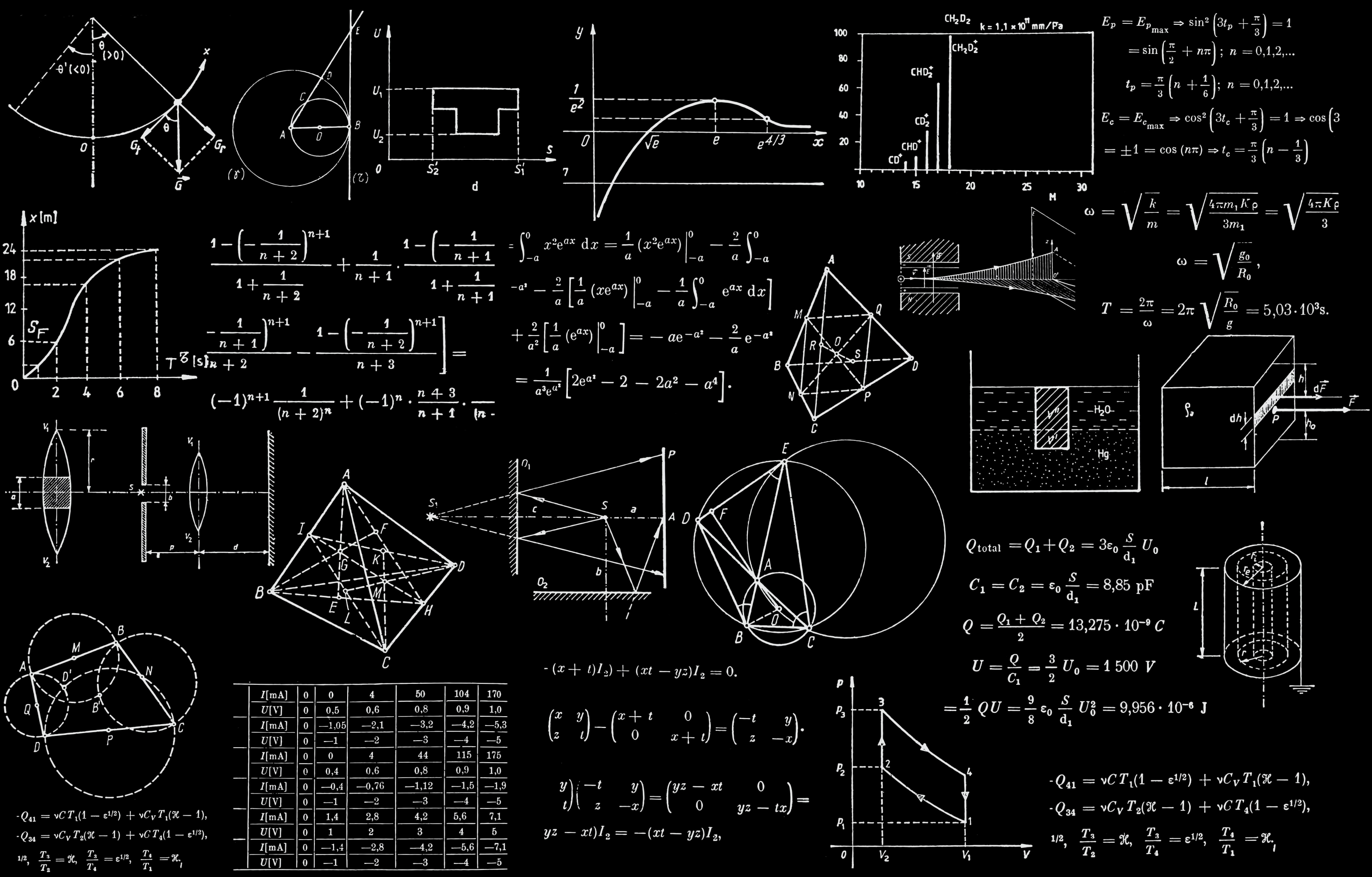
Convolution-Dominated Matrices in Groups of Polynomial Growth

MSc Thesis

Christos Kitsios

Delft University of Technology



TU Delft

Convolution-Dominated Matrices in Groups of Polynomial Growth

by

Christos Kitsios

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Thesis committee:	Prof. dr. D. C. Gijswijt ,	TU Delft
	Dr. M. P. T. Caspers ,	TU Delft, supervisor
	Dr. J. T. van Velthoven,	TU Delft, supervisor

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Preface

This report concludes my thesis project, the final part of my master studies in Applied Mathematics at Delft University of Technology. The project was undertaken under the supervision of Martijn Caspers and Jordy Timo van Velthoven.

I have tried to keep the thesis report as self-contained as possible, in order to help potentially interested readers. I believe that master students with some knowledge in operator algebras can fully comprehend the proofs of the results given in the first part of the report. Regarding the second part, which contains applications of convolution-dominated matrices, some familiarity with representation and frame theory will be helpful to the reader.

I would like to express my gratitude to my supervisors for their support and guidance during the project. Moreover, I would like to thank Dion Gijswijt, Martijn Caspers and Jordy Timo van Velthoven, for taking part in the thesis committee for my graduation.

Christos Kitsios
Delft, July 2022

Abstract

In this thesis, we use a variation of a commutator technique to prove that ℓ^p-stability is independent of $p \in [1, \infty]$ for convolution-dominated matrices indexed by relatively separated sets in groups of polynomial growth. Moreover, from the inverse-closedness of the Schur matrices we deduce a Wiener type Lemma for the matrices in the intersection of the convolution-dominated matrices, $CD_{w_\alpha}(\Lambda)$, over all polynomial weights w_α, where $\alpha \in \mathbb{N}$. Finally, applications of the convolution-dominated matrices are presented. We prove the inverse-closedness of a non-commutative space generated by a discrete series representation restricted to a lattice in a nilpotent Lie group. In addition, we apply the aforementioned result on ℓ^p-stability to show that if $\pi(\Lambda)g$ is a p-frame for the coorbit space $Co(L^p)$ for some $p \in [1, \infty]$, then $\pi(\Lambda)g$ is a q-frame for the coorbit space $Co(L^q)$ for each $q \in [1, \infty]$, where (π, H_π) is a discrete series representation of a group G of polynomial growth, $\Lambda \subseteq G$ is a relatively separated set and $g \in H_\pi \backslash \{0\}$ is such that $V_g g = \langle g, \pi(\cdot)g\rangle$ is in the Amalgam space $W_{w_\alpha}(G)$. Moreover, we prove that the frame operator of the frame $\pi(\Lambda)g$ is invertible on the coorbit spaces $Co(L^p)$ for each $p \in [1, \infty]$, under the assumption that $g \in H_\pi \backslash \{0\}$ is such that $V_g g = \langle g, \pi(\cdot)g\rangle \in W_{w_\alpha}(G)$ for each $\alpha \in \mathbb{N}$.

Contents

Part I

Main

1

Introduction

1.1. Classical Wiener's Lemma

Let us denote with $\mathscr{A}(\mathbb{T})$ the set of periodic continuous functions which possess an absolutely convergent Fourier series,

$$\mathscr{A}(\mathbb{T}) := \left\{ f \in C(\mathbb{T}) \ : \ f(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi nti}, \ \|f\|_{\mathscr{A}} := \|(a_n)_{n\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} < \infty \right\}.$$

Norbert Wiener used the following lemma in his proof of a "Tauberian Theorem" [67].

Wiener's Lemma. If $f \in \mathscr{A}(\mathbb{T})$ such that $f(x) \neq 0$ for all $x \in \mathbb{T}$, then $1/f \in \mathscr{A}(\mathbb{T})$.

Thus, for $f \in \mathscr{A}(\mathbb{T})$, instead of explicitly calculating the Fourier coefficients of the function $1/f$, with the use of Wiener's Lemma we can conclude that the inverse $1/f$ has an absolutely convergent Fourier series by checking whether f vanishes on $\mathbb{T}$.

Naimark [47] observed that Wiener's Lemma describes the relationship between the Banach algebra of continuous functions on the torus, $C(\mathbb{T})$, and its Banach subalgebra of continuous functions with an absolutely convergent Fourier series, $\mathscr{A}(\mathbb{T})$. He observed that for a function $f \in \mathscr{A}(\mathbb{T})$ the assumption $f(x) \neq 0$ for each $x \in \mathbb{T}$ in Wiener's Lemma is equivalent to the invertibility of the function in the Banach algebra $C(\mathbb{T})$, equipped with the pointwise multiplication. Accordingly, Naimark introduced the definition of inverse-closedness by calling the pair $(\mathscr{A}, \mathscr{B})$ *Wiener's pair* when $\mathscr{A}$ is an inverse-closed Banach subalgebra of the Banach algebra $\mathscr{B}$. For $\mathscr{A}$ and $\mathscr{B}$ two Banach algebras with common identity, such that $\mathscr{A} \subseteq \mathscr{B}$, we call $\mathscr{A}$ *inverse-closed* in $\mathscr{B}$ if

$$a \in \mathscr{A} \ , \ \ a^{-1} \in \mathscr{B} \ \ \Rightarrow \ \ a^{-1} \in \mathscr{A}.$$

The algebra $C(\mathbb{T})$ equipped with the pointwise multiplication operation is a Banach algebra and $\mathscr{A}(\mathbb{T})$, equipped with the pointwise multiplication, is a Banach subalgebra of $C(\mathbb{T})$. Hence, with the introduction of the definition of inverse-closedness we can now restate Wiener's Lemma.

Wiener's Lemma. The Banach algebra $\mathscr{A}(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$.

In general, we can state Wiener type lemmas that provide the inverse-closedness of a Banach subalgebra in a Banach algebra.

1.2. Wiener's Lemma for convolution operators on the integers

For an absolutely summable sequence $a \in \ell^1(\mathbb{Z}^d)$ we define the corresponding convolution operator as follows,

$$C_a \ : \ \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d), \ c \longmapsto c * a,$$

where $*$ is the convolution on sequences defined by

$$\begin{aligned} * \ : \quad & \ell^1(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d) && \longrightarrow \ell^2(\mathbb{Z}^d) \\ & \big((a(n))_{n\in\mathbb{Z}^d} \ , \ (b(n))_{n\in\mathbb{Z}^d}\big) && \longmapsto a * b = \left(\sum_{m\in\mathbb{Z}^d} a(m)b(n-m) \right)_{n\in\mathbb{Z}^d} . \end{aligned}$$

We can identify each function $f \in \mathscr{A}(\mathbb{T})$ with the convolution operator $C_a \in \mathscr{B}(\ell^2(\mathbb{Z}))$, where a is the sequence of the Fourier coefficients of f, i.e. $a = (\mathscr{F}f(n))_{n\in\mathbb{Z}}$. This mapping is well defined, since by definition for $f \in \mathscr{A}(\mathbb{T})$ we have that $a = (\mathscr{F}f(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ and hence C_a is an operator in $\mathscr{B}(\ell^2(\mathbb{Z}))$. Thus, we obtain a mapping, which can be shown to be an isomorphism between the Banach algebra $\mathscr{A}(\mathbb{T})$ equipped with the pointwise multiplication and the class of convolution operators $(C_a)_{a\in\ell^1(\mathbb{Z})}$ equipped with the composition of operators. Therefore, we can rephrase Wiener's Lemma in terms of convolution operators.

Wiener's Lemma. The class of convolution operators is inverse-closed in $\mathscr{B}(\ell^2(\mathbb{Z}))$, or, equivalently, if $a \in \ell^1(\mathbb{Z})$ is such that C_a is invertible in $\mathscr{B}(\ell^2(\mathbb{Z}))$, then there exists $b \in \ell^1(\mathbb{Z})$ such that $C_b = (C_a)^{-1}$ in $\mathscr{B}(\ell^2(\mathbb{Z}))$.

Similarly, a Wiener's Lemma can be proved for the class of convolution operators in $\mathscr{B}(\ell^2(\mathbb{Z}^d))$. Furthermore, the previous can be extended for convolution-dominated matrices on $\mathbb{Z}^d$. A matrix $A = (A(i,j))_{i,j\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d\times\mathbb{Z}^d}$ is called *convolution-dominated* if there exists a sequence $d \in \ell^1(\mathbb{Z}^d)$ such that

$$|A(i,j)| \leq d(i-j), \tag{1.1}$$

for each $i,j \in \mathbb{Z}^d$. If $A = (A(i,j))_{i,j\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d\times\mathbb{Z}^d}$ is a convolution-dominated matrix, then for each $c \in \ell^2(\mathbb{Z}^d)$ and each $i \in \mathbb{Z}^d$ we have

$$|Ac(i)| = \left|\sum_{j\in\mathbb{Z}^d} A(i,j)c(j)\right| \leq \sum_{j\in\mathbb{Z}^d} |A(i,j)c(j)| \leq \sum_{j\in\mathbb{Z}^d} |d|\,(i-j)\,|c(j)| = |c| * |d|\,(i).$$

Therefore, A is pointwise dominated by the convolution operator C_d and this explains the name convolution-dominated matrices. Baskakov [4], Gohberg, Kaashoek and Woerdeman [26] and Kurbatov [42] proved a Wiener type Lemma for the class of convolution-dominated matrices, by showing that if A is convolution-dominated and is invertible in $\mathscr{B}(\ell^2(\mathbb{Z}^d))$, then its inverse $A^{-1} \in \mathscr{B}(\ell^2(\mathbb{Z}^d))$ is a convolution-dominated matrix. This result was also proved later by Sjöstrand [60] with a completely different proof, using a commutator technique.

