a predefined number of cycles, say 10^6 or 10^7 cycles, as usually suggested in the standards, or as an asymptotic value according to the own definition of the model. Otherwise, fatigue failure would occur for a finite number of cycles under null loading [2]. Sometimes, the criterion to justify an S-N fatigue model seems to be simply the quality of a particular fitting achieved. This criterion becomes critical when extrapolation for lifetime prediction beyond the scope of the experimental fatigue program is based on speculative empirical models, which do not fulfil a minimum of necessary requirements. Accordingly, the probabilistic definition of the S-N field for the material being studied is crucial to ensure a reliable fatigue design of components, in particular for life prediction or failure hazard under varying loading based on a damage accumulation rule, as required in the structural integrity concept [3,4]. In this work, the fatigue limit is identified with the endurance limit, i.e. the value of the driving force

* Corresponding author.
E-mail address: afc@uniovi.es (A. Fernández Canteli).

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Nomenclature

a	crack size; generic model parameter in the Basquin, Stromeyer, Bastenaire and Weibull models; the extrapolated value of the tangent in the point of inflexion for N = 1 in the Kohout-Věchet model
b	generic model parameter in the Basquin, Stromeyer, Bastenaire and Weibull models; the slope of the tangent in the point of inflexion in the Kohout-Věchet model (log–log scale)
c	generic model parameter in the Basquin, Stromeyer, Bastenaire and Weibull models; number of cycles corresponding to the cross point of the upper asymptote and middle tangent at the inflexion point in the Kohout-Věchet model
cdf	cumulative distribution function
d	number of cycles corresponding to the cross point of the lower asymptote and middle tangent at the inflexion point in the Kohout-Věchet model
k	exponent parameter in the Ravi-Chandran model
n'	cyclic hardening exponent in the Ramberg-Osgood relationship
D	number of cycles determining the turning point (in log–log fit) in the Kohout-Věchet model (log–log scale)
E	Young's modulus
GRV_ε	proposed generalized reference variable (fatigue driving force) defined as $GRV_\varepsilon = \sigma_M\left(\frac{d\varepsilon}{d\sigma}\Big\vert_M\right)$
$GRV_{\varepsilon,0}$	fatigue limit of the proposed generalized reference variable
GRV_σ	alternative generalized reference variable with the stress dimensions, $\left(=E\sigma_M\left(\frac{d\varepsilon}{d\sigma}\Big\vert_M\right)\right)$
$GRV_{\sigma,0}$	fatigue limit of the alternative proposed generalized reference variable
K	exponent parameter in the Ravi-Chandran model
K'	cyclic hardening coefficient in the Ramberg-Osgood relationship
K_ε'	modified K' to obtain GRV_ε
K_σ'	modified K' to obtain GRV_σ
N	generic number of cycles in the S-N field
N_f	generic lifetime in the S-N field
N_0	lower or limit number of cycles in the S-N field
R	stress ratio ($=\sigma_m/\sigma_M$)
U	Strain energy
$\Delta\varepsilon$	generic strain range
$\Delta\sigma$	generic stress range
$\Delta\sigma_u$	$\Delta\sigma$ for $\sigma_M = \sigma_u$
$\Delta\sigma_{up}$	$\Delta\sigma$ for $\sigma_M = \sigma_{up}$ to be distinguished from $\Delta\sigma_u$
$\Delta\sigma_0$	endurance limit in the $\Delta\sigma$-N field identified as the fatigue limit for infinite number of cycles
ϵ	error term in the Pascual-Meeker and d'Angelo and Nussbaumer models
ε	strain
ε_a	strain amplitude
ε_m	minimum strain in the cyclic loading
σ_{lim}	stress related to the proportionality limit in the stress–strain curve
ε_M	maximum strain in the cyclic loading
σ_a	stress amplitude
σ_{el}	elastic limit of the material
σ_{lim}	generic fatigue limit
σ_m	minimum stress in the cyclic loading
σ_{mean}	mean stress in the cyclic loading ($=(\sigma_M+\sigma_m)/2$)
σ_M	maximum stress in the cyclic loading
$\sigma_{M,0}$	endurance limit in the $\sigma_M - N$ field identified as the fatigue limit for infinite number of cycles
σ_p	limit of proportionality of the material, usually identified with σ_{el}
σ_u	ultimate tensile strength
σ_{up}	upper bound as a model parameter in the Class III models
$\frac{d\varepsilon}{d\sigma}\Big\vert_M$	$d\varepsilon/d\sigma$ at the point where $\sigma = \sigma_M$ and, correspondingly. Where $\varepsilon = \varepsilon_M$
$\frac{d\varepsilon}{d\sigma}\Big\vert_m$	$d\varepsilon/d\sigma$ at the point where $\sigma = \sigma_m$ and, correspondingly. Where $\varepsilon = \varepsilon_m$

below which no failure occurs for an infinite number of cycles. Both issues prove to have a great influence on the way the practical design of components is performed. Additionally, other questions may be crucial such as if the model does or does not fulfil certain objective conditions that may be considered ineluctable for the model to be considered a valid one.

Often, the *S-N* fatigue models arise as the definition of a median percentile regression function, see [5]. Frequently, such fatigue models exhibit dimensional inconsistencies in their analytical definition [3]. Furthermore, they may ensure, based on gratuitous premises, the asymptotic approach to the fatigue limit for an infinite lifetime, and define *a posteriori* the percentile functions to account for the variability of the *S-N* field. The rigid starting basis of the model derivation leads to functional contradictions which determine the limits of its possible extension to LCH and VHCF regions and further enhancement to include the stress ratio effect. For instance, if constant scatter along the S-N curve, i.e. homoscedasticity, or linear heteroscedasticity is assumed [6–10], the factual non-linear evolution of the scatter along the S-N field, evidenced in the experimental fatigue campaigns, cannot be replicated, see [11–13].

In this paper the stochastic fatigue model of Castillo-Canteli [3–14] is modified to extend its application validity to the three regions, i.e. LCF, HCF and VHCF, while keeping their compatibility and other important properties. The compatibility condition is not a trivial or secondary requirement but an irrefutable and unavoidable statistical condition for a fatigue model to be valid. As already explained in [3], the compatibility condition implies the equality between the cumulative distributions F(N;$\Delta\sigma$), i.e. the lifetimes for given stress range (or maximum stress), and F($\Delta\sigma$;N), i.e. the stress range (or maximum stress) for given lifetimes, as illustrated in Fig. 1. In other words, this paper provides a unified model of general validity in the three regions.

The rest of the paper is organized as follows. In Section 2, the characteristics of the three fatigue Class models are discussed. In Section 3, some Class III fatigue models entailing the three fatigue domains (LCF, HCF and VHCF), as one supposedly unified solution of the complete *S-N* field, are analyzed from the adequateness of the fitting technique and the type of the data provided from the experimental program, but also from the conceptual and practical viewpoints. In Section 4, the compatible fatigue models are introduced in detail, their limitations analyzed and possible improved alternatives presented, allowing their applicability to the three fatigue domains to be extended. In Section 5, an example of application is introduced. Different Fatigue Class III models as representing integral fatigue proposals are applied to experimental data and the outgoing results discussed. Section 6 is devoted to the discussion on the experimental data assessment and other important questions arising from the previous study about a comparative suitability of the models. Finally, in Section 7 the main conclusions from the paper are drawn.

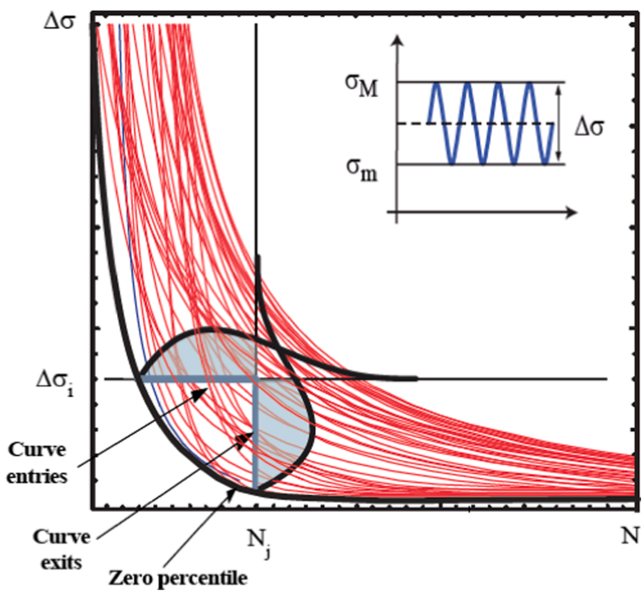

Fig. 1. The compatibility condition required by Castillo-Canteli [3,14].

2. Fatigue models classification

2.1. Class I fatigue models

The paradigmatic Class I case is the Basquin model, see Eq. (1) and [15], which represents a linear relation function between the stress range $\Delta\sigma$ and the number of cycles to failure, N, in log–log scale, as follows (see Fig. 2):

$$N = a\Delta\sigma^b \rightarrow \log N = \log a + b\log\Delta\sigma \tag{1}$$

Despite its limitations, it is the favorite *S-N* model to which many current fatigue studies are referred to define the HCF region of the *S-N* field. Its deterministic version is defined as the mean percentile curve without exhibiting fatigue limit, thus contradicting the experimental evidence of the general fatigue results. Some enhanced alternatives, as bi-linear and tri-linear *S-N* diagrams, are also the common solution in standards and directives. Thus, the utility of these Class I models in the design of components is limited to particular load histories concerned only with the specific stress of the HCF regime implied. The probabilistic definition implies arbitrary proposals for the lifetime distribution along the *S-N* curve around the regression function, which is identified as the mean percentile (see Fig. 2). Because of the obvious incapacity of the Basquin model to follow the trend towards a non-zero horizontal fatigue limit, as experimentally verified, different solutions have been proposed to provide a possible correction to the original proposal. This can be achieved by converting this Class I model into a Class II one by estimating a fixed fatigue limit horizontal *S-N* curve at a certain fixed number of cycles, usually for $N > 10^6$ cycles based, for instance, on the staircase method or simply on arbitrary considerations, as it is the case of most standards, see [16–21].

2.2. Class II fatigue models

The Class II models represent an extension of the Class I ones in which different solutions are proposed to ensure asymptotic matching to the fatigue limit, whereas the condition of existence of an upper limit in the *S-N* field prescribed by the non-linear σ-ε relation and the ultimate stress of the material is not specified. Accordingly, these Class II models are not useful to define the LCF regime. Most Class II models fail to satisfy the compatibility condition and dimensional consistency, which represent unavoidable conditions for a *S-N* fatigue model to be valid. In fact, this requirement is only satisfied by the models proposed by Freudenthal-Gumbel [22], Bolotin [23–25] and Castillo-Canteli [3], see below.

Generally, a probabilistic analysis is added to the data assessment assuming homoscedasticity or linear heteroscedasticity for lifetimes and stress ranges over the HCF but also in the fatigue limit region, see Fig. 3. Most of the standards, directives and recommendations adopt the modality of bilinear or tri-linear S-N curves for the definition of the S-N

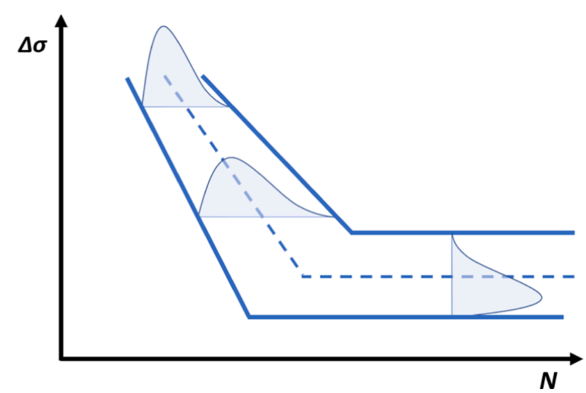

Fig. 3. Schematic representation of the usual variability laws assumed for the *S-N* field models of Class II: linear heteroscedasticity for lifetimes in the HCF region and homoscedasticity for stress ranges in the VHCF region.