Similar results can be obtain for other classes of matrices over the integers. Gröchenig and Leinert [32] proved a variation of Wiener's Lemma for twisted convolution operators. These operators are defined in the same manner as convolution operators, but with the use of a twisted convolution. Given a $\theta > 0$ we define the *twisted convolution* $\natural$ of two sequences $a = (a(n))_{n\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ and $b = (b(n))_{n\in\mathbb{Z}^{2d}} \in \ell^2(\mathbb{Z}^{2d})$ by

$$a\natural b(m,n) = \sum_{k,l\in\mathbb{Z}^d} a(k,l)b(m-k,n-l)\mathrm{e}^{2\pi\theta i(m-k)\cdot l},\ \forall\, m,n \in \mathbb{Z}^d.$$

For a sequence $a = (a(n))_{n\in\mathbb{Z}^{2d}} \in \ell^1(\mathbb{Z}^{2d})$ we define the *twisted-convolution* operator as follows

$$C_a^\natural \;:\; \ell^2(\mathbb{Z}^d) \longrightarrow \ell^2(\mathbb{Z}^d),\ c \longmapsto c\natural a$$

and we observe that $C_a^\natural$ acts on $\ell^2(\mathbb{Z}^d)$ as the matrix given by

$$C_a^\natural = A\left((m,n),(k,l)\right)_{(m,n),(k,l)\in\mathbb{Z}^{2d}} = \left(\mathrm{e}^{2\pi\theta i(m-k)\cdot l}a(m-k,n-l)\right)_{(m,n),(k,l)\in\mathbb{Z}^{2d}}.$$

Thus, for each $m,n,k,l \in \mathbb{Z}^d$ we obtain

$$|A\left((m,n),(k,l)\right)| = |a(m-k,n-l)|$$

and since $a \in \ell^1(\mathbb{Z}^{2d})$ we deduce that $C_a^\natural$ is a convolution-dominated matrix. The inverse-closedness for the class of twisted convolution operators can also be proved by treating the twisted-convolution operators as convolution-dominated matrices and using the Wiener's lemma for the latter, see [28]. Furthermore, results for Gabor frames and the spectral invariance of a non-commutative torus were deduced from this variation of Wiener's Lemma by Gröchenig and Leinert [32]. In a more general setting, Sun [61, 62] studied extensively the inverse-closedness of infinite matrices indexed by the integers and with an off diagonal decay.

A stronger version of the inverse-closedness is the norm-controlled inversion of Banach subalgebras. We say that an inverse-closed Banach subalgebra $\mathscr{A}$ admits a *norm-controlled inversion* in the Banach algebra $\mathscr{B}$, if there exists a function $h : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ such that

$$\left\|A^{-1}\right\|_{\mathscr{A}} \leq h\left(\left\|A^{-1}\right\|_{\mathscr{B}}, \|A\|_{\mathscr{A}}\right), \tag{1.2}$$

for each $A \in \mathscr{A}$ invertible in $\mathscr{B}$. Nikolski [48] introduced the term norm-control and studied the norm-controlled inversion of various Banach, function and measure algebras. Norm-controlled inversion was also studied in different settings, for example the norm-controlled inversion of algebras of infinite matrices was studied by Fang and Shin [14] and by Shin and Sun [58], of convolution algebras by Samei and Shepelska [56], of differential subalgebras by Gröchenig and Klotz [30, 31] and of measure and Fourier-Stieltjes algebras by Ohrysko and Wasilewski [49]. It should be noted that in general norm-controlled inversion is not automatic for inverse-closed subalgebras. In [48] it is proved that the Banach algebra of continuous functions with an absolutely convergent Fourier series, $\mathscr{A}(\mathbb{T})$, does not admit a norm-controlled inversion in the Banach algebra of continuous functions on $\mathbb{T}$. On the other hand, by Wiener's Lemma we have that $\mathscr{A}(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$. Note that the previous also extends to the class of convolution matrices in the operator algebra $\mathscr{B}(\ell^2(\mathbb{Z}))$ by the isomorphism between convolution operators and functions with an absolute convergent Fourier series. Thus, in general we do not expect that an inverse-closed subalgebra admits a norm-controlled inversion. In that way, norm-controlled inversion is a stronger and a quantitative version of inverse-closedness [58].

1.3. Wiener type Lemma on non-commutative groups

The Wiener's Lemma for convolution-dominated matrices on discrete groups of polynomial growth was investigated by Fendler, Gröchenig and Leinert [19] and by Tessera [63]. Conversely, Tessera [64] provided an example of a discrete group, G, of exponential growth, for which the class of convolution-dominated matrices is not inverse-closed in the algebra of bounded operators on $\ell^2(G)$ sequences. In his paper, Tessera shows that the Schur algebra is not inverse-closed in $\mathscr{B}(\ell^2(G))$, however the matrix provided as a counter-example is a convolution matrix. Thus, the result in [64] is a counter-example for the inverse-closedness of convolution-dominated matrices over a group of exponential growth.

Moreover, Tessera [63] showed that for a discrete group G of polynomial growth, if a convolution-dominated matrix A is bounded from below for some $p \in [1,\infty]$, i.e. there exists $C_{A,p} > 0$ such that $\|x\|_{\ell^p(G)} \le C_{A,p} \|Ax\|_{\ell^p(G)}$, then A is bounded from below for each $q \in [1,\infty]$. The previous in combination with the inverse-closedness of the convolution-dominated matrices in discrete groups of polynomial growth, proves that if a convolution-dominated matrix is bounded from below for some $p \in [1,\infty]$, then it has a left inverse in the algebra of convolution-dominated matrices [63].

1.4. Aim of the thesis

Shin and Sun in [59] showed that a variation of Sjöstrand's proof provides both the inverse-closedness and the result on boundedness from below for convolution-dominated matrices indexed by the integers and Tessera [63] claims that this method should also work for groups of polynomial growth. During the project, we have proved, using the commutator technique by Sjöstrand, that if a convolution dominated matrix is bounded from below for some $p \in [1,\infty]$, then it is bounded from below for each $q \in [1,\infty]$. In the special case of discrete groups of polynomial growth, the previous recovers a result given by Tessera [63]. In addition, for matrices indexed by relatively separated sets in homogeneous groups we recover the result on boundedness from below given by Gröchenig, Romero, Rottensteiner and Van Velthoven [27]. However, for matrices indexed by relatively separated sets in locally compact groups of polynomial growth this yields new results. Moreover, using the spectral invariance of the Schur matrices given by Sun [61], we have deduced a Wiener type Lemma for the intersection of all polynomially weighted classes of convolution-dominated matrices in groups of polynomial growth.

Finally, applications of the convolution-dominated matrices and the aforementioned results on non-commutative geometry and frame theory will be presented. For the first application, following Gröchenig and Leinert [32] we prove the spectral invariance of twisted convolution operators on groups of polynomial growth. Afterwards, for a discrete series representation (π, H_π) restricted to a lattice in a nilpotent Lie group we define the non-commutative space

$$\mathscr{A}^1_w = \left\{ A \in \mathscr{B}(H_\pi) \ : \ A = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda),\ a \in \ell^1_w(\Lambda) \right\} \tag{1.3}$$

and prove a Wiener type Lemma for this space in the operator algebra $\mathscr{B}(H_\pi)$. The proof of the previous was inspired by Gröchenig and Leinert [32], who showed a Wiener type Lemma for $\mathscr{A}^1_w$ for the time-frequency shifts $\left(\pi, L^2(\mathbb{R}^d)\right)$. As a second application, for a discrete series representation (π, H_π) of a group of polynomial growth and $g \in H_\pi$ we have studied the frames $\pi(\Lambda)g$, where Λ is a relatively separated subset of the group. Using the result on ℓ^p-stability for convolution-dominated matrices, we have proved that if $\pi(\Lambda)g$ is a p-frame on the coorbit space $Co(L^p)$ for some $p \in [1, \infty]$, then $\pi(\Lambda)g$ is a q-frame on the coorbit space $Co(L^q)$ for each $q \in [1, \infty]$. Moreover, we have investigated the invertibility of the frame operator of $\pi(\Lambda)g$ on the coorbit spaces $Co(L^p)$. For proving the aforementioned result, we show that the Gramian matrix of the frame is a convolution-dominated matrix indexed by Λ and use the Wiener type Lemma for this class of operators.

1.5. Outline

In Chapter 2, we present some important results on Banach algebras and define the spectral invariance for Banach algebras. Groups of polynomial growth and convolution-dominated matrices on such groups are introduced in Chapter 3. In the same chapter, we state several properties and results on groups of polynomial growth, before defining the convolution-dominated operators. In Chapter 4, we present the key lemmas that compose the commutator technique and will be used later in the proof of the result on the boundedness from below. Afterwards, in Chapter 5, we prove that boundedness from below is independent of $p \in [1, \infty]$ for convolution-dominated matrices indexed by relatively separated sets in groups of polynomial growth and deduce the inverse-closedness of the intersection, over all polynomial weights, of the class of convolution-dominated matrices. Part II concerns applications of convolution-dominated matrices in non-commutative geometry and in frame theory. Initially, the background needed in the applications is presented in Chapter 6. Then, we investigate the spectral invariance of a non-commutative space and study frames in coorbit spaces, in Chapters 7 and 8, respectively. Eventually, we conclude the thesis with Chapter 9, where we summarize the significant results and provide recommendation for future research on convolution-dominated matrices.

2

Spectral Invariance

In this chapter, the definitions of spectral invariance and inverse-closedness are provided. Moreover, we present a result by Hulanicki which provides a sufficient condition for the spectral invariance.

2.1. Banach algebras

In this section, we briefly introduce the Banach algebras, C*-algebras and important aspects of these spaces which will be used throughout the paper. Basic notions and properties for Banach and C*-algebras can be found in [9, 46].

A vector space $\mathscr{B}$ with a bilinear map

$$\begin{aligned} \mathscr{B} \times \mathscr{B} &\longrightarrow \mathscr{B} \\ (a, b) &\longmapsto ab, \end{aligned}$$

is called an *algebra*, if for each $a, b, c \in \mathscr{B}$

$$a(bc) = (ab)c. \tag{2.1}$$

The property (2.1) is called *associativity.* If, further, the algebra $\mathscr{B}$ has a unit, i.e. there exist $1_{\mathscr{B}} \in \mathscr{B}$ such that $a1_{\mathscr{B}} = a = 1_{\mathscr{B}}a$, for each $a \in \mathscr{B}$, then $\mathscr{B}$ is said to be a *unital algebra.* We call the vector subspace $\mathscr{A}$ of $\mathscr{B}$ a *subalgebra*, if for each $a, b \in \mathscr{A}$, we have $ab \in \mathscr{A}$. If an algebra $\mathscr{B}$ admits a submultiplicative norm $\|\cdot\|$, i.e. $\|ab\| \leq \|a\| \, \|b\|$ for each $a, b \in \mathscr{B}$, then the pair $(\mathscr{B}, \|\cdot\|)$ is called a *normed algebra.*

A conjugate linear map $a \longmapsto a^*$ on an algebra $\mathscr{B}$ is called an *involution* on $\mathscr{B}$, if $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for each $a, b \in \mathscr{B}$. In that case, we call the pair $(\mathscr{B}, *)$ a $*$*-algebra.* A subalgebra $\mathscr{A}$ of a $*$-algebra $\mathscr{B}$ is called $*$*-subalgebra*, if $a^* \in \mathscr{A}$ for each $a \in \mathscr{A}$. Suppose that $\mathscr{B}$ is a $*$-algebra. Then, an element $a \in \mathscr{B}$ is called *self-adjoint* if $a = a^*$ and *normal* if $aa^* = a^*a$. Moreover, if $\mathscr{B}$ is a unital $*$-algebra, with unit $1_{\mathscr{B}}$, then $a \in \mathscr{B}$ is said to be a *unitary* element of $\mathscr{B}$ if $aa^* = a^*a = 1_{\mathscr{B}}$.

An element $a \in \mathscr{B}$ of a unital algebra $\mathscr{B}$ is *invertible* in $\mathscr{B}$, if there exists $b \in \mathscr{B}$ such that $ab = ba = 1_{\mathscr{B}}$, where $1_{\mathscr{B}}$ is the unit of $\mathscr{B}$. We denote the set of invertible elements of $\mathscr{B}$ by $\operatorname{Inv}\mathscr{B}$. For a unital algebra $\mathscr{B}$, with unit $1_{\mathscr{B}}$, the *spectrum* of an element $a \in \mathscr{B}$ is defined as the set

$$\sigma_{\mathscr{B}}(a) = \{\lambda \in \mathbb{C} \; : \; \lambda 1_{\mathscr{B}} - a \in \operatorname{Inv}\mathscr{B}\}, \tag{2.2}$$

and the *spectral radius* of $a \in \mathscr{B}$ is defined by

$$\mathrm{r}_{\mathscr{B}}(a) = \sup_{\lambda \in \sigma_{\mathscr{B}}(a)} |\lambda| \, . \tag{2.3}$$

We define the Banach algebras and Banach $*$-algebras as follows.

Definition 2.1.1. A complete normed algebra is called a *Banach algebra.* Furthermore, if $\mathscr{B}$, equipped with the norm $\|\cdot\|$, is a complete normed $*$-algebra, such that $\|a^*\| = \|a\|$ for each $a \in \mathscr{B}$, then $\mathscr{B}$ is called a *Banach* $*$*-algebra.* In the case where $\mathscr{B}$ is unital and the unit $1_{\mathscr{B}} \in \mathscr{B}$ is such that $\|1_{\mathscr{B}}\| = 1$, then $\mathscr{B}$ is called a *unital Banach* $*$*-algebra.*

If $\mathscr{A}$ and $\mathscr{B}$ are two Banach algebras, such that $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{A}$ is called a *Banach subalgebra* of $\mathscr{B}$. It should be noted that the norm $\|\cdot\|_{\mathscr{A}}$ of a Banach subalgebra $\mathscr{A} \subseteq \mathscr{B}$ and the norm $\|\cdot\|_{\mathscr{B}}$ of the Banach algebra $\mathscr{B}$ may differ.