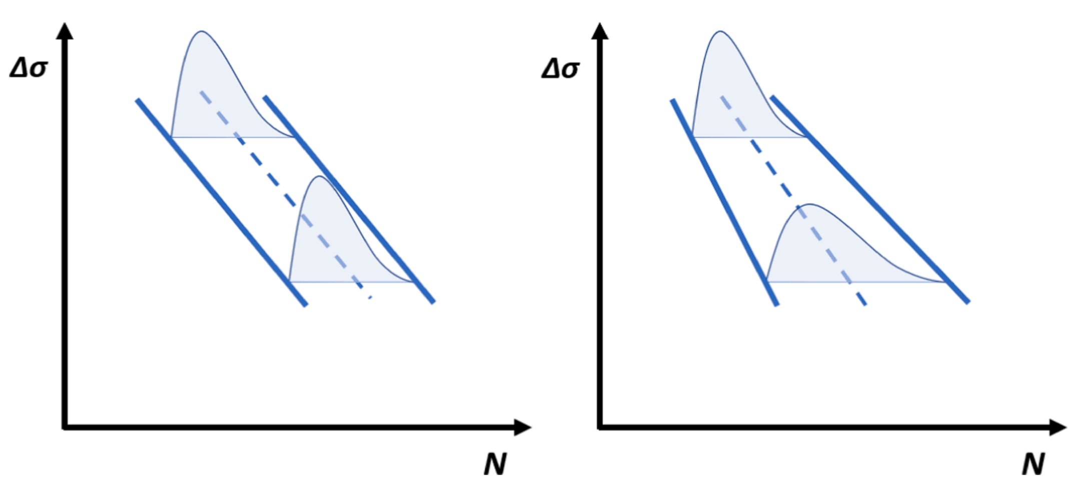

Fig. 2. Schematic representation of homoscedasticity (left) and linear heteroscedasticity (right) for the S-N fields usually assumed in the fatigue models of Class I.

curves [15–20]. The bilinear random fatigue limit model (BRFLM) of D'Angelo and Nussbaumer [26], see Eq. (2), can be included in this group:

$$\log N = \frac{a + b\log\Delta\sigma}{H(\log\frac{\Delta\sigma}{\Delta\sigma_0})} + \epsilon(0, \exp\Delta\sigma), \Delta\sigma > \Delta\sigma_0 \tag{2}$$

where H is the unit step function and $\epsilon(0, \exp(\Delta\sigma))$ is the error term supposed to be normalized and distributed with zero mean and standard deviation equal to exp ($\Delta\sigma$). This model is based on bilinearization of the *S-N* curve and independent Gaussian distributions for the definition of the variability of the lifetime in the HCF region and of the stress range in the VHCF one. The disrupted scatter change and the kinked transition from the HCF to the VHCF regions of the *S-N* curves contradicts the smooth changing observed trend in real fatigue tests.

Paolino *et al.* [27,28] focus their bilinear *S-N* models in the demonstration of the multiplicity of *S-N* fields associated with the different possible fatigue mechanisms intervening in the transition from HCF to VHCF regimes. They confirm the existence of multiple fatigue limits in the VHCF regime according to these fatigue mechanisms refuting the negation of the fatigue limit concept. The *S-N* variability of their models is based on normal distributions, which are not convenient for the estimation of low percentile curves to be applied in practical design.

Alternatively, a more satisfactory way of procuring a continuous conversion from Class I models into Class II ones can be achieved by including the asymptotic matching condition into the model definition of the VHCF region as proposed by different authors, such as Stromeyer [29], see Eq. (3), which manages the expected curvature of the *S-N* field with a simple modification of the Basquin model, see Eq. (1), while Bastenaire [30] proposes a more complex model, see Eq. (4), to ensure the asymptotic matching for the VHCF branch.

$$N = a(\Delta\sigma - \Delta\sigma_0)^b \rightarrow \log N = a + b\log(\Delta\sigma - \Delta\sigma_0), \Delta\sigma > \Delta\sigma_0 \tag{3}$$

$$\log\left(\frac{N}{c}\right) = \frac{a}{\Delta\sigma - \Delta\sigma_0} exp[-b(\Delta\sigma - \Delta\sigma_0)], \Delta\sigma > \Delta\sigma_0 \tag{4}$$

Note that the notation referred to the parameters intervening in the original models may have been rewritten in order to facilitate, hopefully, a comparative analysis among the different models and an easier interpretation of them. The physical meaning of the parameters can be affected by the new notation. For simplicity, the stress amplitude is replaced by the stress range as driving force without the meaning of the formulae being changed.

In a similar way, Leonetti *et al.* [31] ensure the asymptotic matching introducing power coefficients in the equation with the same idea of a denominator singularity:

$$\log N = a\log\left(\frac{\Delta\sigma^b}{(\Delta\sigma - \Delta\sigma_0)^c}\right), \Delta\sigma > \Delta\sigma_0 \tag{5}$$

These authors claim about the importance of the proposed smooth transition from the finite life regime to the infinite life regime proving that this is a more general model that includes the kinked model of D'Angelo and Nussbaumer as a particular case. The Bayesian analysis applied to estimate the variability of the model parameters is a relevant contribution of Leonetti et al. [31] as it points out a methodology of general application to the proposed particular fatigue model. With this solution, the most complete probabilistic information is provided for the application of a fatigue model to practical design.

Pascual and Meeker [5] start from the Stromeyer model to derive the denoted *random fatigue-limit model*:

$$\log N = a + b\log(\Delta\sigma - \Delta\sigma_0) + \epsilon, \Delta\sigma > \Delta\sigma_0 \tag{6}$$

in which ϵ is the error term. The model assumes the probability density function (pdf) of $\log(\Delta\sigma_0)$ to be either the standardized smallest extreme value (SEV) or the normal one, in both cases divided by the standard deviation. Additionally, the variability of the lifetime distribution along the *S-N* curve is considered using the logarithm of the number of cycles, log(*N*). In this way, the model describes the curvature of the *S-N* field provided by the Stromeyer basis and the probabilistic *S-N* field with increasing data scatter for decreasing stress range, as already stated by Freudenthal [32]. Nevertheless, no attention is paid to the distribution of the stress range for a given number of cycles (see words above about the compatibility condition).

In an attempt to allow extreme value criteria to be considered in the definition of the *S-N* field, Strzelecki and Tomaszewski [1] propose an alternative to the Pascual and Meeker model, in which the normal distributions are replaced by a three-parameter Weibull distribution in what concerns the variability of lifetimes along the *S-N* curve and a biparametric Weibull distribution for the variability of the fatigue limit. This last option of replacing a three-parameter Weibull distribution by a biparametric one has no justification taking into account the expected lower bound of the fatigue limit, unless it looks for eliminating one parameter in the model. Nevertheless, Pascual-Meeker and the aforementioned models do not satisfy the compatibility condition between the distribution of the lifetime for given stress range, i.e. $F(N;\Delta\sigma)$, and that for the stress range for a given lifetime, i.e. $F(\Delta\sigma;N)$.

It can be concluded that the satisfaction of compulsory properties rather than the discussion about "the best proposal" to achieve the asymptotic matching to the fatigue limit for a particular sample data is the relevant question for a fatigue model to be valid. Some considerations about the two above distributions are made by Nishijima [34] and Little-Jebe [35], without recognizing the compatibility between both. Compared with the former publication of Freudenthal [32], the later contribution of Freudenthal and Gumbel [22] seems to confirm the presumable contribution of Gumbel to the indefectible focusing of the fatigue problem from the extreme value perspective but also to emphasize the relevance of the recognition of the double analysis of the "*distribution of the fatigue life at constant stress amplitude*" and the "*distribution of fatigue strength at constant number of cycles*" as specific subsection titles in their scientific contribution. This leads to their unequivocal statement that "*the general form of the S-N relation is determined by the condition that the distributions F(Δσ;N) and F(N;Δσ) must be compatible*", see [22] page 150. Surprisingly, this transcendental requirement for a model to be valid remained unadvertised for the practically totality of fatigue model derivations, and were rescued and supported solely, to the authors' knowledge, by Bolotin [23–25] and Castillo *et al.* [3,14,33]. Further discussion will be undertaken in Section 4.

2.3. Class III fatigue models

These models aim to define the three fatigue domains, LCF, HCF and VHCF based on a unified concept of a regression or mean function that enables to reproduce the sigmoidal shape noticed as a trend in the outcomes of experimental campaigns. In this way, two boundary conditions have to be fulfilled consisting in the upper left branch with the condition $\Delta\sigma_{up} = \Delta\sigma_u$, supposedly starting at 0 or 1 cycle (according to the model), and the lower right branch fulfilling the condition of asymptotic matching to the fatigue limit $\Delta\sigma_{lim}$. Two different categories of regression functions are distinguished: those representing non-scaled regression curves, such as the Stüssi [36], and Kohout-Věchet [37] models, and those normalized to the interval [1] which are identified as biparametric Weibull cdfs, such as those of Ravi-Chandran [38,39] and D'Antuono [40] models whereas the Kurek model [41] represents a special case as discussed below. In both cases, the improved version includes a limit number of cycles N_0 as the minimum required to cause LCF failures and an upper bound of the stress range identified with $\Delta\sigma_u$. The possible misinterpretation of these fitting functions as "sample functions", in the sense of stochastic process realizations, and their probabilistic definition deserves an extended analysis of these models in Section 3.

The Class III fatigue models, defined as deterministic regression functions, require an extension to provide the probabilistic definition of the *S-N* field, where definition of the lifetime distribution along the regression curve represents a very involved task due to the sigmoidal shape of the percentile curves. In fact, fulfilment of the required compatibility condition along the whole *S-N* field is mandatory as proved by the functional equations analysis applied to the solution of the Castillo-Canteli model, see [3,42].

2.4. Non-conventional fatigue models

There are models of difficult classification, such as the B-model of Bodanoff-Kozin [43,44] focused on the probabilistic modeling of the evolution of the cumulative damage phenomena based upon the Markov-chain process theory. Although this approach represents a solid basis for the development of models related to the progress of fatigue and fracture phenomena based on their interpretation as stochastic sample functions, i.e. as process realizations, see [45], its application to the definition of the whole probabilistic *S-N* field is not yet clear.

Finally, some comments about the staircase method [46,47], which acts as a well-known auxiliary or complementary procedure in some standards, such as [48–52], to estimate the probabilistic distribution of the fatigue limit albeit for a predetermined number of cycles. Generally, this spurious fatigue limit is misleadingly interpreted as the endurance limit, i.e. the real fatigue limit for an infinite number of cycles in the VHCF regime. An alternative assessment of the staircase method replacing the normal distribution proposed in the original version by a three-parameter Weibull distribution is presented by Castillo *et al.* [53], in which a comparison is performed between the results provided by the staircase method and those from the Castillo-Canteli model, see [3]. Besides some light comments about the unsuitability of the staircase method, as devoted in Schijve [10], Pascual [54] and Snyder *et al.* [55], detailed analyses of the main limitations of the staircase model, as a non-recommendable procedure to estimate fatigue limits even for finite lifetimes, is provided in [53,56].

3. Class III fatigue models as spurious sample function models

The Class III models are the rational evolution of the schematic Basquin-Wöhler model presented in Ciavarella-Monno [57] for subsequent application in the derivation of the Kitagawa-Takahashi diagram, see Fig. 4. In this section, the model proposed by Stüssi [36], Kohout-Věchet [37], Ravi-Chandran [38,39], D'Antuono [40] and Kurek *et al.* [41] are analyzed as a preliminary proposal for modeling the entire *S-N* field by a single equation encompassing the LCF, the HCF and the VHCF domains, see Fig. 5.

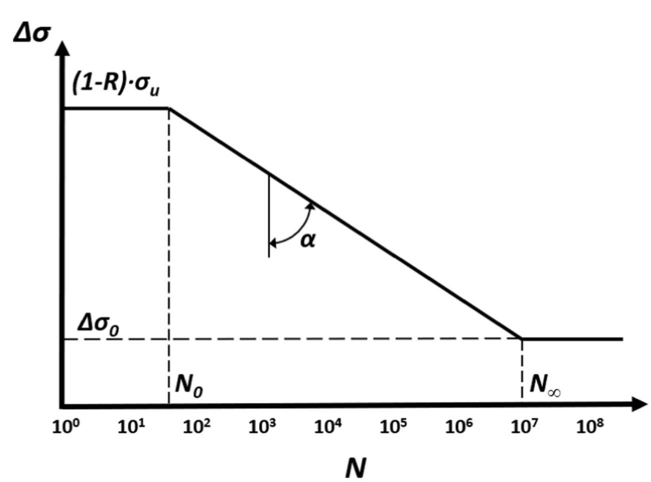

Fig. 4. The schematic Basquin-Wöhler law for a steel. . Adapted from [57]

The solution proposed by Kurek *et al.*, as a three-degree polynomial solution in log–log scale, see [41]:

$$\log\left(\frac{\Delta\sigma}{\sigma_u}\right) = a\log(N) + b\log^2(N) + c\log^3(N) \tag{7}$$

satisfies the upper condition of the LCF regime by allowing σ_a to be identified with $(\sigma_u - \sigma_m)/2$ for $N = 1$. Unlike the original formula of Kurek *et al.*, where $2{\cdot}N_f$ and the stress amplitude, σ_a, are used, respectively, Eq. (7) are referred to N and $\Delta\sigma$. Nevertheless, contrary to the remaining Class III models, which approach asymptotically to the fatigue limit, this model exhibits a minimum value of the regression function for a certain finite number of cycles. This is the consequence of being given by a three-degree polynomial solution, which it is well-known that cannot fit satisfactorily a sigmoidal shape. Consequently, the regression function proves to exist beyond the minimum point, failing to accomplish the condition of asymptotic matching to the fatigue limit, see [2,11–13].

All the remaining Class III regression models referred to in the introductory section fulfil both the upper bound condition represented by $\Delta\sigma = \Delta\sigma_u$, i.e. the trend dictated by the static test, and the condition of the asymptotic matching to the fatigue limit. Nevertheless, they differ in the way the regression function is defined. The models of Stüssi [36] and Kohout-Věchet [37], represent an immediate solution of the regression function, without any previous scaling, that fits directly the experimental results. In the models of Ravi-Chandran [38,39] and D'Antuono [40] the original variation field of the reference variable, i.e. of the driving force between the two bounds, is normalized to the interval [1] allowing a Weibull cumulative distribution function (cdf) to be used as the fitting regression function. The experimental results must be accordingly transformed for the estimation of the model parameters.

In the case of the Stüssi model [36], defined as:

$$\Delta\sigma = \frac{\sigma_u + aN^b\Delta\sigma_0}{1 + aN^b} \rightarrow N = \left[\frac{1}{a}\left(\frac{\sigma_u - \Delta\sigma}{\Delta\sigma - \Delta\sigma_0}\right)\right]^{\frac{1}{b}} \tag{8}$$

the asymptotic matching to the fatigue limit is contrived as a potential singularity of the denominator in Eq. (8), whereas the convergence to the upper bound requires the identification of the monotonic static case as the fatigue case for $N = 0$ (see Fig. 6). Supposedly, the formula is derived for $R = \sigma_m/\sigma_M = 0$, in which case $\Delta\sigma_u = \sigma_u$. The applicability of the Stüssi model to fit results of practical design in the LCF region is investigated by Toasa-Caiza and Ummendorfer [58] and Toasa-Caiza *et al.* [59] while, because its excessive conservatism, the Weibull model, see [3], is recommended instead to fit results in the VHCF regime. An extension of the Stüssi model to account for the effect of the stress ratio, R, is explored by the modified Stüssi model of Toasa Caiza-

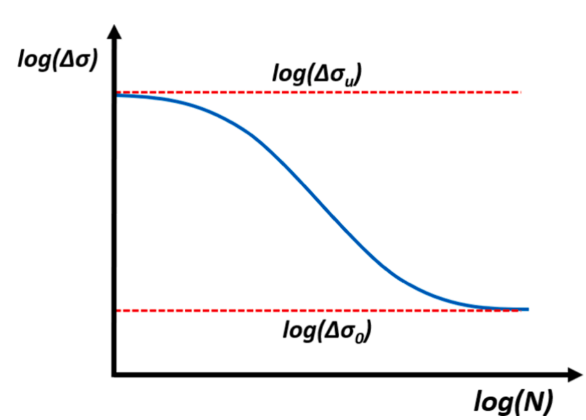

Fig. 5. Schematic Class III model represented as a single median percentile curve.

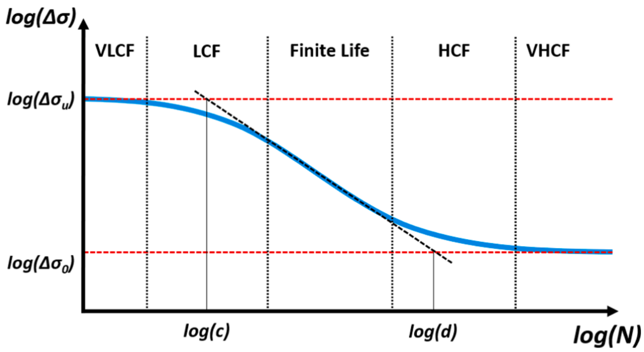

Fig. 6. Adaptation of the original Kohout-Věchet model as referred to stress ranges [36].

Ummendorfer [60] while Castillo and Canteli propose a three-dimensional σ_M-*R*-*N* model to this aim based on cross compatibility [33].

The Kohout-Věchet model [37] is based on the aprioristic assumption of a sigmoidal shaped *S-N* curve that includes two horizontal asymptotes corresponding with the upper and lower bounds, $\Delta\sigma_u$ and $\Delta\sigma_0$, to facilitate the fitting of results in both LCF and VHCF regimes. The authors extend the model even to the very low cycle fatigue (VLCF) regime, which is particularly disputable since it implies continuity between the monotonic failure and fatigue mechanisms (see below).

The equation of the original model is, supposedly, derived for $\sigma_m = 0$ whereas an extension to the general case of stress range is presented here:

$$\Delta\sigma = a\left[\frac{(N+c)d}{N+d}\right]^b \rightarrow log\Delta\sigma = loga + b\left(logd + log\left[\frac{N+c}{N+d}\right]\right) \tag{9}$$

For $N \to \infty$, $\Delta\sigma = \Delta\sigma_0$ and $\Delta\sigma_0 = ad^b$ while for $N \to 0$, $\Delta\sigma = \Delta\sigma_u$ so that $\Delta\sigma_u = ac^b$ from which the model takes the form:

$$\Delta\sigma = \Delta\sigma_0\left(\frac{N+c}{N+d}\right)^b = \Delta\sigma_u\left(\frac{1+\frac{N}{c}}{1+\frac{N}{d}}\right)^b \tag{10}$$

While the fatigue limit, $\Delta\sigma_0$, acts as a model parameter to be assessed from the experimental results, the upper limit, $\Delta\sigma_u$, is derived directly assuming σ_M to be the ultimate tensile stress of the material, σ_u. This allows the number of the model parameters to be reduced to three by considering $a = \Delta\sigma_u/c^b$. The *S-N* curve is additionally defined by two characteristic values of the number of cycles, c and d, which enforce a central symmetry of the model shape implying a severe and unjustified prescription. Furthermore, the scale effect cannot be considered in this model for transferability purposes.

The Kohout-Věchet model is extended to account for temperature effects [61]. Correia *et al.* [62] generalize its application based on single and combined power damage models referred to energy-based driving forces in order to enhance the efficiency of the estimation of the model parameters though pointing out the necessity of a probabilistic assessment for fatigue design.

Ravi-Chandran, see [38,39], has proposed the model:

$$\left(\frac{\Delta\sigma - \Delta\sigma_0}{\Delta\sigma_u - \Delta\sigma_0}\right) = \exp\left(-aN^b\right), \tag{11}$$

based on an asymptotic crack growth rate equation representing asymptotic evolution of the stress range for increasing crack growth. This is accomplished by assuming $1 - \frac{a}{W} = \left[1 - \frac{N}{N_f}\right]^k$, i.e. a function that relates the fractional uncracked section size $(1 - a/W)$ to the fractional remaining fatigue life $(1 - N/N_f)$.

The model represents a biparametric Weibull distribution for minima, where the normalized Δσ interval between the lower bound, $\Delta\sigma_0$, and the upper bound, $\Delta\sigma_u$, is identified with probability, while 1/a and b are the scale and shape parameter, respectively. The model satisfies the monotonic failure condition, $\Delta\sigma = \Delta\sigma_u$ for $N = 0$, and the asymptotic fatigue limit condition, $\Delta\sigma = \Delta\sigma_0 = (\sigma_u - \sigma_m)$, for $N = \infty$, respectively. Note that in the original version, the relations are established as a function of stress amplitudes considering $\sigma_{mean} = 0$, and the conversion of the Weibull cdf into a Gumbel one when log N are considered instead of *N*.

D'Antuono proposes in [40] the use of a biparametic Weibull cdf for minima as the adequate regression function evolved from the Basquin's power law to encompass the three fatigue regimes, LCF, HCF and VHCF as a whole *S-N* field. First, the driving force is normalized between the upper and lower bounds, $\Delta\sigma_u$ and $\Delta\sigma_0$, respectively so that the regression function is first normalized to the interval [1] and then identified as a biparametric Weibull cdf. Using the notch sensitivity factor based on Neuber's building blocks concept allows D'Antuono to smooth the Basquin's law by modifying the spurious fatigue limit while maintaining its associated number of cycles (in this case 10^6 cycles). By extension, a direct conversion of the *S-N* curve using the Weibull model is achieved to take into account the notch effect. The advocated identification of the slope of the sigmoidal *S-N* curve at the inflection point using the Basquin's power-law model contravenes the hypothetical advances represented by the adoption of a Weibull regression function since the simplistic linear log–log scale fitting of the Basquin law just depends on the chosen region for applying that fitting.

Note that the asymptotic fatigue limit implied in most of the Class III models represents an *a priori* model requirement in accordance with the asymptotic matching of the fatigue limit in [2].

A similar normalizing procedure implying identification with generalized extreme value (GEV) cdfs is applied to model cumulative damage processes using stochastic sample functions, as defined in [45]. Nevertheless, fundamental discrepancies can be observed with the above mentioned fitting models. In the damage processes, a sample function fitting the process register from a single specimen is identified after its normalization with a GEV cdf. The choice of the GEV function succeeds either by applying some discriminating statistical criteria or simply comparing the fitting quality observed. The high quality of fitting of sample functions using the GEV cdfs is justified as a subjacent statistical process of progressive damage character, see [45], whereas in the aforementioned Class III models, the data number representing specimen failures is more scarce. On the other hand, the normalized function in the above Class III models happens to be a regression function referred to the mean percentile curve fitted to several random results obtained from different specimen lifetime results, which exhibit a noticeable scatter band.

As a summary of the Class III models, the following comments seem to be pertinent:

i. All of them are supposed to represent mean or median percentile curves as reference functions, so that they are deterministic models, though those of Ravi-Chandran and D'Antuono are represented by Weibull cdfs. Therefore, the stochastic character of the whole *S-N* field is not provided.
ii. The definition of the probabilistic distribution of lifetimes (and correspondingly, of the driving force) along the sigmoidal-shaped regression function in these models entails strong assumptions that can be hardly established. Only the application of Bayesian techniques would represent a plausible solution to this necessary probabilistic improvement of the Weibull function.
iii. No model satisfies the compatibility condition F(*N*; Δσ) = F(Δσ; *N*) along the *S-N* field, see [3,22,24] and all of them present dimensional inconsistencies, which, at least in some cases, can be redressed introducing the adequate normalization into the variables and model parameters.
iv. The probabilistic definition of the *S-N* field is not provided, which requires it to be implemented *a posteriori* once the fatigue curve is

fitted, using for instance the boot-strap method or Bayesian techniques.

v. Only models based on Weibull cdfs are able to handle the size effect to ensure the transferability of laboratory results to practical components design, see [63–65].

vi. Pooling all *S-N* data as coming from a single statistical distribution, using a normalizing variable encompassing both the driving force and the number of cycles, is not feasible. This makes it impossible to improve the reliability of the parameter estimation, see [3].

The biparametric Weibull models for minima of Ravi-Chandran and D'Antuono can be enhanced by assuming three-parameter Weibull or Fréchet cumulative distribution functions (cdf), as the median percentile curves [66,67]. This would imply to recognize the Weibull location parameter, N_0, as the minimum number of cycles necessary to achieve LCF failures and the fact that the upper bound is referred to another model parameter denoted $\Delta\sigma_{up} = (1\text{-}R)\cdot\sigma_{up}$, rather than to $\Delta\sigma_u = (1\text{-}R)\cdot\sigma_u$, see Fig. 7, where σ_{up} is the upper cyclic limit to be distinguished from σ_u so that $\sigma_{up} < \sigma_u$. The adoption of both parameters is justified by the fact that the failure micro-mechanism in the monotonic loading can be hardly identified with that intervening in the LCF regime and because of the experimental discrepancy between the monotonic σ-ε and cyclic σ-ε diagrams with steady changes for the increasing number of applied cycles. Both arguments imply denying the identification of the static failure mechanism for monotonic loading as the effect occurring in a single fatigue cycle (or half a cycle, depending on the model used), and as a consequence, the transition without continuity solution from monotonic loading failure process to that in VLCF, and further on from the latter to that in LCF.