For a closed subalgebra of a unital Banach algebra, we have the following result for the spectrum of elements in the subalgebra. See e.g. [46, Theorem 1.2.8.] for a proof of this theorem. Before stating the theorem, we present a class of sets in the complex plane, $\mathbb{C}$. For a non-empty compact set $V \subseteq \mathbb{C}$, the bounded components of $\mathbb{C}\backslash V$ are called the *holes* of V.

Theorem 2.1.2. Let $\mathscr{A}$ be a closed subalgebra of a unital Banach algebra $\mathscr{B}$. Moreover, suppose that $\mathscr{A}$ contains the unit $1_{\mathscr{B}}$ of $\mathscr{B}$. Then for each $a \in \mathscr{A}$,

$$\sigma_{\mathscr{B}}(a) \subseteq \sigma_{\mathscr{A}}(a).$$

If for $a \in \mathscr{A}$ we further have that $\sigma_{\mathscr{B}}(a)$ has no holes, then

$$\sigma_{\mathscr{B}}(a) = \sigma_{\mathscr{A}}(a).$$

If $\mathscr{A}$ and $\mathscr{B}$ are two algebras and $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ is a linear map, such that

$$\phi(ab) = \phi(a)\phi(b),$$

then ϕ is called a *homomorphism* from $\mathscr{A}$ to $\mathscr{B}$. Furthermore, if $\mathscr{A}$ and $\mathscr{B}$ are unital algebras, we say that a homomorphism $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ is *unital* if $\phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$. A homomorphism ϕ from a $*$-algebra $\mathscr{A}$ to a $*$-algebra $\mathscr{B}$ is called a $*$*-homomorphism* if it preserves the involutions, i.e. $\phi(a^*) = \phi(a)^*$ for each $a \in \mathscr{A}$. For an abelian algebra $\mathscr{B}$ a non-zero homomorphism $\phi : \mathscr{B} \longrightarrow \mathbb{C}$ is called a *character* on $\mathscr{B}$ and the set of all characters on $\mathscr{B}$ is denoted by $X(\mathscr{B})$. We now state a known result in functional analysis that connects the spectrum of an element in an abelian Banach algebra with the characters on the algebra, see e.g. [46, Theorem 1.3.4.] for a proof.

Theorem 2.1.3. Let $\mathscr{B}$ be an abelian unital Banach algebra. Then for each $a \in \mathscr{B}$

$$\sigma_{\mathscr{B}}(a) = \{\phi(a) \ : \ \phi \in X(\mathscr{B})\}. \tag{2.4}$$

Suppose that $\mathscr{B}$ is a Banach $*$-algebra, equipped with the norm $\|\cdot\|$. If for each $a \in \mathscr{B}$ we have $\|a^*a\| = \|a\|^2$, then $\mathscr{B}$ is called a *C*-algebra*. A *C*-subalgebra* is a closed (with respect to the algebra norm) $*$-subalgebra of a C*-algebra. For a C*-algebra $\mathscr{B}$, the pair (π, H_π), where H_π is a Hilbert space and $\pi : \mathscr{B} \longrightarrow \mathscr{B}(H_\pi)$ is a $*$-homomorphism, is called a *representation* of the C*-algebra $\mathscr{B}$. If π is an injective $*$-homomorphism, then the representation is said to be *faithful*.

2.2. Spectral Invariance

Let $\mathscr{A} \subseteq \mathscr{B}$ be two Banach algebras with common identity. We say that $\mathscr{A}$ is *inverse-closed* in $\mathscr{B}$, if

$$a^{-1} \in \mathscr{B} \ \Rightarrow \ a^{-1} \in \mathscr{A}, \tag{2.5}$$

for each $a \in \mathscr{A}$. If $\sigma_{\mathscr{A}}(a) = \sigma_{\mathscr{B}}(a)$ for each $a \in \mathscr{A}$, then we call $\mathscr{A}$ *spectrally invariant* in $\mathscr{B}$, where

$$\sigma_{\mathscr{A}'}(a) := \{\lambda \in \mathbb{C} \ : \ \lambda 1_{\mathscr{A}} - a \text{ is invertible in } \mathscr{A}'\},$$

denotes the spectrum of a in the unital Banach algebra $\mathscr{A}'$. By the previous definitions we deduce that inverse-closedness and spectral invariance of a unital Banach subalgebra are equivalent.

Lemma 2.2.1. Given $\mathscr{A} \subseteq \mathscr{B}$ two Banach algebras with common identity, we have that

$$\mathscr{A} \text{ is inverse-closed in } \mathscr{B} \iff \sigma_{\mathscr{A}}(a) = \sigma_{\mathscr{B}}(a) \text{ for each } a \in \mathscr{A}.$$

The spectral invariance of a Banach subalgebra in a Banach algebra is not automatic in general. We provide an example of a Schur matrix which is invertible in $\mathscr{B}(\ell^2(\mathbb{N}))$, but not in the Schur algebra. This example is inspired by the discussion in [63] and shows that the Schur algebra is not inverse closed in $\mathscr{B}(\ell^2(\mathbb{N}))$. For the following example we first introduce the Schur algebra and the Schur norm. The Schur algebra is the Banach $*$-algebra that contains all operators in $\mathscr{B}(\ell^2(\Gamma), \ell^2(\Lambda))$ which belong in both $\mathscr{B}(\ell^1(\Gamma), \ell^1(\Lambda))$ and $\mathscr{B}(\ell^\infty(\Gamma), \ell^\infty(\Lambda))$. In general, for countable index sets Λ and Γ we denote the norm of $T \in \mathbb{C}^{\Lambda\times\Gamma}$ in $\mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$ by $\|T\|_{\ell^p(\Gamma) \to \ell^p(\Lambda)}$ and the Schur norm of $T \in \mathbb{C}^{\Lambda\times\Gamma}$ by

$$\|T\|_{Schur(\Gamma\to\Lambda)} := \max\left\{ \sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} |T_{\lambda,\gamma}| \,,\; \sup_{\gamma\in\Gamma} \sum_{\lambda\in\Lambda} |T_{\lambda,\gamma}| \right\}. \tag{2.6}$$

The Schur algebra of matrices indexed by the countable sets Λ and Γ is defined by

$$\mathscr{S}(\Gamma,\Lambda) := \left\{ A \in \mathscr{B}(\ell^2(\Gamma), \ell^2(\Lambda)) \;:\; \|A\|_{Schur(\Gamma\to\Lambda)} < \infty \right\}, \tag{2.7}$$

and for $\Lambda = \Gamma$ we denote $\mathscr{S}(\Lambda,\Lambda) = \mathscr{S}(\Lambda)$. Using the Schur test, see e.g. [22, Theorem 6.18], we obtain for each $T \in \mathbb{C}^{\Lambda\times\Gamma}$ and each $p \in [1,\infty]$

$$\|T\|_{\ell^p(\Gamma) \to \ell^p(\Lambda)} \le \|T\|_{Schur(\Gamma\to\Lambda)}\,, \tag{2.8}$$

hence if $T \in \mathscr{S}(\Gamma,\Lambda)$, then $T \in \mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$ for each $p \in [1,\infty]$.

Note that for $A = (A(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda}\,, B = (B(\lambda,\lambda'))_{\lambda,\lambda'\in\Lambda} \in \mathbb{C}^{\Lambda\times\Lambda}$ we have

$$\begin{aligned}
\sup_{\lambda\in\Lambda} \sum_{\lambda'\in\Lambda} |(AB)(\lambda,\lambda')| &= \sup_{\lambda\in\Lambda} \sum_{\lambda'\in\Lambda} \left| \sum_{k\in\Lambda} A(\lambda,k)B(k,\lambda') \right| \\
&\le \sup_{\lambda\in\Lambda} \sum_{\lambda'\in\Lambda} \sum_{k\in\Lambda} |A(\lambda,k)|\,|B(k,\lambda')| \\
&\le \sup_{\lambda\in\Lambda} \sum_{\lambda'\in\Lambda} \sup_{k\in\Lambda} |B(k,\lambda')| \sum_{k\in\Lambda} |A(\lambda,k)| \\
&\le \left(\sup_{\lambda\in\Lambda} \sum_{k\in\Lambda} |A(\lambda,k)| \right) \left(\sup_{k\in\Lambda} \sum_{\lambda'\in\Lambda} |B(k,\lambda')| \right).
\end{aligned}$$

Similarly, we have

$$\sup_{\lambda'\in\Lambda} \sum_{\lambda\in\Lambda} |(AB)(\lambda,\lambda')| \le \left(\sup_{k\in\Lambda} \sum_{\lambda\in\Lambda} |A(\lambda,k)| \right) \left(\sup_{\lambda'\in\Lambda} \sum_{k\in\Lambda} |B(k,\lambda')| \right).$$

Combining the previous, we obtain for each $A, B \in \mathbb{C}^{\Lambda\times\Lambda}$

$$\|AB\|_{Schur(\Lambda\to\Lambda)} \le \|A\|_{Schur(\Lambda\to\Lambda)}\, \|B\|_{Schur(\Lambda\to\Lambda)}\,. \tag{2.9}$$

With the following example we prove that $\mathscr{S}(\mathbb{N})$ is not inverse-closed in $\mathscr{B}(\ell^2(\mathbb{N}))$.

Example 2.2.2. Let D be the dilation operator on $\ell^2(\mathbb{N})$, such that

$$\begin{aligned}
D \,:\quad \ell^2(\mathbb{N}) \qquad &\longrightarrow \ell^2(\mathbb{N}) \\
x = (x_i)_{i\in\mathbb{N}} &\longmapsto \left(\frac{x_1}{2}, \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_2}{2}, \dots \right).
\end{aligned}$$

We then compute the adjoint of D in $\mathscr{B}(\ell^2(\mathbb{N}))$,

$$\begin{aligned}
D^* \,:\quad \ell^2(\mathbb{N}) \qquad &\longrightarrow \ell^2(\mathbb{N}) \\
x = (x_i)_{i\in\mathbb{N}} &\longmapsto \left(\frac{x_{2i-1} + x_{2i}}{2} \right)_{i\in\mathbb{N}}
\end{aligned}$$

and define

$$A = I - D^* \in \mathscr{B}(\ell^2(\mathbb{N})) \;:\; x = (x_i)_{i\in\mathbb{N}} \;\longmapsto\; \left(x_i - \frac{x_{2i-1} + x_{2i}}{2} \right)_{i\in\mathbb{N}}.$$

The operator $A \in \mathscr{B}(\ell^2(\mathbb{N}))$ can be interpreted as the matrix $A = (A(i,j))_{i,j\in\mathbb{N}}$ acting on $\ell^2(\mathbb{N})$ by $Ax = \left(\sum_{j\in\mathbb{N}} A(i,j)x_j\right)_{i\in\mathbb{N}}$ for each $x = (x_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$. For each $n \in \mathbb{N}$ let $e_n \in \ell^2(\mathbb{N})$ be the sequence such that $e_n(m) = 1$ for $n = m$ and $e_n(m) = 0$ for $n \neq m$. Then for $i, j \in \mathbb{N}$ we obtain

$$\begin{aligned}\langle Ae_j, e_i\rangle &= \left\langle \left(e_j(n) - \frac{e_j(2n-1)+e_j(2n)}{2}\right)_{n\in\mathbb{N}}, e_i \right\rangle = e_j(i) - \frac{e_j(2i-1)+e_j(2i)}{2} \\ &= \begin{cases} 1, & j = i \\ -1/2, & j = 2i-1 \\ -1/2, & j = 2i \\ 0, & \text{else} \end{cases}.\end{aligned}$$

Thus, the matrix elements $A(i,j)$ of A are given by

$$A(i,j) = \langle Ae_j, e_i\rangle = \begin{cases} 1, & j = i \\ -1/2, & j = 2i-1 \\ -1/2, & j = 2i \\ 0, & \text{else} \end{cases}, \tag{2.10}$$

for each $i, j \in \mathbb{N}$. Then, we obtain

$$\sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |A(i,j)| = \sup_{i\in\mathbb{N}} |A(i,i)| + |A(i,2i-1)| + |A(i,2i)| = \sup_{i\in\mathbb{N}} 1 + 1/2 + 1/2 = 2$$

and

$$\begin{aligned}\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |A(i,j)| &= \max\left\{ \sup_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}} |A(i,2k)|, \sup_{k\in\mathbb{N}} \sum_{i\in\mathbb{N}} |A(i,2k-1)| \right\} \\ &= \max\left\{ \sup_{k\in\mathbb{N}} |A(2k,2k)| + |A(k,2k)|, \sup_{k\in\mathbb{N}} |A(2k-1,2k-1)| + |A(k,2k-1)| \right\} \\ &= 3/2.\end{aligned}$$

It follows that

$$\|A\|_{Schur(\mathbb{N}\to\mathbb{N})} = \max\left\{ \sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |A(i,j)|, \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |A(i,j)| \right\} = 2, \tag{2.11}$$

hence $A \in \mathscr{S}(\mathbb{N})$, i.e. A is a Schur matrix. Furthermore, since for $x \in \ell^2(\mathbb{N})$

$$\begin{aligned}\|D^*x\|^2_{\ell^2(\mathbb{N})} &= \sum_{i\in\mathbb{N}} |(D^*x)_i|^2 = \sum_{i\in\mathbb{N}} \left| \frac{x_{2i-1}+x_{2i}}{2} \right|^2 \\ &= \sum_{i\in\mathbb{N}} \frac{1}{4}\left(|x_{2i-1}|^2 + |x_{2i}|^2 + 2\Re(x_{2i-1}\overline{x_{2i}}) \right) \\ &\leq \frac{3}{4} \|x\|^2_{\ell^2(\mathbb{N})},\end{aligned}$$

we have $\|D^*\|_{\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})} < 1$. Thus, $A = I - D^*$ is invertible in $\mathscr{B}(\ell^2(\mathbb{N}))$ with

$$A^{-1} = (I - D^*)^{-1} = \sum_{n=0}^{\infty} (D^*)^n$$

in $\mathscr{B}(\ell^2(\mathbb{N}))$. We will show that A is not invertible in the Schur algebra. Let us assume that A is invertible in the Schur algebra, i.e. $A^{-1} \in \mathscr{S}(\mathbb{N})$. By Equation (2.8) we obtain that $A^{-1} \in \mathscr{B}(\ell^\infty(\mathbb{N}))$. In order to prove our claim we define the sequences

$$\phi_n = (n, n, n, \ldots) \in \ell^\infty(\mathbb{N}).$$

Since A is invertible in $\mathscr{B}(\ell^\infty(\mathbb{N}))$, then it is bounded from below and hence there exists $C_A > 0$ such that

$$n = \|\phi_n\|_{\ell^\infty(\mathbb{N})} \leq C_A \|A\phi_n\|_{\ell^\infty(\mathbb{N})} = 0. \tag{2.12}$$

By contradiction we have that A is not invertible in $\mathscr{B}(\ell^\infty(\mathbb{N}))$ and thus is not invertible in the Schur algebra. We conclude that the Schur Algebra $\mathscr{S}(\mathbb{N})$ is not spectrally invariant in $\mathscr{B}(\ell^2(\mathbb{N}))$.

In the previous example, we have defined a matrix $A \in \mathscr{S}(\mathbb{N})$, such that A is invertible in $\mathscr{B}(\ell^2(\mathbb{N}))$. Since A is invertible in $\mathscr{B}(\ell^2(\mathbb{N}))$, then it is bounded from below for $p = 2$, i.e. there exists $C_{2,A} > 0$ such that

$$\|c\|_{\ell^2(\mathbb{N})} \leq C_{2,A} \|Ac\|_{\ell^2(\mathbb{N})}, \ \forall c \in \ell^2(\mathbb{N}).$$

Moreover, we have shown that A is not bounded from below for $p = \infty$, see Equation (2.12). Thus, the previous example shows that if a matrix is bounded from below for some $p \in [1, \infty]$, then it is not automatically bounded from below for each $q \in [1, \infty]$.

In the case of C*-algebras, spectral invariance is automatic. Let $\mathscr{B}$ be a unital C*-algebra and $\mathscr{A}$ be a C*-subalgebra of $\mathscr{B}$, containing the unit of $\mathscr{B}$. Let $a = a^* \in \mathscr{A}$ be a self-adjoint element of $\mathscr{A}$. Then, since a is self-adjoint we have $\sigma_{\mathscr{B}}(a) \subseteq \mathbb{R}$. Since $\sigma_{\mathscr{B}}(a)$ is a compact subset of the real line in $\mathbb{C}$ we have that $\mathbb{C}\backslash\sigma_{\mathscr{B}}(a)$ has only one component which is unbounded. We deduce that $\sigma_{\mathscr{B}}(a)$ contains no holes and hence $\sigma_{\mathscr{B}}(a) = \sigma_{\mathscr{A}}(a)$, by Theorem 2.1.2. The previous can be extended for each $a \in \mathscr{A}$ as follows. Let $a \in \mathscr{A}$ and $z \notin \sigma_{\mathscr{B}}(a)$. Then $z1_{\mathscr{B}} - a \in \operatorname{Inv}\mathscr{B}$ and $(z1_{\mathscr{B}} - a)^*(z1_{\mathscr{B}} - a), (z1_{\mathscr{B}} - a)(z1_{\mathscr{B}} - a)^* \in \operatorname{Inv}\mathscr{B}$. Since $(z1_{\mathscr{B}} - a)^*(z1_{\mathscr{B}} - a)$ and $(z1_{\mathscr{B}} - a)(z1_{\mathscr{B}} - a)^*$ are self-adjoint elements of $\mathscr{A}$, by the equality of the spectrum for self-adjoint elements we obtain that $(z1_{\mathscr{B}}-a)^*(z1_{\mathscr{B}}-a), (z1_{\mathscr{B}}-a)(z1_{\mathscr{B}}-a)^* \in \operatorname{Inv}\mathscr{A}$. Now, since $\left((z1_{\mathscr{B}} - a)^*(z1_{\mathscr{B}} - a)\right)^{-1} (z1_{\mathscr{B}}-a)^*$ is a left inverse of $(z1_{\mathscr{B}} - a)$ in $\mathscr{A}$ and $(z1_{\mathscr{B}} - a)^* \left((z1_{\mathscr{B}} - a)(z1_{\mathscr{B}} - a)^*\right)^{-1}$ is a right inverse of $(z1_{\mathscr{B}} - a)$ in $\mathscr{A}$ we deduce that $(z1_{\mathscr{B}} - a) \in \operatorname{Inv}\mathscr{A}$. Thus, $z1_{\mathscr{B}} \notin \sigma_{\mathscr{A}}(a)$. It follows that $\sigma_{\mathscr{A}}(a) \subseteq \sigma_{\mathscr{B}}(a)$. The inclusion $\sigma_{\mathscr{B}}(a) \subseteq \sigma_{\mathscr{A}}(a)$ comes from Theorem 2.1.2. Thus we conclude that $\mathscr{A}$ is spectrally invariant in $\mathscr{B}$, or equivalently $\mathscr{A}$ is inverse-closed in $\mathscr{B}$.

Often Hulanicki's Lemma [38] is used for proving the spectral invariance. We now present a version of the result and a proof given by Fendler, Gröchenig, Leinert, Ludwig and Molitor-Braun, see [18, Proposition 6.1].

Theorem 2.2.3 (Hulanicki)**.** Let S be a $*$-subalgebra in the Banach $*$-algebra $\mathscr{A}$. Assume that there exists a faithful $*-$representation (π, H) of $\mathscr{A}$,

$$\pi \ : \ \mathscr{A} \longrightarrow \mathscr{B}(H),$$

such that

$$\|\pi(x)\|_{\mathscr{B}(H)} = \lim_{n\to\infty} \|x^n\|_{\mathscr{A}}^{1/n},$$

for each $x = x^* \in S$. Moreover, suppose that $\mathscr{A}$ has a unit $1_{\mathscr{A}} \in \mathscr{A}$ and $\pi(1_{\mathscr{A}}) = Id_H$. Then, for each $x = x^* \in S$ we have

$$\sigma_{\mathscr{A}}(x) = \sigma_{\mathscr{B}(H)}(\pi(x)).$$

Proof. Let $x = x^* \in S$ and let B be the commutative $\|\cdot\|_{\mathscr{A}}-$closed $*-$subalgebra of $\mathscr{A}$ generated by x and $1_{\mathscr{A}}$. We define the norms $\nu : x \mapsto \lim_{n\to\infty} \|x^n\|_{\mathscr{A}}^{1/n}$ and $\lambda : x \mapsto \|\pi(x)\|_{\mathscr{B}(H)}$ on $\mathscr{A}$.

Since B is generated by the self-adjoint elements x and $1_{\mathscr{A}}$, then we have that every element of B is self-adjoint. Then, by assumption, $\nu(y) = \lambda(y)$ for each $y = y^* \in S$ and $\nu(1_{\mathscr{A}}) = \lambda(1_{\mathscr{A}})$. In particular, $\nu(y) = \lambda(y)$ for each $y \in B$. Hence, ν and λ are equivalent norms on B.

We define B^λ to be the completion of B with respect to λ. Note that for each $x \in B$ we have

$$\lambda(x^*x) = \|\pi(x^*x)\|_{\mathscr{B}(H)} = \|\pi(x)^*\pi(x)\|_{\mathscr{B}(H)} = \|\pi(x)\|^2_{\mathscr{B}(H)} = \lambda(x)^2. \tag{2.13}$$

Hence, B^λ is a C*-algebra. By definition B^λ is isomorphic to $\overline{\pi(B)}^\lambda \subseteq \mathscr{B}(H)$ and thus

$$\sigma_{B^\lambda}(x) = \sigma_{\overline{\pi(B)}^\lambda}(\pi(x)). \tag{2.14}$$

We denote with $X(B)$ and $X(B^\lambda)$ the set of characters on B and B^λ, respectively. Define the map

$$\begin{aligned} \psi : X(B^\lambda) \ &\longrightarrow X(B) \\ \phi \ &\longmapsto \ \phi|_B. \end{aligned}$$

We have that this map is well-defined, since $B \subseteq B^{\lambda}$. Moreover, B^{λ} is the completion of B with respect to λ, hence B is $\lambda-$dense in B^{λ}. Combining the previous and the equivalence of ν and λ we have that ψ is continuous. Using the same arguments we obtain that ψ is injective. On the other hand, by the density and the equivalence of the norms for every $\phi \in X(B)$ we can define $\tilde{\phi}$ by $\tilde{\phi}(a) = \lim_{n\to\infty} \phi(a_n)$ for each $a \in B^{\lambda}$, where $(a_n)_{n\in\mathbb{N}} \subseteq B$ is a sequence such that $\lambda(a_n - a) \xrightarrow{n\to\infty} 0$. Note that if $\phi \in X(B)$, then $\|\phi\| \leq 1$ since ϕ is a character on a unital abelian Banach algebra, see e.g. [46, Theorem 1.3.3]. If $(a_n)_{n\in\mathbb{N}} \subseteq B$ and $(b_n)_{n\in\mathbb{N}} \subseteq B$ are two sequences converging to $a \in B^{\lambda}$ with respect to λ, then using the equality of the norms ν and λ on B we have

$$|\phi(a_n) - \phi(b_n)| \leq \nu(a_n - b_n) = \lambda(a_n - b_n) \xrightarrow{n\to\infty} 0.$$

Hence $\tilde{\phi}(a)$ is well defined for each $a \in B^{\lambda}$. Moreover, since $\phi \in X(B)$, then for each $a, b \in B^{\lambda}$ and $(a_n)_{n\in\mathbb{N}} \subseteq B$, $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $\lambda(a_n - a) \xrightarrow{n\to\infty} 0$ and $\lambda(b_n - b) \xrightarrow{n\to\infty} 0$ we obtain

$$\tilde{\phi}(a)\tilde{\phi}(b) = \lim_{n\to\infty} \phi(a_n)\phi(b_n) = \lim_{n\to\infty} \phi(a_n b_n) = \tilde{\phi}(ab).$$

Thus, the map

$$\begin{aligned} \tilde{\psi} : X(B) &\longrightarrow X(B^{\lambda}) \\ \phi &\longmapsto \tilde{\phi}, \end{aligned}$$

is well-defined and, similarly to ψ, we can show that the map $\tilde{\psi}$ is continuous and injective. Moreover, we obtain $\tilde{\psi} \circ \psi(\phi) = \phi$ for each $\phi \in X(B^{\lambda})$ and $\psi \circ \tilde{\psi}(\phi) = \phi$ for each $\phi \in X(B)$. Thus, $X(B^{\lambda})$ and $X(B)$ are homeomorphic, that is, there exists a bijective, continuous map from $X(B^{\lambda})$ to $X(B)$, with a continuous inverse. From this homeomorphism and since $x \in B \cap B^{\lambda}$ we obtain

$$\{\phi(x) \ : \ \phi \in X(B)\} = \left\{\phi(x) \ : \ \phi \in X(B^{\lambda})\right\}. \tag{2.15}$$

Moreover, since B is a commutative Banach algebra, we obtain from Theorem 2.1.3

$$\sigma_B(x) = \{\phi(x) \ : \ \phi \in X(B)\}. \tag{2.16}$$

Similarly, we obtain

$$\sigma_{B^{\lambda}}(x) = \left\{\phi(x) \ : \ \phi \in X(B^{\lambda})\right\}. \tag{2.17}$$

Combining the previous we deduce that

$$\sigma_B(x) = \{\phi(x) \ : \ \phi \in X(B)\} = \left\{\phi(x) \ : \ \phi \in X(B^{\lambda})\right\} = \sigma_{B^{\lambda}}(x). \tag{2.18}$$

Now, since B^{λ} is a C*-algebra and $x = x^*$ we have that the spectrum $\sigma_{B^{\lambda}}(x)$ is real. From Equation (2.18), it follows that $\sigma_B(x)$ is real.

Additionally, since B is a closed subalgebra of $\mathscr{A}$, we have $\sigma_{\mathscr{A}}(x) \subseteq \sigma_B(x)$, by Theorem 2.1.2. Hence $\sigma_{\mathscr{A}}(x) \subseteq \mathbb{R}$ and it follows that $\sigma_{\mathscr{A}}(x)$ has no holes. Moreover, $\mathscr{A}$ and B have a common unit element and hence, by applying Theorem 2.1.2, we obtain

$$\sigma_B(x) = \sigma_{\mathscr{A}}(x).$$

Since $\sigma_{\overline{\pi(B)}^{\lambda}}(\pi(x)) = \sigma_{B^{\lambda}}(x) \subseteq \mathbb{R}$ and $\overline{\pi(B)}^{\lambda}$ is a closed subalgebra of $\mathscr{B}(H)$, then we obtain

$$\sigma_{\overline{\pi(B)}^{\lambda}}(\pi(x)) = \sigma_{\mathscr{B}(H)}(\pi(x)),$$

by Theorem 2.1.2. We now conclude that

$$\sigma_{\mathscr{A}}(x) = \sigma_B(x) = \sigma_{B^{\lambda}}(x) = \sigma_{\overline{\pi(B)}^{\lambda}}(\pi(x)) = \sigma_{\mathscr{B}(H)}(\pi(x)). \tag{2.19}$$

□

Let $A \in \mathscr{B}(H)$, where H is a Hilbert space. Moreover, suppose that the range $M = \mathrm{Ran}(A) \subseteq H$ of A is a closed subspace of H,A is bijective from M onto M and there exists an operator $B \in \mathscr{B}(M)$ such that

$$AB = BA = I_M, \tag{2.20}$$

i.e. B is the inverse of A in $\mathscr{B}(M)$. Let $P \in \mathscr{B}(H)$ be the orthogonal projection onto M and $A^\dagger$ be the trivial extension of B from M to H, i.e. $A^\dagger = B$ on M and $A^\dagger = 0$ on $H \backslash M$. Then the operator $A^\dagger$ is called the *pseudoinverse* of A and satisfies

$$AA^\dagger = A^\dagger A = P. \tag{2.21}$$

Using the Hulanicki's Lemma, it can be shown that if an element in an inverse-closed subalgebra has a pseudoinverse, then its pseudoinverse belongs in the subalgebra. We will present a proof of this result given by Fornasier and Gröchenig [23], which uses Hulanicki's lemma 2.2.3. This result will be used in Chapter 8 in order to show that the pseudoinverse of the Gramian matrix of a frame is a convolution-dominated matrix.