4. The extended Weibull regression model: Towards a general concept of the reference parameter in the *s-n* field

The probabilistic *S-N* Weibull model developed by Castillo and Canteli [3], given by Eq. (12):

$$p = 1 - \exp\left[\left(-\frac{\left(\log\frac{N}{N_0}\right)\left(\log\frac{\Delta\sigma}{\Delta\sigma_0}\right) - \lambda}{\delta}\right)^{\beta}\right]; \left(\log\frac{N}{N_0}\right)\left(\log\frac{\Delta\sigma}{\Delta\sigma_0}\right) \geq \lambda \quad (12)$$

arises from (a) physical conditions, (b) some requirements supported by the statistical extreme value theory (weakest link principle) and the compatibility condition $F(N;\Delta\sigma) = F(\Delta\sigma;N)$ (see Fig. 1), which allows a functional equation to be established [42]. Its solution provides the probabilistic definition of the *S-N* field as hyperbolic percentile curves. The relevance of the compatibility condition has been already emphasized in the Introduction and in Refs [3,33].

The fortresses of the model are, among others:

- its dimensionless form of being analytically defined,
- derivation based on demanding strong and necessary statistical conditions,
- reduction of the *S-N* field to a single cumulative distribution function of the normalized variable $V = \log(N/N_0)\log(\Delta\sigma/\Delta\sigma_0)$,
- enhanced reliability in the model assessment due to feasible pooling of all data as pertaining to a single distribution due to the normalizing property,
- its robustness as a result of the probabilistic definition of the *S-N* field by means of percentile curves,
- definition of the asymptotic fatigue limit as a model parameter not included as an initial premise in the derivation of the model but resulting from the solution of a functional equation,
- direct physical interpretation of the model parameters,
- simple way to account for the scale effect,
- easy consideration of the runouts in the evaluation,
- possible estimation of confidence intervals using the boot-strap method or Bayesian techniques,
- extension of the model to the strain-based approach and to any other fatigue analysis implying adequate choice of the driving force, as energy-based parameters,
- application to the cumulative damage design for life prediction under varying load with the definition of a probabilistic Palmgren-Miner rule,
- friendly and easy application to the practical data fatigue assessment using the free-use program ProFatigue [68,69].

The model is built on the premise that the stress range, $\Delta\sigma$, as driving force, is valid without any value limitation up to infinity. Since the maximum stress in the fatigue tests cannot exceed the ultimate strength due to physical reasons, the applicability of the model, when using $\Delta\sigma$ as driving force, expires as soon as $\Delta\sigma$ becomes $> \Delta\sigma_u$, or more precisely, as soon as $\Delta\sigma$ implies $\sigma_M > \sigma_p$, i.e. when σ_M exceeds the limit threshold of linear proportionality of the stress–strain diagram. In this work, for the sake of simplicity, the limit of proportionality is identified with the elastic limit of the material.

As a result, the model shows the following two apparent weaknesses, which are the consequence of the acceptance of the validity of the stress range as the driving force when the maximum stress exceeds σ_{lim}, i.e. the σ-ε proportionality threshold:

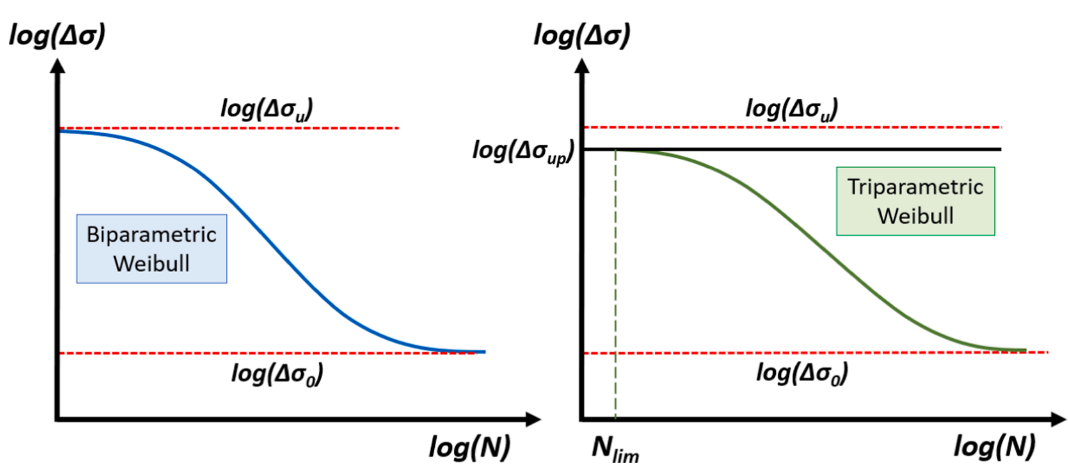

Fig. 7. Two-parameter Weibull Class III model and the alternative three-parameter Weibull model including location parameter and the upper bound represented by $\Delta\sigma_{up} \leq \Delta\sigma_u$.

- the asymptotic character of the $\Delta\sigma$ for $N = N_0$ is not consistent with the existence of an upper bound of the stress range related to the ultimate strength of the material, which makes impossible its application in the LCF region,
- the difficult physical interpretation of the vertical asymptote of the model, representing the lower limit of the number of cycles below which no damage accumulation begins.

As an advantageous probabilistic alternative to the Morrow equation, see [70], the Castillo-Canteli model is applied to the strain-based approach in the form of $\Delta\varepsilon$-N or ε_a-N fields, see [71]. In its derivation, the same requirements as those used in the S-N case are applied, but the limitations related to the upper bound of the stress range are in this case roughly avoided without practically incidence on the solution. In fact, no maximum strain is prescribed, and large, though only finite strain values are accepted as the alternative trend to the asymptotic strain values for $N = N_0$.

Finally, note that the variability of the fatigue limit, as an estimated model parameter, can be estimated, the same as those of the remaining model parameters, using the maximum likelihood method or the Bayesian technique. The latter allows the distribution of the model parameters to be determined as random variables rather than deterministic ones, see [31,72].

After analyzing the experimental results of a number of experimental fatigue programs, the adequacy of the stress range, $\Delta\sigma$, as a ubiquitous and universal driving force in the fatigue analysis, must be put under question in those fatigue programs in which the applied maximum stress exceeds the proportionality limit in the stress–strain curve. In such cases, it would be advisable to search for a driving force alternative to the conventional stress range $\Delta\sigma$, as a more adequate fatigue reference variable able to account for the potential influence of the non-linear relation between stresses and strains. In this respect, the consideration of the Smith-Watson-Topper parameter as a driving force in the model of Castillo-Canteli provides a satisfactory solution when applied to the probabilistic fatigue analysis in notches [73–76]. Among other energetic alternatives already envisaged, a new parameter, $GP = \Delta\sigma\cdot\varepsilon_M$, was proposed by the authors in [77], which provides fairly satisfactory results in the probabilistic assessment. Nevertheless, the lack of theoretical justification of this driving force and the critical comments of Dowling concerning its questionable suitability to consider the mean stress effect [78,79] suggest the search of a new alternative.

The proposed solution requires the extension and reinterpretation of the reference or generalized parameter (GP) in an attempt to include the effects of the non-linearity of the σ-ε relation into the fatigue assessment. In fact, the stress range $\Delta\sigma$, though masterfully recognized by Wöhler [80] as the traditionally accepted fundamental parameter from the initial fatigue studies to the present, it does not seem to represent the adequate parameter for the fatigue analysis when the maximal stress surpasses the linear elastic domain. The generalized local model (GLM) proposed by Muñiz-Calvente et al. [63–65] provides a consequent statistical treatment in the evaluation of experimental fatigue results and recognition of the limitations of the procedures currently employed in the evaluation of fatigue data including the size effect and its subsequent application to the components design. One of the fundamental contributions of this methodology is the feasibility of generalizing the concept of the reference generalized parameter (GP), i.e., the driving force, in agreement with the model chosen.

According to the above considerations, a new generalized reference variable denoted GRV_ε, as given in Eq. (13), is proposed based on the maximum stress, σ_M, rather than on the stress range, $\Delta\sigma$, as the driving force, which includes the influence of the tangent modulus of the σ-ε curve both on the maximum cyclic stresses and strains.

$$GRV_\varepsilon = \frac{\sigma_M}{\left(\left.\frac{d\sigma}{d\varepsilon}\right|_M\right)} = \sigma_M\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right) \tag{13}$$

The GRV_ε can be interpreted as a strain-based approach. In fact, when $\sigma_M < \sigma_p$, $\left.\frac{d\varepsilon}{d\sigma}\right|_M = 1/E$ so that $GRV_\varepsilon = \sigma_M/E = \varepsilon_M$. i.e., the maximum strain applied during the test, whereas when $\sigma_M > \sigma_p$, the GRV_ε represents the magnification of the maximum strain, ε_M, due to the monotonically decreasing slope of the σ-ε curve and its influence on the factor $\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right)$. This can be interpreted as the enhancing effect of the non-linearity between σ and ε on the fatigue damage:

Assuming the Ramberg-Osgood relationship to be applicable for the σ-ε relation:

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}} \tag{14}$$

the proposed driving force, $GRV_\varepsilon = \sigma(d\varepsilon/d\sigma)$, becomes

$$GRV_\varepsilon = \sigma_M\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right) = \frac{\sigma_M}{E} + \frac{1}{n'}\left(\frac{\sigma_M}{K'}\right)^{1/n'} = \frac{\sigma_M}{E} + \left(\frac{\sigma_M}{K'_\varepsilon}\right)^{1/n'} \tag{15}$$

Eq. (15) confirms that the new parameter GRV_ε represents a certain enhanced strain from a hypothetical Ramberg-Osgood equation in which a new parameter is included, given as $K'_\varepsilon = (n')^{n'} K'$, in the original relationship. This evinces that the GRV_ε represents a simple mapping of ε_M based on an σ-ε law affine to the original one of the material. While in the elastic domain the mapping should preserve the initial value of the variable ε_M, in the plastic region, the maximum strain ε_M is enhanced by the factor $\left.\frac{d\varepsilon}{d\sigma}\right|_M$ trending asymptotically to infinity (in the hypothesis of null hardening at the ultimate tensile strength) for the maximum of the σ-ε_M function.

This suggests a possible alternative generalized parameter, GRV_σ, simply defined as:

$$GRV_\sigma = \sigma_M\frac{E}{\left(\left.\frac{d\sigma}{d\varepsilon}\right|_M\right)} = E\sigma_M\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right) \tag{16}$$

This evinces the similitude between the stress-based approach using the new parameter GRV_σ as driving force and the strain-based approach when the strain range $\Delta\varepsilon$ is used as the reference parameter, the former allowing the influence of the plastic phase to be emphasized.

In this case, when $\sigma_M < \sigma_p$ then $\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right) = \frac{1}{E}$ so that the new driving force, GRV_σ, coincides with the maximum stress,σ_M, and so do both S-N fields, the modified and the conventional ones, without the assessment of the S-N field being affected by the change of the driving force. On the contrary, if $\sigma_M > \sigma_p$, the GRV_σ values, representing the transformed original σ_M data, may be fitted to the hyperbolic shape of the model proposed by Castillo-Canteli while maintaining the probabilistic definition of the S-N field by the percentile curves. In this case, the fulfilment of the compatibility condition between the cdfs $F(N;\sigma_M)$ and $F(\sigma_M;N)$ is extended even to the LCF region.

When the Ramberg-Osgood Eq. (14) is used to define the σ-ε relation, the GRV_σ becomes:

$$GRV_\sigma = E\sigma_M\left(\left.\frac{d\varepsilon}{d\sigma}\right|_M\right) = \sigma_M + \frac{E}{n'}\left(\frac{\sigma_M}{K'}\right)^{1/n'} = \sigma_M + \left(\frac{\sigma_M}{K'_\sigma}\right)^{1/n'} \tag{17}$$

Similar considerations as above can be drawn from Eq. (17), which proves that the new parameter GRV_σ represents a certain enhanced stress from a hypothetical Ramberg-Osgood equation where $K'_\sigma = (n'/E)^{n'} K'$.

In a general fatigue program, data for different values of $R = \sigma_m/\sigma_M$ are usually involved, see the application example in Section 5. In this case, when $\sigma_M < \sigma_p$, the fatigue characterization of a material can be described in terms of σ_M as primary variable, and R, as secondary variable, rather than in terms of $\Delta\sigma$ and R, see [33]. The extension to the non-linear portion of the σ-ε is straightforward by replacing σ_M by GRV_σ given by Eq. (16), whereas for the moment the conventional definition for R as σ_m / σ_M is maintained.