Theorem 2.2.4. Suppose that $\mathscr{A}$ is an inverse-closed Banach subalgebra of $\mathscr{B}(H)$. Let M be a closed subspace of the Hilbert space H. Let $A \in \mathscr{A}$, such that $A = A^*$, $\ker(A) = M^\perp$ and $A : M \longrightarrow M$ is invertible. Then the pseudoinverse $A^\dagger$ of A belongs in $\mathscr{A}$.

Proof. Let P be the orthogonal projection onto $M \subseteq H$. Define

$$S = \{B \in \mathscr{A} \ : \ B = PBP\}, \tag{2.22}$$

with norm $\|\cdot\|_S := \|\cdot\|_{\mathscr{A}}$. Since $A \in S$ we have that S is non-empty. Moreover, S is a $*$-subalgebra of $\mathscr{A}$.

We define the map $h : S \longrightarrow \mathscr{B}(M)$, $h(B) = B|_M$. We have that h is a $*$-representation of S. Furthermore, if $h(B) = 0$ for $B \in S$, then $B|_M = 0$ and from $B \in S$ we have that $M^\perp \subseteq \ker(B)$, hence $B = 0$. Thus h is a faithful $*$-representation of S.

Let S_0 be the closed commutative $*$-subalgebra of S generated by A. Then $\overline{h(S_0)}$, with the closure taken in the operator norm $\|\cdot\|_{\mathscr{B}(M)}$, is generated by $A|_M$. Note that since $\overline{h(S_0)}$ is a closed $*$-subalgebra of the C*-algebra $\mathscr{B}(M)$, then $\overline{h(S_0)}$ is a a C*-subalgebra. By the invertibility of $A|_M$ in $\mathscr{B}(M)$ and since $\overline{h(S_0)}$ is a C*-subalgebra of $\mathscr{B}(M)$, we have that $(A|_M)^{-1} \in \overline{h(S_0)}$ by the inverse-closedness of C*-subalgebras. Thus, $Id_M = A|_M(A|_M)^{-1} \in \overline{h(S_0)}$. It follows that there exists $E \in S_0$ such that $h(E) = Id_M$, see e.g. [23, Lemma 3.3]. Since $E \in S$ and $Id_M = h(E) = E|_M$, we deduce that $E = P$.

By the spectral invariance of $\mathscr{A}$ in $\mathscr{B}(H)$ for each $B \in \mathscr{A}$ we have

$$\sigma_{\mathscr{B}(H)}(B) = \sigma_{\mathscr{A}}(B).$$

Moreover, from the choice of norm on S we obtain

$$\mathrm{r}_{\mathscr{A}}(B) = \lim_{n\to\infty} \|B^n\|_{\mathscr{A}}^{1/n} = \lim_{n\to\infty} \|B^n\|_S^{1/n},$$

for each $B \in S$. Thus, combining the previous for each $B = B^* \in S$ we have

$$\lim_{n\to\infty} \|B^n\|_S^{1/n} = \mathrm{r}_{\mathscr{A}}(B) = \mathrm{r}_{\mathscr{B}(H)}(B). \tag{2.23}$$

Then, since $M^\perp \subseteq \ker(B)$ for each $B = B^* \in S$ we obtain

$$\lim_{n\to\infty} \|B^n\|_S^{1/n} = \mathrm{r}_{\mathscr{B}(H)}(B) = \|B\|_{\mathscr{B}(H)} = \|B\|_{\mathscr{B}(M)} = \|B|_M\|_{\mathscr{B}(M)}. \tag{2.24}$$

Thus,

$$\lim_{n\to\infty} \|B^n\|_S^{1/n} = \|B\|_{\mathscr{B}(M)} = \|h(B)\|_{\mathscr{B}(M)}, \tag{2.25}$$

for each $B = B^* \in S$. Applying Hulanicki's Lemma (Theorem 2.2.3) we deduce

$$\sigma_S(B) = \sigma_{\mathscr{B}(M)}(h(B)) = \sigma_{\mathscr{B}(M)}(B|_M), \tag{2.26}$$

for each $B = B^* \in S$.

Since $A : M \longrightarrow M$ is invertible, i.e. A is invertible in $\mathscr{B}(M)$ we have that $0 \notin \sigma_{\mathscr{B}(M)}(A|_M) = \sigma_S(A)$. Thus, there exists $B \in S$ such that B is the inverse of A in S, i.e. $BA = AB = P$. Since $B \in S$, we have that $B = 0$ on $H \backslash M$ and it follows that B is exactly the pseudoinverse $A^\dagger$ of A. We conclude that $A^\dagger \in \mathscr{A}$. □

From the previous Theorem we deduce that if a subalgebra is inverse-closed in $\mathscr{B}(H)$, it is also pseudoinverse-closed.

Corollary 2.2.5. Suppose that $\mathscr{A}$ is an inverse-closed Banach subalgebra of $\mathscr{B}(H)$, where H is a Hilbert space. Then $\mathscr{A}$ is pseudoinverse-closed in $\mathscr{B}(H)$, i.e. if $A \in \mathscr{A}$ has a pseudoinverse $A^\dagger \in \mathscr{B}(H)$, then $A^\dagger \in \mathscr{A}$.

3

Convolution-dominated Matrices

In this chapter, we present locally compact groups of polynomial growth and then define relatively separated sets in such groups. Afterwards, we introduce a class of integrable functions, known as Amalgam space, on a locally compact group which is essential to the definition of convolution-dominated matrices that follows.

3.1. Locally compact groups

Initially, we define the notion of compactly generated locally compact groups. The basic properties presented here can be found in [21].

Definition 3.1.1. A topological group whose topology is locally compact and Hausdorff is called a *locally compact group*. We say that a topological group G is *compactly generated*, if there exists a relatively compact symmetric neighbourhood, $U \subseteq G$, of the identity such that

$$G = \bigcup_{n \in \mathbb{N} \cup \{0\}} U^n,$$

where $U^n := \{u = u_1 u_2 ... u_n \; : \; u_i \in U\}$, for $n \in \mathbb{N}$ and $U^0 = \{e\}$, for the identity element $e \in G$.

In a locally compact group, G, which is compactly generated by a symmetric unit neighbourhood $U \subseteq G$, we define the *word metric* by

$$\mathrm{d}(x,y) = \inf\left\{n \in \mathbb{N} \cup \{0\} \; : \; x^{-1}y \in U^n\right\}, \tag{3.1}$$

for $x, y \in G$. We have that the word metric is left invariant from its definition and from the symmetry of U, we have that the word metric is symmetric. It can be furthermore shown that $\mathrm{d}(x,y) = 0$ if and only if $x^{-1}y \in U^0 = \{e\}$, which is equivalent to $x = y$. Moreover we have

$$\mathrm{d}(x,y) \leq \mathrm{d}(x,z) + \mathrm{d}(z,y),$$

for each $x, y, z \in G$, hence the word metric defines a (left invariant) metric on G. We denote with $B(x,r)$ the balls of radius r and center $x \in G$ with respect to the word metric, i.e.

$$B(x,r) := \{z \in G \; : \; \mathrm{d}(x,z) < r\}. \tag{3.2}$$

Hence, for $r \in \mathbb{N}$ we have

$$\begin{aligned}
\overline{B(x,r)} &= \{y \in G \; : \; \mathrm{d}(x,y) \leq r\} \\
&= \left\{y \in G \; : \exists\, n \in \mathbb{N}\,,\, n \leq r \text{ s.t. } x^{-1}y \in U^n \right\} \\
&= \left\{ y \in G \; : \; x^{-1}y \in \bigcup_{n \in \mathbb{N}\,,\, n \leq r} U^n = U^r \right\} \\
&= \{y \in G \; : \; y \in xU^r\} = xU^r,
\end{aligned} \tag{3.3}$$

where $\overline{V}$ denotes the closure, of the measurable set $V \subseteq G$, with respect to the word metric.

Definition 3.1.2. A *Radon measure* μ on a locally compact group G is a measure defined on the $\sigma-$algebra $\mathbf{B}(G)$ of the Borel measurable sets of G such that:

- μ is locally finite, i.e. for every $x \in G$ there is a neighbourhood of x with finite measure,
- μ is inner regular, i.e. for every open $V \in \mathbf{B}(G)$ we have

$$\mu(V) = \sup_K \{\mu(K) \ : \ K \subset V \, , \ K \text{ compact}\} \, ,$$

- μ is outer regular, i.e. for every $V \in \mathbf{B}(G)$ we have

$$\mu(V) = \inf_K \{\mu(K) \ : \ V \subset K, \ K \in \mathbf{B}(G), \ K \text{ open}\} \, .$$

A nonzero Radon measure, μ, on a locally compact group G is called *left* (resp. *right*) *Haar measure* if μ is left (resp. right) translation invariant, i.e. $\mu(xV) = \mu(V)$ (resp. $\mu(Vx) = \mu(V)$) for every measurable set $E \subseteq G$ and $x \in G$. An important result for locally compact groups is the existence of a Haar measure, see e.g. [21, Theorem 2.10].

Theorem 3.1.3. Every locally compact group G has a unique (up to a constant) left Haar measure μ.

Let G be a locally compact group and μ be the (left) Haar measure on G. For each $x \in G$ we define

$$\begin{aligned} \mu_x \ : \ \mathbf{B}(G) &\longrightarrow \mathbb{C} \\ V \quad &\longmapsto \mu(Vx), \end{aligned} \tag{3.4}$$

where $\mathbf{B}(G)$ is the Borel $\sigma-$algebra of (G, μ). Then, by the uniqueness (up to a constant) of the Haar measure we have that there exists $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$. The function $\Delta \ : \ G \longrightarrow (0, \infty)$ is called the modular function of G. It can be shown that the modular function is a homomorphism. We call a locally compact group *unimodular* if its modular function $\Delta \equiv 1$ and in that case the Haar measure is also right-invariant. Some classes of unimodular groups are the Abelian groups, the compact groups and the locally compact groups of polynomial growth.

3.1.1. Groups of polynomial growth

We now have all the components needed in order to define locally compact groups of polynomial growth. Throughout this section we assume that G is a compactly generated group.

Definition 3.1.4. A compactly generated group G is called a group of *polynomial growth*, if for some generating neighbourhood $U \subseteq G$ of the identity, there exist constants $C_G > 0$, $D_G \in \mathbb{N}$ such that

$$\mu(U^n) \leq C_G n^{D_G}, \tag{3.5}$$

for all $n \in \mathbb{N}$. The minimal exponent D_G such that (3.5) holds is called the *order of growth* of the group G. Furthermore, we say that a compactly generated group G has *strict polynomial growth*, if there exists a symmetric generating neighbourhood $U \subseteq G$ of the identity, and constants $C_G > 0$, $D_G \in \mathbb{N}$ such that

$${C_G}^{-1} n^{D_G} \leq \mu(U^n) \leq C_G n^{D_G},$$

for all $n \in \mathbb{N}$.

A fundamental result on groups of polynomial growth that will be used extensively later is that in such groups strict polynomial growth is automatic. A proof of this combines results from [35] and [43], see [20, Lemma 2.3.] for details.

Theorem 3.1.5. ([35], [43]) Every locally compact group of polynomial growth has strict polynomial growth.

Some trivial examples of groups of polynomial growth are the Euclidean spaces $\mathbb{R}^d$ and the integers $\mathbb{Z}^d$. For connected Lie groups, Jenkins [40] provides a simple characterization for polynomial growth. A connected Lie group G is said to be *type R* if for each X in the Lie algebra $\mathbf{g}$ of G the adjoint representation $ad(X) : \mathbf{g} \longrightarrow \mathbf{g}$ has imaginary eigenvalues [40]. Then for type R groups we have the following result by Jenkins, see [40, Theorem 1.4].

Theorem 3.1.6 ([40]). A connected Lie group G has polynomial growth if and only if G is type R.

A Lie algebra **g** is said to be *nilpotent* if for each $X \in \mathbf{g}$ the adjoint representation $ad(X) : \mathbf{g} \longrightarrow \mathbf{g}$ is nilpotent, i.e. $ad(X)^k = 0$ for some $k \in \mathbb{N}$. Moreover, we call a Lie group G *nilpotent* if its Lie algebra is nilpotent. From this definition we deduce that each connected nilpotent Lie group is a type R group and thus, by Theorem 3.1.6, it has polynomial growth.

Groups of polynomial growth are shown to be unimodular groups. Therefore, non-unimodular groups do not have polynomial growth.

Lemma 3.1.7. Every locally compact group of polynomial growth is unimodular.

Proof. Let G be a locally compact group of polynomial growth, with $U \subseteq G$ a generating unit neighbourhood and μ the Haar measure on G. If $V \subseteq G$ is a measurable subset of the group, then there exists an integer $m = m(V) \in \mathbb{N}$, such that $V \subseteq U^m$, hence

$$1 = \lim_{n\to\infty} \mu(V)^{1/n} \le \lim_{n\to\infty} \mu(V^n)^{1/n} \le \lim_{n\to\infty} \mu(U^{nm})^{1/n} \le \lim_{n\to\infty} (C(nm)^{D_G})^{1/n} = 1,$$

by the polynomial growth. Thus, for every measurable $V \subseteq G$ we have $\lim_{n\to\infty} \mu(V^n)^{1/n} = 1$.

Let $\Delta : G \longrightarrow (0,\infty)$ be the modular function of G and let $x \in G$ and $V \subseteq G$ be a measurable subset such that $x \in V$. Then, for each $n \in \mathbb{N}$ we get

$$\mu(V)\Delta(x)^{n-1} = \mu(Vx^{n-1}) \le \mu(V^n),$$

since the modular function is an homomorphism and $x \in V$. Hence,

$$\left(\mu(V)\Delta(x)^{n-1}\right)^{1/n} \le \mu(V^n)^{1/n},$$

and we deduce that

$$\Delta(x) = \lim_{n\to\infty} (\mu(V)\Delta(x)^{n-1})^{1/n} \le \lim_{n\to\infty} \mu(V^n)^{1/n} = 1. \tag{3.6}$$

Thus, for each $x \in G$ we have $\Delta(x) \le 1$, but since Δ is a homomorphism we have also $\Delta(x)^{-1} = \Delta(x^{-1}) \le 1$. We conclude that $\Delta(x) = 1$ for each $x \in G$. □

We say that a metric space (G, d) is a *doubling metric space* if there exists $C > 0$, such that for all $x \in G$ and $r > 0$

$$\mu(B(x, 2r)) \le C\mu(B(x, r)).$$

Moreover, we say that a set $V \subseteq G$ has the *doubling property*, if there exists $C = C(V) > 0$, such that for all $x \in G$ and $r \in \mathbb{N}$

$$\mu(xV^{2r}) \le C\mu(xV^r). \tag{3.7}$$

Using the strict polynomial growth, we deduce that groups of polynomial growth have the doubling property.

Lemma 3.1.8. Every locally compact group, G, of polynomial growth with the word metric is a doubling metric space.

Proof. Let $U \subseteq G$ be a generated unit neighbourhood of G. By Theorem 3.1.5, G has strict polynomial growth and hence there exist constants C_1, $C_2 > 0$, $D_G \in \mathbb{N}$ such that

$$C_1 n^{D_G} \le \mu(U^n) \le C_2 n^{D_G}, \tag{3.8}$$

for all $n \in \mathbb{N}$. From Equation (3.3) and the left invariance of the metric, for each $r \in \mathbb{N}$ and $x \in G$ we obtain

$$\mu\left(\overline{B(x, r)}\right) = \mu\left(xU^r\right) = \mu\left(U^r\right).$$

From Equation (3.8), we have for each $n \in \mathbb{N}$

$$\mu\left(\overline{B(x, 2n)}\right) = \mu\left(U^{2n}\right) \le C_2(2n)^{D_G} \le C_2 C_1^{-1} 2^{D_G} \mu\left(U^{2n}\right) = C_2 C_1^{-1} 2^{D_G} \mu\left(\overline{B(x, n)}\right).$$

Thus, for each $n \in \mathbb{N}$ using the inner regularity of the Haar measure μ we deduce

$$\mu\left(B\left(x, 2n\right)\right) = \sup_{m\in\mathbb{N}} \mu\left(\overline{B\left(x, 2(n - \frac{1}{m}\right)}\right) \le C_3 \sup_{m\in\mathbb{N}} \mu\left(\overline{B\left(x, n - \frac{1}{m}\right)}\right) \le C_3 \mu\left(B\left(x, n\right)\right),$$

where $C_3 = C_2 C_1^{-1} 2^{D_G} > 0$. □

3.1.2. Discrete sets

We define relatively separated sets, a class of discrete sets in locally compact groups, which will be used as the index sets of matrices on such groups.

Definition 3.1.9. Let G be a locally compact group, $\Lambda \subseteq G$ be a set in the group and let $V \subset G$ be a relatively compact unit neighbourhood. The set Λ is called a *relatively separated set* in G, if

$$\mathrm{Rel}_V\left(\Lambda\right) := \sup_{x \in G} \#\left(\Lambda \cap xV\right) < \infty. \tag{3.9}$$

Let $\Lambda \subseteq G$ be a relatively separated set in the locally compact group G and let $V, V' \subset G$ be relatively compact unit neighbourhoods. By relative compactness, there exists a finite number of $x_i \in G,\ i \in I_{V,V'}$, such that $V \subseteq \bigcup_i x_i V'$. Hence,

$$\begin{aligned}\mathrm{Rel}_V\left(\Lambda\right) &= \sup_{x \in G} \#(\Lambda \cap xV) \leq \sup_{x \in G} \#\left(\Lambda \cap \bigcup_i x x_i V'\right) \leq \sum_i \sup_{x \in G} \#(\Lambda \cap x x_i V') \\ &\leq \sum_i \sup_{x \in G} \#(\Lambda \cap xV') = \#(I_{V,V'})\, \mathrm{Rel}_{V'}\left(\Lambda\right).\end{aligned} \tag{3.10}$$

This shows that if (3.9) holds for some relatively compact unit neighbourhood $V \subset G$, then its holds for all relatively compact unit neighbourhoods.

The following simple characterization provides a necessary and sufficient condition for relatively separated sets, see [25].

Lemma 3.1.10. Let $\Lambda \subseteq G$ be a set in the locally compact group G with Haar measure μ. Then Λ is a relatively separated set if and only if for each measurable relatively compact set $U \subseteq G$ and $n \in \mathbb{N}$ there exists $C_{U,n} > 0$, such that

$$\#\left(\Lambda \cap xU^n\right) \leq C_{U,n}\mu\left(xU^n\right). \tag{3.11}$$

Proof. Let $\Lambda \subseteq G$ be a relatively separated set and let $n \in \mathbb{N}$. Then using Equation (3.10) we obtain

$$\#\left(\Lambda \cap xU^n\right) \leq \sup_{x \in G} \#\left(\Lambda \cap xU^n\right) = \mathrm{Rel}_{U^n}\left(\Lambda\right) \leq C_0\, \mathrm{Rel}_U\left(\Lambda\right),$$

where $C_0 = C_0(U, n) > 0$. Hence,

$$\begin{aligned}\#\left(\Lambda \cap xU^n\right) &\leq C_0\, \mathrm{Rel}_U\left(\Lambda\right) = C_0 \frac{\mu(U^n)}{\mu(U^n)} \mathrm{Rel}_U\left(\Lambda\right) \\ &= C_{U,n}\mu(U^n) = C_{U,n}\mu(xU^n),\end{aligned}$$

by the left invariance of the measure, where the constant $C_{U,n} > 0$ is given by $C_{U,n} := C_0 \frac{\mathrm{Rel}_U(\Lambda)}{\mu(U^n)}$. Conversely, let $\Lambda \subseteq G$ be such that

$$\#\left(\Lambda \cap xU^n\right) \leq C_{U,n}\mu\left(xU^n\right),$$

for a measurable relatively compact set $U \subseteq G$ and $n \in \mathbb{N}$. Then taking supremum over all $x \in G$ and using the left invariance of the Haar measure we obtain

$$\begin{aligned}\mathrm{Rel}_{U^n}\left(\Lambda\right) &= \sup_{x \in G} \#\left(\Lambda \cap xU^n\right) \leq \sup_{x \in G} C_{U,n}\mu\left(xU^n\right) \\ &= C_{U,n} \sup_{x \in G} \mu\left(U^n\right) = C_{U,n}\mu\left(U^n\right) < \infty.\end{aligned}$$

Thus, $\Lambda \subseteq G$ is a relatively separated set. □

In compactly generated groups we have that relatively separated sets are countable discrete sets.

Lemma 3.1.11. Let G be a compactly generated set and $\Lambda \subseteq G$ be a relatively separated set. Then Λ is countable.

Proof. Suppose that $U \subseteq G$ is a generating neighbourhood of the compactly generated G, i.e. $G = \bigcup_{n=0}^{\infty} U^n$. Then by the relative separation of Λ there exist $C_n > 0$ for each $n \in \mathbb{N} \cup \{0\}$ such that

$$\mathrm{Rel}_{U^n}(\Lambda) = C_n < \infty. \tag{3.12}$$

Hence,

$$\#(\Lambda \cap U^n) \leq \sup_{x \in G} \#(\Lambda \cap xU^n) = \mathrm{Rel}_{U^n}(\Lambda) = C_n < \infty. \tag{3.13}$$

Since $G = \bigcup_{n=0}^{\infty} U^n$ we deduce $\Lambda = \Lambda \cap G = \bigcup_{n=0}^{\infty} (\Lambda \cap U^n)$. Thus Λ is countable as the countable union of the finite sets $\Lambda \cap U^n$. □

We present a covering result for locally compact groups where relatively separated sets arise naturally. This result will be used extensively in the following results. The proof presented here is based on a covering lemma given by Anker [2].

Lemma 3.1.12. Let $W \subseteq G$ be a relatively compact symmetric unit neighbourhood with non-empty interior in the locally compact group G. Then there exists a relatively separated set $Y \subseteq G$ such that

1. $\{x_k W^2\}_{x_k \in Y}$ is a cover of G.
2. the sets $\{x_k W\}_{x_k \in Y}$ are pairwise disjoint.
3. every $x \in G$ belongs to at most $\frac{\mu(W^5)}{\mu(W)}$ sets xW^2, $x \in Y$.
4. $\mathrm{Rel}_{W^2}(Y) \leq \frac{\mu(W^5)}{\mu(W)}$.

Proof. Let $X := \left\{ \{yW\}_{y \in V} \;:\; V \subseteq G,\ \{yW\}_{y \in V} \text{ are pairwise disjoint} \right\}$. We define the partial order $\leq_X$ on X, such that

$$\{yW\}_{y \in V_1} \leq_X \{yW\}_{y \in V_2} \iff V_1 \subseteq V_2,$$

for each $\{yW\}_{y \in V_1}, \{yW\}_{y \in V_2} \in X$. Then $P := (X, \leq_X)$ is a partially order pair. Suppose that

$$P' = \left(\left\{ \{yW\}_{y \in V_i},\ V_i \in I \right\}, \leq_X \right)$$

is a chain in P, where $I \subseteq \mathscr{P}(G)$ and $\mathscr{P}(G) = \{V \subseteq G\}$ is the power set of G. We claim that P' has an upper bound. Set $V' = \bigcup_{V_i \in I} V_i$ and let $a, b \in V'$. Then there exist $V_i, V_j \in I$ such that $a \in V_i$ and $b \in V_j$. Since P' is a chain in P, we have V_i and V_j are comparable and without loss of generality we suppose that $V_i \leq_X V_j$. Then $V_i \subseteq V_j$ and hence $a, b \in V_j$. Thus, aW and bW are disjoint and we deduce that $\{yW\}_{y \in V}$ are pairwise disjoint. We have that $\{yW\}_{y \in V}$ is an upper bound of P' since $\{yW\}_{y \in V} \in X$ and $V_i \leq_X V$ for each $V_i \in I$. We have shown that an arbitrary chain in P has an upper bound, and hence we can apply Zorn's Lemma.

By Zorn's Lemma, there exist a maximal subset $Y \subset G$, such that the set $\{xW\}_{x \in Y}$ consists of pairwise disjoint sets. Let $g \in G$. By the maximality of $\{xW\}_{x \in Y}$ we have that gW meets at least one set zW, for some $z \in Y$. Hence, by the symmetry of W we have $g \in zWW^{-1} = zWW \subseteq zW^2$, and we deduce that

$$G = \bigcup_{x \in Y} xW^2, \tag{3.14}$$

or, equivalently, we have that $\{xW^2\}_{x \in Y}$ is a cover of G.

Let $g \in G$. Assume that g belongs to $m \geq 1$ sets of the set $\{xW^2\}_{x \in Y}$. Suppose that $g \in x_i W^2$ for $i = 1, 2, \ldots, m$, where $x_i \in Y$ for every $i = 1, 2, \ldots, m$. Then for $i \in \{1, 2, , ..m\}$ we have that $g \in x_1 W^2 \cap x_i W^2$. Hence, there exist $v_1, v_i \in W^2$ such that $g = x_1 v_1 = x_i v_i$ and $x_i = x_1 v_1 v_i^{-1}$. If $y \in x_i W$, then there exists $z \in W$ such that $y = x_i z$ and we obtain $y = x_1 v_1 v_i^{-1} z \in x_1 W^2 W^{-2} W$. Hence $x_i W \subseteq x_1 W^2 W^{-2} W$ for every $i \in \{1, 2, , ..m\}$ and since the sets $\{x_i W\}_i$ are pairwise disjoint and the measure is left invariant we have

$$\begin{aligned} m\mu(W) = \sum_{i=1}^{m} \mu(W) = \sum_{i=1}^{m} \mu(x_i W) = \mu\left(\cup_{i=1}^{m} x_i W\right) \\ \leq \mu\left(x_1 W^2 W^{-2} W\right) \leq \mu\left(x_1 W^5\right) = \mu\left(W^5\right). \end{aligned}$$

Thus every $g \in G$ belongs to at most $m \leq \frac{\mu(W^5)}{\mu(W)}$ sets xW^2, $x \in Y$. Moreover, using the symmetry of W we obtain

$$\mathrm{Rel}_{W^2}(Y) = \sup_{x \in G} \#\left(Y \cap xW^2\right) = \sup_{x \in G} \#\left\{x_k \in Y \ : \ x \in x_k W^2\right\} \leq m < \infty,$$

and we deduce that Y is a relatively separated set. □

We now adapt the previous result to a covering lemma for groups of polynomial growth. In this case, we cover the group by translations of the set U^{2N}, where U is the generating neighbourhood of the group. Furthermore, using the strict polynomial growth we show that the maximum number of covering set that each element belongs to can be chosen to be independent of the power N.