Assuming that the curve σ-ε reaches a maximum, the *GP* would adopt a singular value at this point, thus justifying the asymptotic trend of the model.

Finally, note that the conventional way of defining the strain energy can be transformed as a function of the proposed generalized parameter, *GP*:

$$U = \int_0^{\varepsilon M} \sigma d\varepsilon = \int_0^{\sigma M} \left(\sigma \frac{d\varepsilon}{d\sigma}\right) d\sigma = \int_0^{\sigma M} GP(\sigma, \varepsilon) d\sigma \quad (18)$$

where $GP(\sigma, \varepsilon) = GP(\sigma)$ for given σ-ε relation. This proves that the objective function, in this case the strain energy, can be indistinctly defined based on either the maximum stress σ_M or the mapped *GP* value.

Accordingly, the left and right hand integrals of Eq. (18) are equivalent and represent alternative ways of defining the amount of energy implied in the fatigue loading and unloading process irrespective of the integral being referred to $d\varepsilon$ or to $d\sigma$.

Incidentally, the limits of the integral being referred to stresses and not to strains seem to correspond better to the real control conditions established in the fatigue tests when the stress-based approach, i.e. the *S-N* field, is selected for fatigue characterization.

Therefore, nothing may be objected to the formal replacement of $\Delta\sigma$ by GRV_σ as the driving force when applied to the assessment of the *S-N* field, according to the model in Eq. (12)

$$p = 1 - \exp\left[\left(-\frac{(\log N/N_0)(\log \frac{GRV}{GRV_{\sigma,0}}) - \lambda}{\delta}\right)^{\beta}\right]; (\log N/N_0)(\log GRV/GRV_{\sigma,0}) \geq \lambda \quad (19)$$

which entails GRV_σ and N as the new model variables and the same meaning of the model parameters as in Eq. (12) except for $\Delta\sigma_0$, which is replaced by $GRV_{\sigma,0}$.

Nevertheless, this simple change allows the two limitations of the model proposed by Castillo and Canteli [3] to be overcome and, as a result, the probabilistic prediction of the *S-N* field to be extended to the LCF region, implying stresses beyond the linear elastic region of the σ-ε curve.

In an experimental program, a feasible solution in the search of a suitable definition of the shape of the curve σ-ε, and therefore of the reference parameter would consist in estimating the $d\varepsilon/d\sigma$ derivatives directly from the experimental monotonic σ-ε curve or preferably from the cyclic one. Since the cyclic curve is varying over the test and a unified methodology for the definition of the cyclic curve from loading-unloading tests seems far from being agreed, see [81], other options, as the use of the hysteresis loop, should be considered to ensure a reliable value of the $d\varepsilon/d\sigma$ derivative just from the specimen being tested. This seems to be in agreement with the Z_{dD} and P_J damage parameters proposed by Heitmann [82] and Vormwald [83], respectively, as driving forces in the fatigue Wöhler field analysis.

The credibility of the resulting model lies mainly on the compatibility on which it is based, allowing statistical and functional contradictions to be avoided. Among the number of fatigue models proposed in the literature, only those of Freudenthal-Gumbel [22], Bolotin [23–25] and Castillo-Canteli [3,14,33] models fulfil this important statistical requirement for the definition of the probabilistic *S-N* field. The two former models represent only one of the two possible solutions from the functional equation, in which the zero percentile hyperbola degenerates into the asymptotes in contradiction with the experimental evidence, see [3] pages 49–53. This is the reason why these compatible models are discarded as suitable models. Rigor and statistical consistency are the basis of the model derivation as confirmed by later extensions to strain-, and fracture mechanics-based approaches [3,33], which proves the interrelation among the three traditional approaches, stress-, strain- and fracture mechanics based approaches to the fatigue problem. In addition, the modified version of the model referred to $\sigma_M \left(\frac{d\varepsilon}{d\sigma}\Big|_M\right)$ instead to $\Delta\sigma$, or even σ_M, provides a unified probabilistic solution applicable over the three domains LCF, HCF and VHCF after evidencing the unsuitability of the stress-based approach, and the convenience of its substitution by an advanced "mixed stress–strain" based approach, possibly derived directly from the hysteretic cycle of the material.

5. Example of application

The experimental program on P355NL1 steel reported in [84] is used to compare the suitability of both the improved Castillo-Canteli model and the different Class III models reported above to fit fatigue data.

The mechanical properties of the fatigue results cover the LCF and HCF regions for three different stress ratios R = 0, −0.5 and −1 whereas VHCF data are unfortunately missed what impedes a reliable estimation of the fatigue limit (see Fig. 8). In the present data assessment, the maximum value of the stress, σ_M, is considered the driving force. This facilitates the comparison of results, whose R values are equal or less than zero, by considering in this case $\sigma_{M,0}$ and σ_u instead of $\Delta\sigma_0$ and $\Delta\sigma_u$, respectively, as boundary limits in the fitting process. Such a change does not affect the general procedure, but only simplifies it. Accordingly, the figures are referred to the maximum stress value on the ordinate axis.

5.1. Assessment using the Class III fatigue models

First, the models, of Kohout-Věchet and Stüssi are evaluated, followed by the biparametric Weibull mean function as that representing

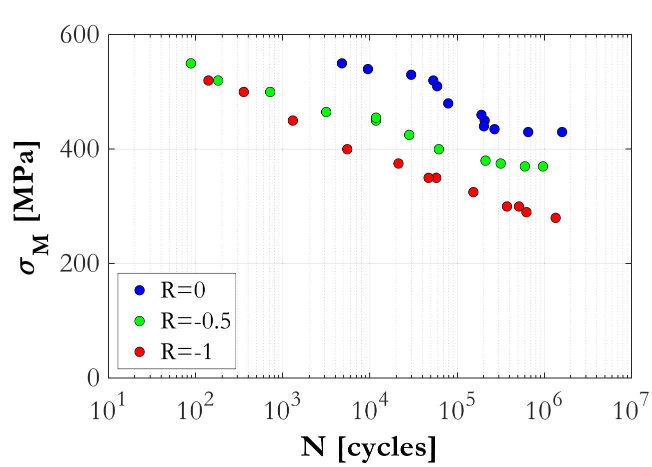

Fig. 8. Original fatigue results from the experimental campaign reported in [84] for three different R ratios.

the proposals of Ravi-Chandran and D'Antuono. As a second alternative, the improved three-parameter Weibull and Fréchet cdf versions proposed by the authors are also included. Latter extreme value distributions exhibit a location parameter ≥ 0 and tend asymptotically to infinity. Finally, the modified probabilistic Weibull model of Castillo-Canteli is applied to the assessment of the probabilistic *S-N* analysis of the above data. As explained in Section 4, the new GRV_σ resulting from the consideration of the tangent modulus of the σ-ε curve is used to replace the traditional stress range, $\Delta\sigma$, as the driving force. The results are discussed with particular attention to the upper and lower bounds of the regression curve and the limit number of cycles, N_0.

5.1.1. Assessment using current Class III fatigue models

The experimental values obtained under zero load ratio are initially considered. The Kohout-Věchet model, see Fig. 9-a is previously fitted providing the estimated parameter values shown in Table 1, among them both upper and lower bounds, i.e., $\Delta\sigma_0$ and $\Delta\sigma_u$. The latter is subsequently used in the evaluation of the Stüssi model so that only two additional parameters, namely $b_{\text{Stüssi}}$ and $a_{\text{Stüssi}}$ are required to complete the fitting procedure in the second case, see Fig. 9-b. The same bounds are also assumed in the evaluation of the 2-parameter Weibull distribution model according to Ravi-Chandran and D'Antuono, see Fig. 9-c and Table 2.

5.1.2. Improved Class III fatigue models

In the second alternative the improved model proposed in Section 3, see Fig. 7, based on five parameters, two representing the free upper and lower bounds, and the remaining three ones corresponding to the Weibull and Fréchet distributions is used. Nevertheless, due to the possible influence of the inconvenient test strategy followed in the experimental campaign implying absence of experimental results at the VHCF domain and, consequently, the fatigue limit region, two assessments are performed. The first one, assuming free both lower and upper model limits, and the second one assuming the lower limit, i.e. the fatigue limit, to be given as the one previously determined from the ProFatigue software [68,69]. In this way, a more realistic solution of the whole *S-N* field is achieved, see Fig. 10 and Table 3.

Table 1
Mechanical properties and cyclic hardening parameters of the P355NL1 steel used in the example of application, from [84].

Ultimate tensile strength, σ_u [MPa]	568
Monotonic yield strength, σ_y [MPa]	418
Young's modulus, E [GPa]	205.2
Poisson's ratio, ν	0.275
cyclic strain hardening coefficient, K' [MPa]	948.35
cyclic strain hardening exponent, n'	0.1533

5.1.3. Scrutiny of results

At first glance, all the above referred models seem to fulfil adequately the requirements to admit a successful data fitting without any of them representing clear superiority in terms of the regression coefficient R^2 values over the other ones. The values from Table 2, see Fig. 9, let conclude that the regression coefficient R^2 is practically independent of the model applied and the number of parameters, so that the assessment does not provide a definitive criterion to decide which of the proposals is the most convenient to be used for lifetime prediction. The same can be said about the results from the 3P-Weibull and Fréchet models in Table 3, Fig. 10 (a) and (b), though they point out the possible limit number of cycles in the LCF regime. After proceeding similarly as above to the assessment of the remaining stress ratios, $R = -0.5$ and -1, Fig. 11 summarizes the results of [84] using the different Class III models for the three different stress ratios, $R = 0$, -0.5 and -1 illustrating the similitude among the evaluations. Nevertheless, the results from the

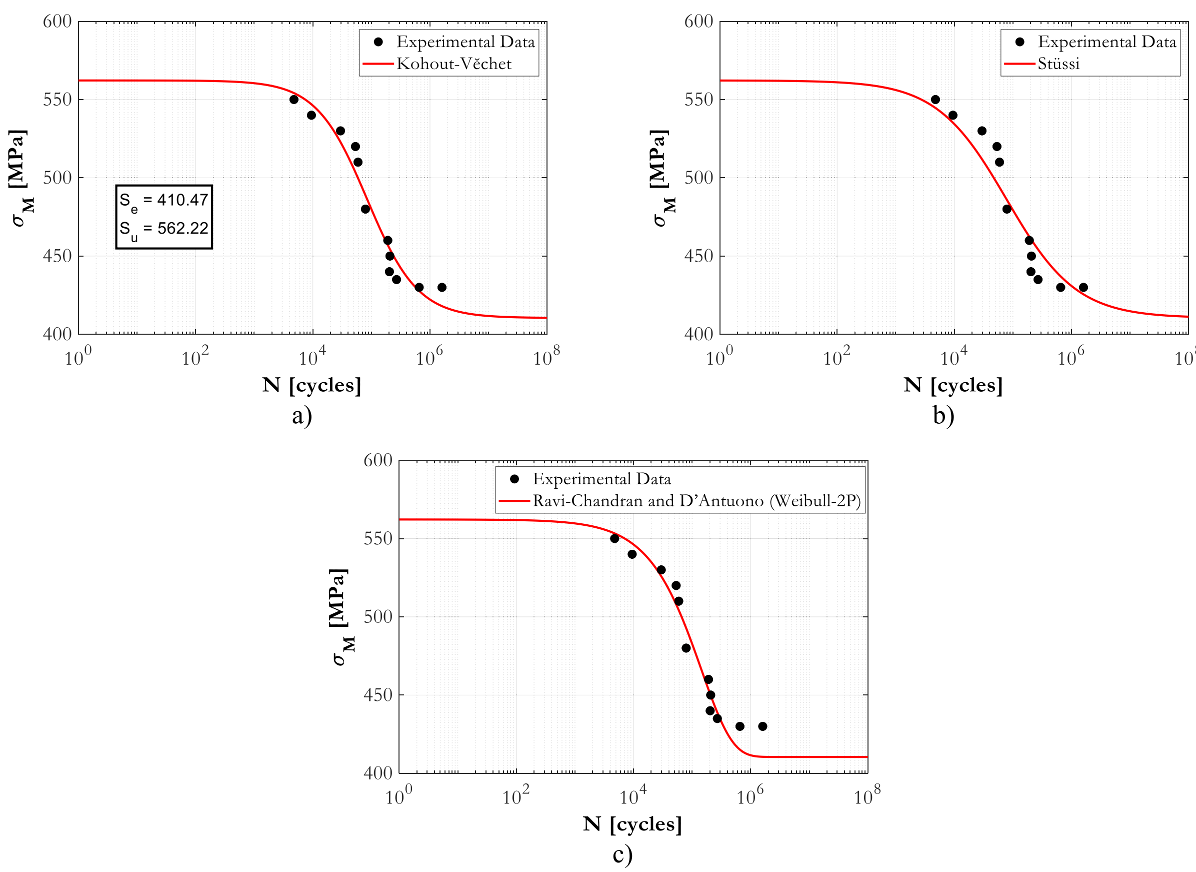

Fig. 9. Fiting of the experimental data in [84] for $R = 0$ using: (a) the Kohout-Věchet model; (b) the Stüssi model and (c) the Ravi-Chandran and D'Antuono models.

Table 2
Results from the assessment of the experimental data R = 0 for the material tested in [84] according to the different Class III fatigue models considered, see Fig. 9.

Class III models	$\Delta\sigma_0$ [MPa]	$\Delta\sigma_u$ [MPa]	a	b	c	d	R^2
Kohout-Věchet	410.47	562.22		−4.3107	95386.1	102,607	0.95264
Stüssi	*(410.47)	*(562.22)	$2.7580\cdot10^{-4}$	0.72772			0.90582
Weibull biparametric (Ravi-Chandran, D'Antuono)	*(410.47)	*(562.22)	$5.7065\cdot10^{-5}$	0.82273			0.94937

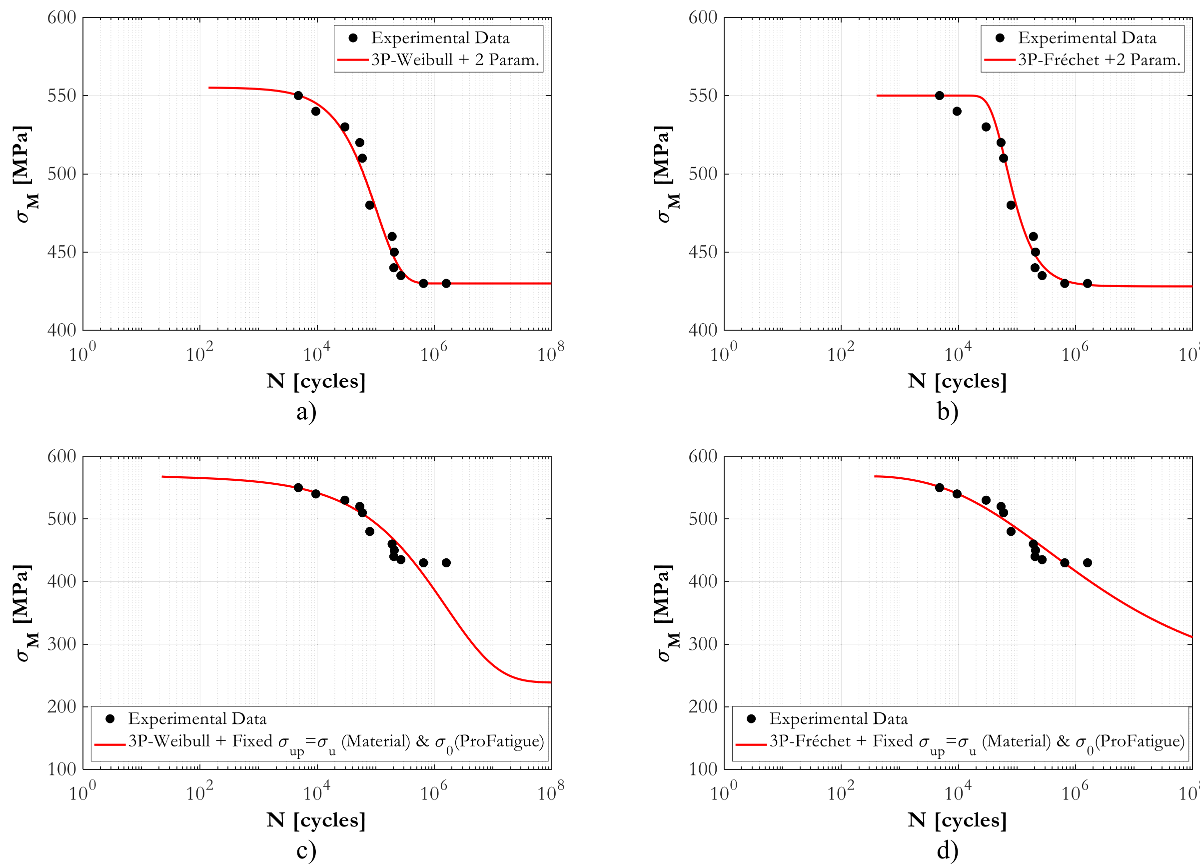

Fig. 10. Fitting of the experimental data in [84] for R = 0 using the five-parameter model. Top: For both limits free; (a) 3P-Weibull distribution; (b) 3P-Fréchet distribution. Bottom: For fixed upper and lower limits from ProFatigue assessment; (c) 3P-Weibull distribution; (d) 3P-Fréchet distribution.

Table 3
Results from the assessment of the experimental data R = 0 for the material tested in [84] according to the improved Class III fatigue model, see Fig. 10.

Improved Class III models	$\Delta\sigma_0$ [MPa]	$\Delta\sigma_{up}$ [MPa]	λ [cycles]	δ [cycles]	β	R^2
3P-Weibull + 2 additional parameters (lower and upper bound), Fig. 10-a	430.0	555.1	140	104,243	1.0355	0.9750
3P-Fréchet + 2 additional parameters (lower and upper bound), Fig. 10-b	428.1	550.0	400	65,399	1.5097	0.9684
3P-Weibull ($\Delta\sigma_u$ as the strength and $\Delta\sigma_0$ from ProFatigue), Fig. 10-c	238.7	568.0	22	1,586,775	0.4896	0.6831
3P-Fréchet ($\Delta\sigma_u$ as the strength and $\Delta\sigma_0$ from ProFatigue), Fig. 10-d	238.7	568.0	366	361,369	0.2476	0.8848

alternative 3P Weibull and Fréchet models in Table 3, Fig. 10 (c) and (d), taking into account a more realistic fatigue limit provided by the ProFatigue software [69], illustrates the hazard of an incorrect definition of that lower limit when the experimental campaign is not suitably performed omitting to cover the lower domain of the HCF region.

Nonetheless, the limitations of all former Class III model fittings in what concerns reliability for fatigue lifetime prediction in the LCF regime (but also in HCF and VHCF) is notorious due to the lack of probabilistic information provided by them and the weak basis on which the estimation of both lower and, particularly, upper model bounds is sustained.

In fact, the scarce number of data available and their unfavorable distribution, just clustered in the upper HCF region and omitted in the lower HCF region, represents a serious limitation to estimate the fatigue limit and, hence, to achieve a reliable determination of the complete *S-N* regression sigmoidal curve independently of resorting to the 3- or the 5-parameter models. Even assuming a high-quality data fit, the uncertainty of predictable lifetimes is remarkable due to the quasi-horizontal trend of the regression function when approaching to the upper bound, i.e. just in the critical region of low number of cycles. This implies a wide lifetime band in the LCF regime for very small variations of the applied stress range. The uncertainty of the lifetime estimation increases with the variability inherent to the deterministic character of the model, despite the choice of Weibull and Fréchet cdfs to fit the regression line, paradoxically. All this proves that the discussed Class III models are unsuitable to define the LCF region in a probabilistic, reliable way.

Finally, note that these empirical *S-N* models have to be distinguished from those stochastic models represented by sample functions in

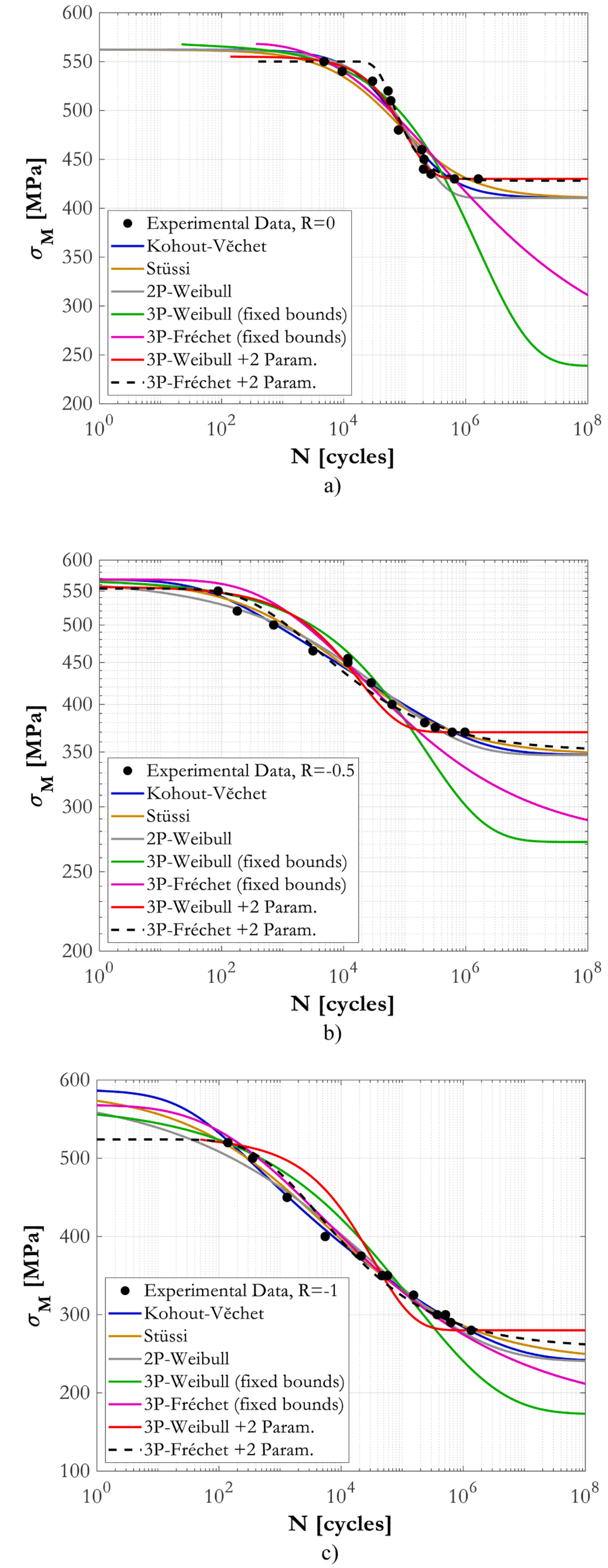

Fig. 11. Comparison among the Class III *S-N* fields obtained for the material tested in [84] for stress ratios R = 0, −0.5 and −1 using different *S-N* models (Kohout-Věchet, Stüssi, 2P-Weibull and 3P Weibull and Fréchet).

the sense defended by Bogdanoff-Kozin [43,44] and pointed out by the authors in [45]. In the former models, a regression line is fitted from random, statistically independent terminal results obtained from different tests whereas in the latter case the continuous record of damage is determined from a single specimen over the test though, generally, only a fraction of the process is feasible to be monitored, see [45].

5.2. Assessment using the improved Weibull model of Castillo-Canteli

In the search of a reliable alternative to the above Class III regression models to define the LCF region, the improved version of the probabilistic Weibull regression *S-N* model proposed by Castillo and Canteli, see Section 4, is applied to the assessment of the same experimental data reported in [84]. In this way, the probabilistic definition of the LCF, HCF and VHCF regions as percentile curves is achieved. For the sake of simplicity, the analysis is performed for the generalized reference variable, GRV_σ, as derived from σ_M according to Eq. (16), as a more suitable driving force than $\Delta\sigma$. The original experimental results represented as σ_M-N are showed in Fig. 8. The σ-ε curve of the material is assumed to follow a Ramberg-Osgood equation, see Eq. (14) and Fig. 12 (Left) with parameters $K' = 948.35$ MPa and $n' = 0.1533$, and $E = 205$ GPa, from which the stress gradients $d\sigma/d\varepsilon$ are first obtained and then applied to Eq. (16) to yield the GRV_σ values. Fig. 12 (Right) represents the magnification factor, $E(d\varepsilon/d\sigma)$, that allows the results of the maximum stress, σ_M, to be converted into the corresponding transformed values of GRV_σ helping us to interpret how the transformation evolves as a function of the σ_M value.