Lemma 3.1.13. Fix $N \in \mathbb{N}$. Let G be a locally compact group of polynomial growth generated by the unit neighbourhood $U \subset G$, with order of growth equal to $D_G > 0$. Then there exists a relatively separated and countable set $X_N \subset G$, such that:

1. $\left\{x_k U^{2N}\right\}_{x_k \in X_N}$ is a cover of G.
2. the sets $\left\{x_k U^{N}\right\}_{x_k \in X_N}$ are pairwise disjoint.
3. every $x \in G$ belongs to at most $C_G^2 5^{D_G}$ sets $x_k U^{2N}$, $x_k \in X_N$, where $C_G > 0$ is a constant given by the growth of U, i.e. $C_G^{-1} n^{D_G} \leq \mu(U^n) \leq C_G n^{D_G}$ for each $n \in \mathbb{N}$.
4. $\mathrm{Rel}_{U^{2N}}(X_N) \leq {C_G}^2 5^{D_G}$.

Proof. From Theorem 3.1.5, we have that G has strict polynomial growth and hence there exist constants $C_G > 0$, $D_G \in \mathbb{N}$ such that

$$C_G^{-1} n^{D_G} \leq \mu(U^n) \leq C_G n^{D_G}, \tag{3.15}$$

for all $n \in \mathbb{N}$.

By Lemma 3.1.12 for $W = U^N$, there exists a relatively separated set X_N in G such that the sets $\left\{x_k U^N\right\}_{x_k \in X_N}$ are pairwise disjoint and $\left\{x_k U^{2N}\right\}_{x_k \in X_N}$ is a cover of G. Moreover, every $x \in G$ belongs to at most $\frac{\mu(U^{5N})}{\mu(U^N)}$ sets $x_k U^{2N}$, $x_k \in X_N$. Using Equation (3.15), we obtain

$$\frac{\mu\left(U^{5N}\right)}{\mu(U^N)} \leq \frac{C_G (5N)^{D_G}}{C_G^{-1} N^{D_G}} = C_G^2 5^{D_G}. \tag{3.16}$$

From the previous, we conclude that there exists a set $X_N \subset G$ such that $\left\{xU^{2N}\right\}_{x \in X_N}$ is a cover of G, the sets $xU^N,\ x \in X_N$ are pairwise disjoint and every $g \in G$ belongs to at most ${C_G}^2 5^{D_G}$ sets $xU^{2N},\ x \in X_N$. Moreover, using the symmetry of U we obtain

$$\mathrm{Rel}_{U^{2N}}(X_N) = \sup_{x \in G} \#\left(X_N \cap xU^{2N}\right) = \sup_{x \in G} \#\left\{x_{k,N} \in X_N \ : \ x \in x_{k,N} U^{2N}\right\} \leq {C_G}^2 5^{D_G} < \infty,$$

Finally, since X_N is relatively separated in a group of polynomial growth we deduce from Lemma 3.1.11 that X_N is countable. □

The following result provides a similar estimate as in Equation (3.11), but with a constant independent of the power of the neighbourhood. See [25, Lemma 3.4.] for a similar proof.

Lemma 3.1.14. Let $\Lambda \subseteq G$ be a relatively separated set in the locally compact group G with Haar measure μ and $\rho \in \mathbb{N}$. Then for each relatively compact symmetric unit neighbourhood $U \subseteq G$ with the doubling property there exists $D_0 := D_0(\mathrm{Rel}_U(\Lambda), \rho) > 0$, such that for each $R \geq 0$, and each $x \in G$ we have

$$\#\left(\Lambda \cap xU^R\right) \leq D_0\ \mu(xU^{R+\rho})$$

Proof. From Lemma 3.1.10, since Λ is relatively separated, for $\rho \in \mathbb{N}$ there exists $C_\rho > 0$ such that

$$\#\left(\Lambda \cap zU^{2\rho}\right) \leq C_\rho \mu(zU^{2\rho}),\ z \in G.$$

Note that the constant $C_\rho > 0$ given by Lemma 3.1.10 depends on $\mathrm{Rel}_U(\Lambda)$, U and ρ. Recall from Equation (3.7) that, since U has the doubling property, there exists $C = C(U) > 0$ such that

$$\mu(xU^{2\rho}) \le C\,\mu(xU^\rho)$$

By applying Zorn's Lemma, using a similar argument as in Lemma 3.1.12, there exists a relatively separated set $X_0 \subseteq xU^R$ such that $\{yU^\rho\}_{y\in X_0}$ is a maximal disjoint set and $\{yU^{2\rho}\}_{y\in X_0}$ is a cover of xU^R. Then we obtain

$$\begin{aligned}\#\left(\Lambda \cap xU^R\right) &\le \#\left(\bigcup_{y\in X_0}\left(\Lambda\cap yU^{2\rho}\right)\right) \le \sum_{y\in X_0}\#\left(\Lambda\cap yU^{2\rho}\right)\\ &\le C_\rho\sum_{y\in X_0}\mu(yU^{2\rho}) \le CC_\rho\sum_{y\in X_0}\mu(yU^\rho),\end{aligned}$$

by the doubling property. Furthermore, we obtain

$$\#\left(\Lambda\cap xU^R\right) \le CC_\rho\,\mu\left(\bigcup_{y\in X_0} yU^\rho\right) \le CC_\rho\,\mu(xU^{R+\rho}) \le D_0\,\mu(xU^{R+\rho}),$$

where $D_0 := CC_\rho$ and for the second inequality we used that if $y \in X_0 \subseteq xU^R$, then $yU^\rho \subseteq xU^{R+\rho}$ and that the sets $\{yU^\rho\}_{y\in X_0}$ are disjoint. □

In locally compact groups, we can cover open, relatively compact unit neighbourhoods V^{2n} by a finite number of translations of V^n. Moreover, if V is a doubling neighbourhood, then this number can be chosen to be independent of n, as the following shows. The proof of this was provided by Van Velthoven and Voigtlaender.

Lemma 3.1.15. Let G be a locally compact group with Haar measure μ and let $V \subseteq G$ be an open relatively compact, symmetric unit neighbourhood. Furthermore, assume that V is a doubling neighbourhood, i.e. there exists $C > 0$ such that

$$\mu(V^{2n}) \le C\mu(V^n),$$

for each $n \in \mathbb{N}$. Then there exists $K := K(V) \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, there exist $x_1, x_2, \ldots, x_K \in G$ such that

$$V^{2n} \subseteq \bigcup_{i=1}^{K} x_i V^n. \tag{3.17}$$

Proof. Fix $n \in \mathbb{N}$. By Lemma 3.1.12 there exists a relatively separated set $X \subset G$, such that the sets $\{xV^n\}_{x\in X}$ are pairwise disjoint and $\{xV^{2n}\}_{x\in X}$ is a cover of G. Set

$$I := \left\{x \in X \;:\; V^{4n}\cap xV^{2n} \neq \emptyset\right\}.$$

Since $\{xV^{2n}\}_{x\in X}$ is a cover of G, we have

$$V^{4n} \subseteq \bigcup_{x\in I} xV^{2n}. \tag{3.18}$$

Using that $\{xV^n\}_{x\in X}$ are pairwise disjoint and the left invariance of the Haar measure μ, we obtain

$$|I|\,\mu(V^n) = \sum_{i=1}^{|I|}\mu(V^n) = \sum_{x\in I}\mu(xV^n) = \mu\left(\bigcup_{x\in I} xV^n\right).$$

If $x \in I$, then $x \in V^{4n}V^{-2n} \subseteq V^{6n}$ by the symmetry of V and $xV^n \subseteq V^{6n}V^n \subseteq V^{7n}$ and hence

$$|I|\,\mu(V^n) = \mu\left(\bigcup_{x\in I} xV^n\right) \le \mu\left(V^{7n}\right) \le \mu\left(V^{8n}\right) \le C^3\mu\left(V^n\right),$$

where for the last inequality we have used the doubling property of V. From the previous we conclude that $|I| \leq C^3$ and thus there exist $y_1, y_2, \ldots, y_{\lceil C^3 \rceil} \in X$, such that

$$V^{4n} \subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^{2n}. \tag{3.19}$$

Let $\epsilon_n \in \{0, 1\}$ and $m \in \mathbb{N}$ with $n+\epsilon_n = 2m$. Then by applying the previous for m there exist $y_1, y_2, \ldots, y_{\lceil C^3 \rceil} \in X$, such that

$$V^{4m} \subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^{2m},$$

and hence

$$\begin{aligned} V^{2n} \subseteq V^{2(n+\epsilon_n)} = V^{4m} &\subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^{2m} = \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^{n+\epsilon_n} \\ &\subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^{1+2\epsilon_n} V^{n-1} \subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^3 V^{n-1}. \end{aligned}$$

Since V^3 is relatively compact and V is a unit neighbourhood, there exists $T \in \mathbb{N}$ and $z_1, \ldots, z_T \in G$ such that

$$V^3 \subseteq \bigcup_{i=1}^{T} z_i V.$$

Thus,

$$V^{2n} \subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i V^3 V^{n-1} \subseteq \bigcup_{i=1}^{\lceil C^3 \rceil} y_i \bigcup_{j=1}^{T} z_j V V^{n-1} = \bigcup_{i=1}^{\lceil C^3 \rceil} \bigcup_{j=1}^{T} y_i z_j V^n.$$

We conclude that there exist $K := \lceil C^3 \rceil T \in \mathbb{N}$ and $x_1, \ldots, x_K \in G$ such that

$$V^{2n} \subseteq \bigcup_{i=1}^{K} x_i V^n.$$

Finally, we observe that K does not depend on the choice of $n \in \mathbb{N}$, which proves the claim.

□

For a locally compact group G of polynomial growth, with generating neighbourhood U we have from Theorem 3.1.5 and Lemma 3.1.8 that U verifies the assumptions of the previous Lemma. Thus, there exists $K := K(U) \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, there exist $x_1, x_2, \ldots, x_K \in G$ with

$$U^{2n} \subseteq \bigcup_{i=1}^{K} x_i U^n. \tag{3.20}$$

Now, for a relatively separated set $\Lambda \subseteq G$ we have

$$\begin{aligned} \mathrm{Rel}_{U^{2n}}(\Lambda) &= \sup_{x \in G} \#\left(\Lambda \cap xU^{2n}\right) \leq \sup_{x \in G} \#\left(\Lambda \cap \bigcup_{i=1}^{K} x x_i U^n\right) \\ &\leq \sum_{i=1}^{K} \sup_{x \in G} \#\left(\Lambda \cap x x_i U^n\right) \leq \sum_{i=1}^{K} \sup_{x \in G} \#\left(\Lambda \cap x U^n\right) \\ &= \sum_{i=1}^{K} \mathrm{Rel}_{U^n}(\Lambda) = K \, \mathrm{Rel}_{U^n}(\Lambda). \end{aligned}$$

This proves the following Lemma.

Lemma 3.1.16. Let G be a locally compact group of polynomial growth, with generating neighbourhood U and let $\Lambda \subseteq G$ be a relatively separated set. Then there exists $K > 0$ such that for each $n \in \mathbb{N}$

$$\mathrm{Rel}_{U^{2n}}(\Lambda) \leq K \, \mathrm{Rel}_{U^n}(\Lambda). \tag{3.21}$$

Lattices

A lattice in a locally compact group is a discrete set and has the group structure. In this subsection, we give a definition for lattices and show that a lattice is also a relatively separated set in the group.

Definition 3.1.17. Let G be a locally compact group and $\Lambda \subseteq G$ be a discrete subgroup of G. The subgroup Λ is called a *lattice* in G if there exists a finite Borel measure ν on the quotient space G/Λ, which is G-invariant, that is, for each $g \in G$ and any open subset $W \subseteq G/\Lambda$ we have $\nu(gW) = \nu(W)$. Moreover, if the quotient space G/Λ is compact, then the lattice Λ is called *uniform.*

Let Λ be a lattice in the locally compact group G. Since Λ is a discrete group, then the subspace topology is the discrete topology. Hence, there exists an open, non-empty subset V_e of G, such that $\Lambda \cap V_e = \{e\}$, where e is the unit of G. By local compactness we can choose a relatively compact open unit neighbourhood W and then by taking the intersection $W \cap V_e$ we deduce that $W \cap V_e$ is an open, non-empty unit neighbourhood and $\Lambda \cap (W \cap V_e) = \{e\}$. Thus, without loss of generality we choose V_e to be relatively compact. Let $V \subseteq V_e$ be a unit neighbourhood such that $V^{-1}V \subseteq V_e$. Now, suppose that $x \in G$ and $\lambda_1,\ \lambda_2 \in \Lambda \cap xV$. Then, $\lambda_1 \in \lambda_2 V^{-1}V$ and we deduce that $\lambda_2^{-1}\lambda_1 \in V^{-1}V \subseteq V_e$. Since Λ is a lattice and in particular a subgroup we have that $\lambda_2^{-1}\lambda_1 \in \Lambda \cap V_e = \{e\}$. Hence, $\lambda_1 = \lambda_2$ and we deduce for each $x \in G$ that $\#\left(\Lambda \cap xV\right) \leq 1$. Thus,

$$\mathrm{Rel}_V\left(\Lambda\right) := \sup_{x\in G} \#\left(\Lambda \cap xV\right) \leq 1, \tag{3.22}$$

and we conclude that Λ is a relatively separated set in G. Furthermore, it can be shown that a uniform lattice in a locally compact group of polynomial growth is also a group of polynomial growth equipped with the counting measure.