This conversion is understood as a mapping of the original σ_M-N field to the asymptotic GRV_σ-N field, where the highest possible σ_M value, i.e. σ_{up}, would correspond to $d\varepsilon/d\sigma = \infty$. Nevertheless, this will not be the case when the cyclic Ramberg-Osgood equation is used because it represents a monotonically non-decreasing parabolic function, which does not reach a maximum value at σ_{up}. In fact, to achieve the complete reconversion of the GRV_σ-N field into the conventional sigmoidal σ_M-N field showing both horizontal upper and lower limits, as those evaluated in Section 5.1, a σ-ε curve exhibiting an absolute maximum at σ_u would be required.

Figs. 13 and 14 show, respectively, the σ_M-N field resulting from the original Weibull model of Castillo-Canteli [3] and the GRV_σ-N field resulting from the proposed improved model for the three stress ratios $R = -1$, -0.5 and 0, for p = 0.01, 0.50 and 0.99, see Tables 4 and 5. The choice of these percentiles for the probabilistic *S-N* field representation could seem contradictory since, as well known, only low or very low percentiles are of interest in practical design. In this case, it obeys only to an attempt to give an overview of the variability field of the fatigue properties of this material. In fact, any desired percentile curve is easily obtained from the analytical expression, Eq. (19), by substituting the model parameters, see Table 6, already estimated using the ProFatigue software [68,69].

In the assessment of the present experimental campaign, the Ramberg-Osgood equation, apart from impeding to attain an asymptotic value of GRV_σ, as already mentioned, avoid any coincidence between both original and the transformed *S-N* fields even for small σ_M values. Even if the discrepancy among those values is unnoticeable for $\sigma_M < 100$ MPa, this is not applicable since unfortunately this value is below the fatigue limit, at least for $R = -1$ and -0.5. Nonetheless, this has no repercussion on the estimation of the model parameters. The relatively high values of the Weibull shape parameters allow the Weibull distribution to be approached as a Gumbel distribution, proving that their variability has no significant influence between the reciprocal results from both the original and the transformed S-N fields. In fact, despite the diverse fatigue limits obtained from the original and transformed results, the predicted lifetimes in a broad scope of the VHCF region (from 10^6 to 10^{10} cycles) are approximately the same, thus confirming the lower values fatigue limit as mentioned above. The equally low lifetime limit, N_0, obtained for the three stress ratios, point to the inconvenient

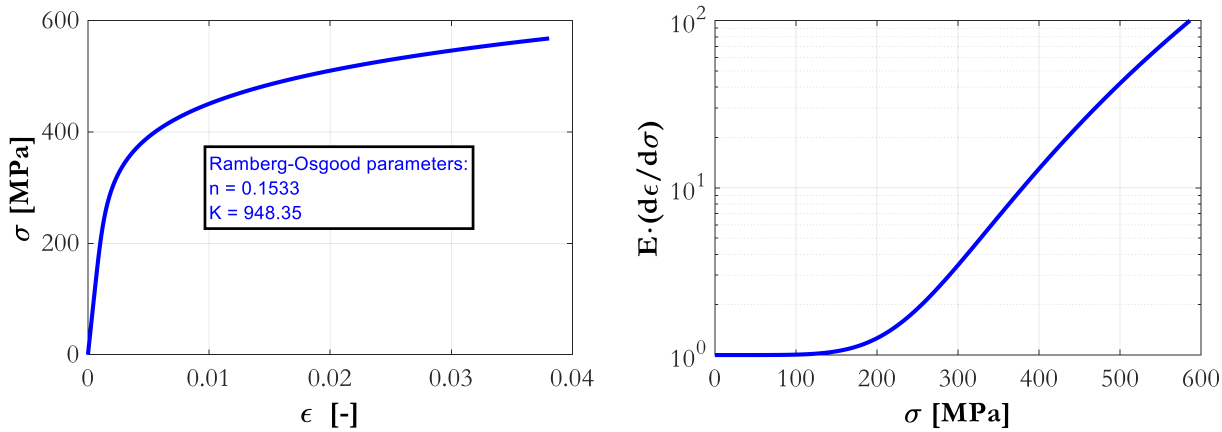

Fig. 12. (Left) Cyclic Ramberg-Osgood curve for the material tested in the experimental campaign reported in [84] and (Right) E·(dε/dσ)–σ curve derived from the Ramberg-Osgood equation.

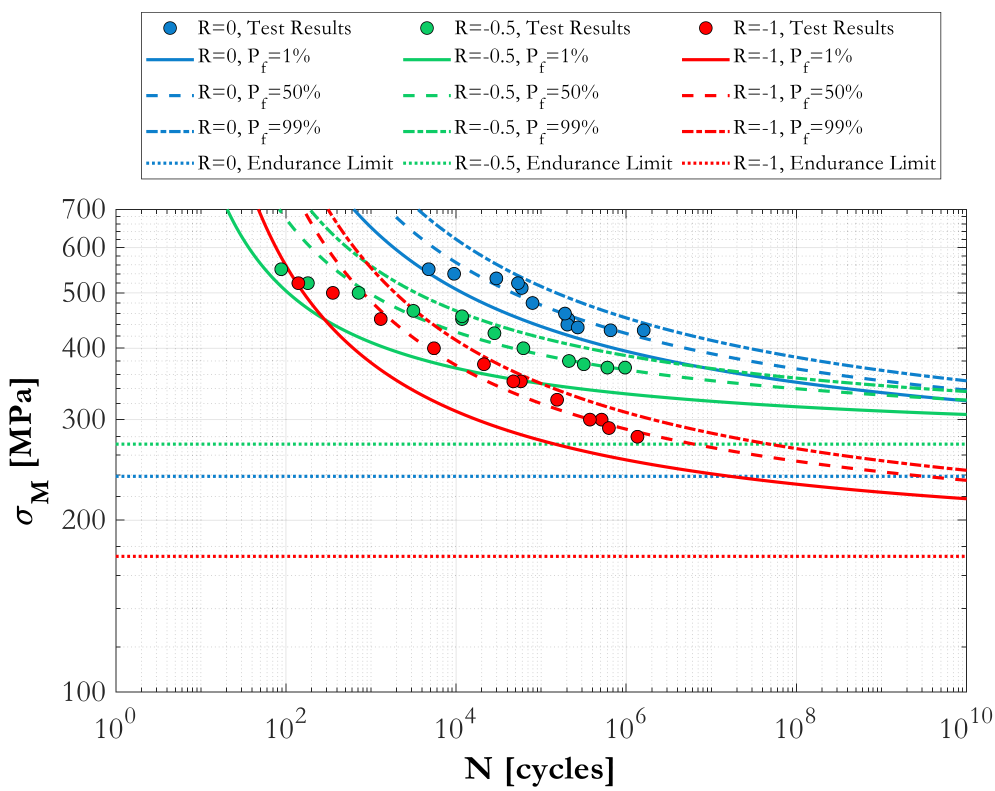

Fig. 13. Assessment of the fatigue properties of the material tested in [84] for the stress ratios R = 0, −0.5 and −1, and p = 0.01, 0.50 and 0.99 using the ProFatigue software [68,69]. σ_M-N field for the original fatigue data according to [3].

test planning.

The limit number of cycles, represented by the N_0 parameter, happens to be near zero according to the ProFatigue evaluation of this particular data set, contrary to what happens in the most practical cases in which it turns to be of thousands denoting the particularity of the estimated parameters in this example of application. This, besides the unreliable estimation of both upper and lower bounds of the sigmoidal regression line, provides the possible explanation of the negligible difference between the two- and three-parameter Class III models assessed in Section 5.1.

Finally, note that the comparison between Figs. 13 and 14 points out that the transformed results practically maintain their respective position though shifted up to higher values of the new driving force GRV_σ. This can be again assigned to the Ramberg-Osgood equation from which an almost linearly relation between the factor $E\left(\frac{d\varepsilon}{d\sigma}\right)$ and σ is observed, see in Fig. 12 (Right), implying a constant translation at the logarithmic scale irrespective of the σ value.

Planning a future experimental campaign based on a suitable test strategy implying the suitable data distribution in the three regions LCF, HCF and VHCF) is envisaged.

6. Discussion

From the comparative evaluation of the experimental data, it follows that satisfactory fittings are obtained for the Class III models without remarkable differences irrespective of the model used, see Fig. 11, for the three stress ratios, R, investigated in the practical example. There are

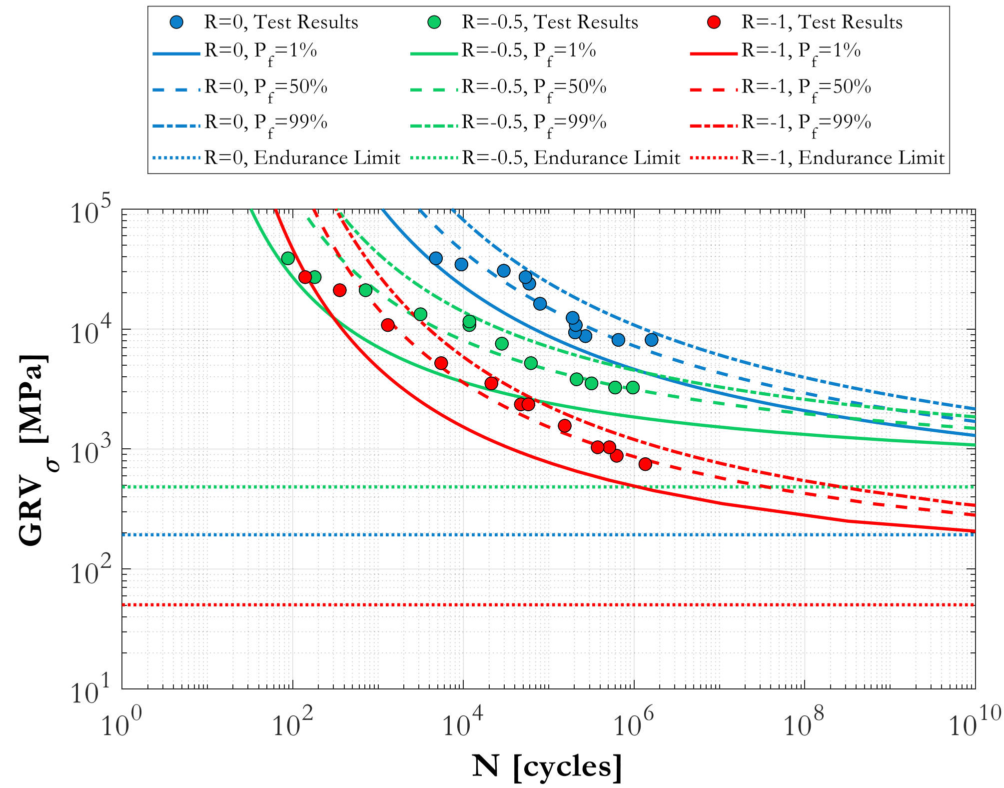

Fig. 14. Assessment of the fatigue properties of the material tested in [84] for the stress ratios R = 0, −0.5 and −1, and p = 0.01, 0.50 and 0.99 using the ProFatigue software [68,69]. GRV_σ -N field for the relocated fatigue data according to the proposed new Weibull model.

Table 4
Parameters of the Castillo-Canteli σ_M-N field model for the material tested in [84] (without considering relocation according to the proposed model), see Fig. 13.

Castillo-Canteli parameters (from ProFatigue software)	$R = 0$	$R = -0.5$	$R = -1$
B	0	0	0
N_0 [cycles]	1	1	1
C	5.48	5.60	5.15
σ_0 [MPa]	238.70	271.78	172.89
β	4.38	11.04	15.42
δ	1.74	4.35	7.29
λ	6.29	0	0

Table 5
Parameters of the Castillo-Canteli GRV_σ -N field model for the material tested in [84] with data relocation according to the proposed model, see Fig. 14.

Castillo-Canteli parameters (from ProFatigue software)	$R = 0$	$R = -0.5$	$R = -1$
B	0	0	0
N_0 [cycles]	1	1	1
C	5.26	6.18	3.92
GRV_0 [MPa]	192.85 (σ_0 = 172.78)	482.96 (σ_0 = 251.38)	50.35 (σ_0 = 50.34)
β	4.21	6.88	11.58
δ	10.64	16.85	26.41
λ	40.34	9.9	13.66

no criteria to decide which model would be more reliable due, first, to the impossibility to define precisely either upper and lower tails of the equations proposed; second, because the probabilistic assessment of these Class III models, pertaining to the stress-based approach, is yet pending, while the modified Stüssi model proposed by Toasa-Caiza and Ummendorfer [58–60] deserves further analysis, and third, because the $\Delta\sigma$ is considered to be an inadequate driving force to be used as a driving force in the LCF analysis. It suffices to compare the variations of the $\Delta\sigma$ and $\Delta\varepsilon$ fields implied in the non-linear region of the σ-ε material curve to verify the disproportion in the sensitivities expected for each of both variables. This suggests the convenience of substituting the stress-based approach, when $\Delta\sigma$ acts as the driving force, by the strain-based approach or, preferably, by the proposed generalized reference variable, GRV_σ to represent better the effect of the non-linearity in the σ- ε relation.

It follows that the class III models are not recommended for LCF failure prediction in components' design. Note that the key objective in the evaluation of experimental results is to guarantee the probabilistic transferability from the experimental results in the laboratory rather than to achieve a suitable, or even optimal data fitting. Only this way, the structural integrity principles will be accomplished in the practical design of components irrespective of the shape and size of the specimens tested or in service.

In any case, the capability of the different former regression Class III models in Section 3 to fit the experimental data from the campaign referred in [84] are disturbed by the following issues:

(a) The number of results located in the LCF region is generally scarce in the experimental campaigns focused on Class III models. This impedes a reliable estimation of the upper tail of the *S-N* curve. On the other side, when the LCF region is the specific region to characterize and adequately covered by experimental results, usually, the lack of results concerns the HCF and VHCF regions. In this case, inaccurate definition of the fatigue limit is unavoidable impairing a reliable estimation of the model parameters and so of the integral *S-N* field. It follows that a correct strategy in the fatigue test planning is crucial to achieve a reliable model.

Table 6
Predicted maximum stress values (MPa) for $p = 1\%$, 5%, 50% in the VHCF region from the original σ_M-N and GRV_σ -N fields, respectively.

From σ_M-N	R = 0			R = −0.5			R = −1		
N [cycles]	p = 1%	p = 5%	p = 50%	p = 1%	p = 5%	p = 50%	p = 1%	p = 5%	p = 50%
10^7	368,0	374,3	391,3	323,1	332,4	351,1	241,2	250,4	268,2
10^8	348,8	354,0	368,1	316,0	323,9	339,8	231,3	239,0	253,8
10^9	334,6	339,0	351,0	310,6	317,5	331,3	223,9	230,5	243,1
10^{10}	323,6	327,5	337,9	306,3	312,4	324,7	218,1	223,9	234,9
From GRV_σ -N	R = 0			R = −0.5			R = −1		
N [cycles]	p = 1%	p = 5%	p = 50%	p = 1%	p = 5%	p = 50%	p = 1%	p = 5%	p = 50%
10^7	363,6	370,1	387,8	323,6	332,2	351,3	228,7	241,2	263,2
10^8	342,6	348,2	363,3	314,9	322,4	339,0	208,7	221,0	242,3
10^9	326,5	331,4	344,6	308,1	314,8	329,5	190,9	203,3	224,4
10^{10}	313,7	318,1	329,9	302,7	308,8	322,0	175,2	187,4	208,6

(b) The possible Class III model improvement proposed by applying three-parameter Weibull and Fréchet solutions, which include N_0 and $\Delta\sigma_{up}$ as additional parameters, represents a conceptual advance but only if a sufficient number of data is available. Otherwise, i.e. when the number of results is scarce, it is not a recommendable alternative.
(c) It is not clear how the size effect can be included in the Class III models in order to ensure transferability from the laboratory fatigue results to design.
(d) A probabilistic definition of the Class III models studied in Section 3, for the improved suggested Weibull and Fréchet version with the two aforementioned parameters, N_0 and $\Delta\sigma_{up}$, could be achieved using the Bayes technique. Nevertheless, the limitations of such a regression model due to the lack of fulfilment of the compatibility condition, cannot be obviated.

The nearly horizontal upper tail of the sigmoidal regression function in the Class III models studied in Section 3 reveals a likely high variability of fatigue lifetimes even for small variations of the stress range, see Fig. 15. It follows that descriptive (data evaluation) and predictive (lifetime prediction) analyses in the practical LCF design risk being unreliable using this kind of models. This way, the improved Weibull model of Castillo-Canteli, based on the substitution of the stress range by a generalized reference variable (GRV_σ) as the driving force, while preserving the probabilistic requirements in its derivation, comes up as the more convincing alternative to achieve the assessment of a complete GRV_σ -N field encompassing the LCF region. Note that a redefinition of the traditional $\Delta\sigma$-N field from the GRV_σ -N ones is superfluous in the practical design. In fact, the equivalence of lifetime prediction using either the stress-based GRV_σ -N or the strain-based GRV_ε -N approach is recognized as a single and advantageous unified methodology, which suggests obviating the traditional stress-based approach.

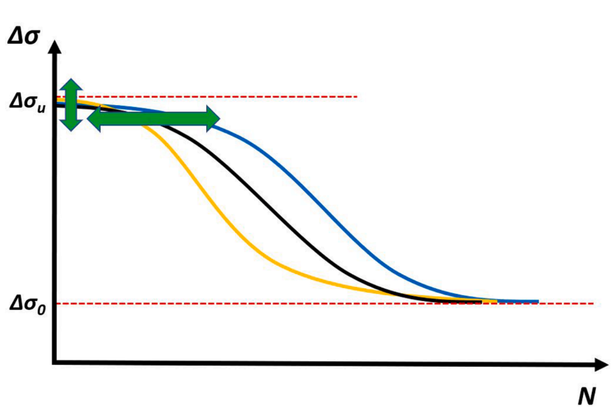

Fig. 15. Schematic illustration of the sensitivity of lifetime due to small stress range variations.

The substitution of the conventional stress range, as the driving force, by the new GRV_σ parameter in the Castillo-Canteli model allows the model to overcome the two limitations implied in its basic version, namely, to justify the asymptotic behaviour represented by the hyperbolic solution, and the presence of a limit number of cycles, N_0. As a result, the admissibility of a possible continuous transition from the monotonic to LCF failure over the VLCF regime must be rejected as failures being generated according to different specific mechanisms. In this way, the model provides the whole probabilistic definition of the S-N field, including the LCF region, through the definition of percentile curves extended to the LCF region. Given the limitations exhibited by the Ramberg-Osgood equation, the choice of an adequate σ-ε function assumed for the calculation of the derivative $d\varepsilon/d\sigma$ turns out to be key point for the reliable calculation of the GRV_σ values.

Lastly, some comments are added regarding a possible comparison among different fatigue models to decide which one fits “the best”, (goodness of fit test), see [1,85]. Some basic statistical requirements on the models are a minimal prerequisite for the models to be accepted as candidates for comparison. Otherwise, the models are compared under inconsistent criteria. If compulsory constraints are relaxed and overlooked for some of the models, their assessment disposes about higher freedom to improve the fit but in an irregular, fraudulent and misleading way, particularly if only a restricted domain of the whole model is chosen for the assessment. If the number of results is scarce or ill-balanced without covering the whole potential existence field (as usually happens and is the case of the experimental campaign evaluated here, see Section 5.1.2) the fulfilment of additional requisites, such as the compatibility condition, may represent an additional constrain for an advanced compatible model compared with a constraint-free elementary model. In any case, the compatibility condition is essential for a consistent S-N fatigue model but rarely fulfilled, see [29–31]. This is just the case when a small pseudo-linear piece in the HCF region is scrutinized where the data do not still exhibit yet clearly the trend to the S-N incurvation. In fact:

- The comparative analysis must be extended to the whole potential field of application of the phenomenon i.e. not limited to a particular domain covered by the chosen fatigue data ignoring the complete field of existence of the models. This represents a tricky way to distort the objectivity of the comparison by limiting it to a “conveniently chosen” restricted domain of application (as would be limited to the HCF region). Accordingly, the capability of extrapolation of the model beyond the test scope is an important criterion of validity, particularly, for prediction in structural design.
- The comparison of the models must be referred to the real low failure percentiles considered in the real components’ design (p = 0.05, 0.01, or even lower) and not for p = 0.50, i.e. the mean percentile curve in the S-N field, which reference is irrelevant to judge the model quality as associated to a safe design avoiding catastrophic events. This is why the extreme value statistics are suggested by the

authors, see [3], but also by Freudenthal-Gumbel [22] and Bolotin [23–25].

- The comparison cannot be referred indistinctly to models pertaining to Class I and Class II or even Class III, and the same applies for comparing probabilistic with deterministic models.
- Both the test planning strategy applied and the number of data in the sample may exert a noticeable influence on the assessment and therefore on the test goodness results, see [3]. In fact, the testing strategy responds to the specific fatigue model trying to be confirmed.
- Further abilities of the model, as its capability to be further extended to include the influence of secondary parameters, such as the stress ratio *R*, while maintaining the functional structure is surely a reliability criterion that reinforces its robustness and credibility.
- According to the statistical theory, the randomness concomitant to any experimental sample data do not ensure that the correct distribution provides necessarily the best fitting for a particular sample. This is why the analysis is complemented with a relative evaluation of the interpretation of the results referring to, for example, the Kolmogorov-Smirnoff statistic, which allows the goodness of fit to be relativized when comparing a sample with a reference cdf, or two samples each other.

This proves that a judgment about the hypothetic best fatigue model is difficult if not impossible, in particular when different models pertaining to the above mentioned different Classes are mixed in promiscuity. In this respect, we point out the inadequate comparison sometimes performed in the literature based on the presumable objectivity provided from the comparison of the R^2 coefficient, see [1,85], where none of the foregoing remarks have been observed. Even respecting strength statistical criteria such comparisons are of limited value and only the application to the real practical design provides criteria for the acceptation or non-acceptation of a model.

7. Conclusions

The main conclusions derived from this work are the following:

- The fatigue models used in the stress-based approach are classified according to the LCF, HCF and VHCF domains of the *S-N* field to which they can be applied. The attention is focused on models labelled Class III, which allow data fitting to be extended to all fatigue regimes.
- The Class III models of Kohout-Věchet, Ravi-Chandran and D'Antuono provide, in principle, satisfactory fitting of fatigue results in the three domains. Nevertheless, the inadequacy of $\Delta\sigma$, as the reference driving force, and the inherent deterministic definition of the S-N field does not ensure a reliable *S-N* field to be used in a practical fatigue design. The same applies to the Stüssi model, which additionally proves to be too conservative in the VHCF region.
- Two additional limitations of the Class III models analyzed are corrected with the consideration of the proposed three-parameter Weibull model which provides new model parameters such as: a) the limiting lifetime, N_0, which represents the minimum number of cycles for LCF failures to occur, and b) the upper bound of the S-N field $\Delta\sigma_{up}$, to be distinguish from $\Delta\sigma_u$, i.e. the stress range resulting from the ultimate stress σ_u. This modified model points out the inadequacy of the conventional driving force $\Delta\sigma$, and the discontinuity in the transition from the monotonic to the LCF zone, whose tests evidence the failure mechanisms diversity.
- The application of the former Class III models to the whole *S-N* field assessment of one experimental program confirms, on one side, the satisfactory applicability of the former Class III models by fitting data to the LCF domain with descriptive purposes. On the other side, it also evidences the limitations of these models when a probabilistic failure prediction is considered, which can be assigned to the sigmoidal shape of the S-N field due to the driving force selected. In fact, the compatibility condition is not fulfilled. From this, it follows the need for a definitive replacement of $\Delta\sigma$ as the driving force for the S-N assessment and, therefore, the need to renounce the stress-based approach in the evaluation of the LCF results for failure prediction.
- The probabilistic Weibull model of Castillo-Canteli, which should be classified as a Class II model when the conventional $\Delta\sigma$ is taking as the driving force, can be extended towards a Class III model when the generalized reference resulting from the variable $GRV_\sigma = E{\cdot}\sigma{\cdot}(d\varepsilon/d\sigma)$, instead of $\Delta\sigma$, or even σ_M is adopted as the reference driving force. This makes this model a compatible one to be applied for the probabilistic prediction of fatigue failure over the whole S-N field, i.e. over the three fatigue domains, LCF, HCF and VHCF. In this case, the limiting condition $\sigma_M \leq \sigma_{up}$ is ensured and the asymptotic fatigue limit is included as a model parameter, while the normalization of the *S-N* field implies more reliable assessment through data pooling and easier evaluation of cumulative damage under varying load.
- The adoption of advanced driving forces as the one proposed to improve the model of Castillo-Canteli, suggests the traditional classification of stress- and strain-based approaches to be unified as a unique mixed approach which simultaneously incorporates the advantages of both approaches to be applied in a components' design.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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