Lemma 3.1.18. Let G be a locally compact group of polynomial growth and Λ is a uniform lattice in G. Then $(\Lambda,\ \mathscr{P}(\Lambda),\ \mu_C)$ is a locally compact group of polynomial growth, where $\mathscr{P}(\Lambda) := \{V \subseteq \Lambda\}$ is the power set of Λ and

$$\begin{aligned} \mu_C\ &: \mathscr{P}(\Lambda) \longrightarrow [0,\infty] \\ \Gamma\ &\longmapsto \#\Gamma := \#\left\{x \in G\ :\ x \in \Gamma\right\}, \end{aligned} \tag{3.23}$$

is the counting measure on Λ. Moreover, the order of growth of Λ is equal to the one of G.

Proof. Since Λ is a uniform lattice in the compactly generated group we deduce that Λ is compactly generated, see e.g. [10, Proposition 4.C.11], with generating neighbourhood $V \subseteq \Lambda$. Suppose that $U \subseteq G$ is the generating neighbourhood of G and G has order of growth equal to $D_G \in \mathbb{N}$, i.e. there exists $C_G > 0$ such that $\mu(U^n) \leq C_G n^{D_G}$, where μ is the Haar measure on G. Then, since G is generated by the neighbourhood U, there exists $n_0 \in \mathbb{N}$ such that $V \subseteq U^{n_0}$. Using Lemma 3.1.14 we have

$$\begin{aligned} \mu_C(V^n) &= \#\left(\Lambda \cap V^n\right) \leq \#\left(\Lambda \cap (U^{n_0})^n\right) \leq \#\left(\Lambda \cap U^{n_0 n}\right) \\ &\leq D_0\mu\left(U^{n_0 n+1}\right) \leq D_0 C_G (n_0 n+1)^{D_G} \leq D_0 C_G (2n_0)^{D_G} n^{D_G}, \end{aligned}$$

where $D_0 := D_0(\mathrm{Rel}_U\left(\Lambda\right)) > 0$. Thus, for each $n \in \mathbb{N}$ we have

$$\mu_C(V^n) \leq C_\Lambda n^{D_G},$$

and we deduce that Λ is a locally compact group of polynomial growth with order of growth equal to D_G.

□

There are classes of locally compact groups for which any lattice is automatically uniform, see [3]. Examples of such groups are the nilpotent Lie groups and connected solvable Lie groups [3, 45]. On the other hand, it should be noted that not every nilpotent Lie group admits a lattice, an example of such group is given in [52, Remark 2.14.].

3.2. Amalgam spaces

The Amalgam functions is a class of integrable functions on a locally compact group, that will be used in the definition of convolution-dominated matrices. In order to define this class, we initially introduce the local maximal functions. Most properties presented in this section can be found in [27, 54, 66]. See also [24, 36] for more on Amalgam spaces.

Throughout this section, G denotes a compactly generated group with generating neighbourhood $U \subseteq G$ and (left) Haar measure μ. Moreover, we equip G with the word metric d.

3.2.1. Local Maximal Functions

For defining the local maximal functions, we denote the space of all measurable functions $f : G \longrightarrow \mathbb{C}$ by $L^0(G)$ and the space of all locally essentially bounded functions on G by

$$L^\infty_{loc}(G) := \left\{ f \in L^0(G) \ : \ f\chi_K \in L^\infty(G) \text{ for every } K \subseteq G \text{ compact} \right\} \tag{3.24}$$

The local maximal functions are defined as follows.

Definition 3.2.1. Fix an open, symmetric, relatively compact unit neighbourhood Q in G. For a function $f \in L^\infty_{loc}(G)$ we define the *left* and *right maximal functions* of f by

$$\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(x) := \operatorname*{ess\,sup}_{y\in Q} |f(xy)| \quad \text{and} \quad \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f(x) := \operatorname*{ess\,sup}_{y\in Q} |f(yx)| \,.$$

Define the involution of a function $f \in L^0(G)$ by $f^\vee(x) = f(x^{-1})$. Note that if $f \in L^\infty_{loc}(G)$, then for $x \in G$, we obtain

$$\begin{aligned}
\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}(f^\vee)(x) &= \operatorname*{ess\,sup}_{y\in Q} |f^\vee(xy)| = \operatorname*{ess\,sup}_{y\in Q} \left|f(y^{-1}x^{-1})\right| \\
&= \operatorname*{ess\,sup}_{y\in Q} \left|f(yx^{-1})\right| = \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}(f)(x^{-1}) = \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}(f)^\vee(x),
\end{aligned}$$

where for the third equality we have used that Q is a symmetric unit neighbourhood. Thus, if $f \in L^\infty_{loc}(G)$, then

$$\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}(f^\vee) = \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}(f)^\vee. \tag{3.25}$$

The local maximal functions of a function provide an estimate of the function. We present a proof for this estimate, given in [66].

Lemma 3.2.2. For any $Q \subset G$ as in Definition 3.2.1 and for each $f \in L^\infty_{loc}(G)$ we have

$$|f(x)| \leq \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(x) \ \text{ and } \ |f(x)| \leq \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f(x),$$

μ-almost every $x \in G$.

Proof. Let $V \subseteq G$ be an open, symmetric, relatively compact unit neighbourhood such that $VV \subseteq Q$. Since G is compactly generated, there exist a generating neighbourhood U such that

$$G = \bigcup_{n\in\mathbb{N}} U^n.$$

Using the relative compactness of U^n we can cover the sets U^n by a finite number of sets xV for $x \in G$, hence $U^n \subseteq \bigcup_{k=1}^{N_n} x_{k,n}V$. Thus, $G = \bigcup_{k\in\mathbb{N}} x_k V$ for a countable sequence $(x_k)_{k\in\mathbb{N}}$. We observe that for, μ-almost every $x \in x_k V$,

$$|f(x)| \leq \operatorname*{ess\,sup}_{y\in V} |f(x_k y)| \,,$$

and, for every $z \in z'V$,

$$\operatorname*{ess\,sup}_{y\in V} |f(zy)| \leq \operatorname*{ess\,sup}_{y\in VV} |f(z'y)| \leq \operatorname*{ess\,sup}_{y\in Q} |f(z'y)| = \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(z').$$

If $x \in x_k V$, then $x_k \in xV^{-1} = xV$ and combining the previous we have for μ-almost every $x \in x_k V$

$$|f(x)| \leq \operatorname*{ess\,sup}_{y\in V} |f(x_k y)| \leq \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(x). \tag{3.26}$$

From $G = \bigcup_{k\in\mathbb{N}} x_k V$ and Equation (3.26) we conclude that for μ-almost every $x \in G$

$$|f(x)| \leq \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(x). \tag{3.27}$$

Furthermore, using the previous and Equation (3.25) we obtain for μ-almost every $x \in G$

$$\begin{aligned}|f(x)| &= \left|f^\vee(x^{-1})\right| \leq \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} f^\vee(x^{-1}) = \operatorname*{ess\,sup}_{y\in Q}\left|f^\vee(x^{-1}y)\right| \\ &= \operatorname*{ess\,sup}_{y\in Q}\left|f(y^{-1}x)\right|.\end{aligned}$$

Thus, using the symmetry of Q we obtain

$$|f(x)| = \operatorname*{ess\,sup}_{y\in Q}\left|f(y^{-1}x)\right| = \operatorname*{ess\,sup}_{y\in Q}|f(yx)| = \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f(x). \tag{3.28}$$

□

3.2.2. Amalgam Function Spaces

For the rest we fix an open, symmetric, relatively compact unit neighbourhood Q in the group G, unless stated otherwise. We denote by $\operatorname{Rel}\Lambda := \operatorname{Rel}_Q(\Lambda)$ the relatively separated constant of the set Λ with respect to Q.

Definition 3.2.3. A normed space $(Y, \|\cdot\|_Y)$ is called a *function space* on a measure space (G, Σ, μ), if Y is a subspace of $L^0(G,\mu)$. If Y is furthermore complete, then is called a *Banach function space.*

Definition 3.2.4. A function space $(Y, \|\cdot\|_Y)$ on (G, Σ, μ) is called *solid*, if for each measurable function $f \in L^0(G,\mu)$ such that $|f| \leq |g|$ μ-almost everywhere for some $g \in Y$, we have $f \in Y$, with $\|f\|_Y \leq \|g\|_Y$.

For $p \in [1,\infty]$ the Lebesgue space $L^p(G)$ is defined by

$$L^p(G) := \left\{ f \in L^0(G) \;:\; \|f\|_{L^p(G)} < \infty \right\}, \tag{3.29}$$

where

$$\|f\|_{L^p(G)} = \left(\int_G |f|^p \,\mathbf{d}\mu\right)^{1/p}, \tag{3.30}$$

when $p \in [1,\infty)$, and

$$\|f\|_{L^p(G)} = \sup_G |f|, \tag{3.31}$$

when $p = \infty$. We call a measurable weight $w : G \longrightarrow [1,\infty)$ *submultiplicative*, if $w(xy) \leq w(x)w(y)$, for each $x, y \in G$. For a measurable, submultiplicative weight w on G and $p \in [1,\infty]$ the weighted $L^p_w(G)$ space of functions on G is defined by

$$L^p_w(G) := \left\{ f \in L^0(G) \;:\; \|fw\|_{L^p(G)} < \infty \right\} \tag{3.32}$$

and it can be shown to be solid Banach function spaces.

We now define the Amalgam spaces.

Definition 3.2.5. Let $(Y, \|\cdot\|_Y)$ be a solid function space on (G,μ). The *left* and *right Wiener Amalgam spaces* with local component L^∞ and global component Y are defined by

$$W_Q^L(L^\infty, Y) := \left\{ f \in L^\infty_{loc}(G) \;:\; \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} f \in Y \right\}, \tag{3.33}$$

$$W_Q^R(L^\infty, Y) := \left\{ f \in L^\infty_{loc}(G) \;:\; \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f \in Y \right\}, \tag{3.34}$$

with norms

$$\|f\|_{W_Q^L(L^\infty, Y)} = \left\|\mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} f\right\|_Y \tag{3.35}$$

$$\|f\|_{W_Q^R(L^\infty, Y)} = \left\|\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f\right\|_Y \tag{3.36}$$

respectively.

We mainly focus our attention to the study of the left and right Wiener Amalgam spaces $W_Q^L(L^\infty, Y)$ and $W_Q^R(L^\infty, Y)$, with local component $L^\infty(G)$ and global component $Y = L^1_w(G)$, where $w : G \to [1,\infty)$ is a measurable, submultiplicative weight and $L^1_w(G)$ is the weighted Lebesgue function space. In some cases, we also consider the Amalgam spaces with global component $W_Q^L(L^\infty, L^1_w(G))$ or $W_Q^R(L^\infty, L^1_w(G))$, i.e. the spaces $W_Q^L(L^\infty, W_Q^R(L^\infty, L^1_w(G)))$ and $W_Q^R(L^\infty, W_Q^L(L^\infty, L^1_w(G)))$.

It can be shown that the Amalgam spaces $W_Q^L(L^\infty, L^1_w(G))$ and $W_Q^R(L^\infty, L^1_w(G))$ are embedded in $L^1_w(G)$. From Lemma 3.2.2 and since $L^1_w(G)$ is a solid function space, it follows

$$\|f\|_{L^1_w(G)} \leq \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1_w(G)} \quad \text{and} \quad \|f\|_{L^1_w(G)} \leq \left\|\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f\right\|_{L^1_w(G)}. \tag{3.37}$$

Furthermore, the left Amalgam space $W_Q^L(L^\infty, L^1_w(G))$ is embedded in $L^\infty(G)$.

Lemma 3.2.6. Let $w : G \longrightarrow [1,\infty)$ be a measurable, submultiplicative weight. For each $f \in L^\infty_{loc}(G)$ we have

$$\|f\|_{L^\infty(G)} \leq C_Q \|f\|_{W_Q^L(L^\infty, L^1_w(G))}, \tag{3.38}$$

where $C_Q > 0$ and hence $W_Q^L(L^\infty, L^1_w(G)) \hookrightarrow L^\infty(G)$.

Proof. If we choose a symmetric open unit neighbourhood V, such that $VV \subseteq Q$, then for each $x \in G$ and $v \in V$ we get $xV = xvv^{-1}V \subseteq xvQ$. Hence, for each $f : G \to \mathbb{C}$ measurable we have

$$\|f\|_{L^\infty(xV)} \leq \|f\|_{L^\infty(xvQ)} = \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(xv).$$

Averaging over V and using the left invariance of the Haar measure μ, we obtain

$$\|f\|_{L^\infty(xV)} \leq \frac{1}{\mu(V)} \int_V \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(xv)\mathbf{d}\mu(v) = \frac{1}{\mu(V)} \int_{xV} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f(v)\mathbf{d}\mu(v) \leq \frac{1}{\mu(V)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1(G)}. \tag{3.39}$$

Since G is compactly generated, then there exist $\{x_n\}_{n\in\mathbb{N}}$, such that $G = \bigcup_{n\in\mathbb{N}} x_n V$. This can be done by covering the sets U^n by translations of the set V (see proof of Lemma 3.2.2 for details). Hence,

$$\|f\|_{L^\infty(G)} = \sup_{n\in\mathbb{N}} \|f\|_{L^\infty(x_n V)} \leq \frac{1}{\mu(V)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1(G)}.$$

By assumption, the weight w satisfies $w \geq 1$, thus

$$\|f\|_{L^\infty(G)} \leq \frac{1}{\mu(V)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1_w(G)} = \frac{1}{\mu(V)} \|f\|_{W_Q^L(L^\infty, L^1_w(G))}$$

and we conclude that $W_Q^L(L^\infty, L^1_w(G)) \hookrightarrow L^\infty(G)$. □

The following estimates will be useful for the upcoming results. We present a proof by Romero, van Velthoven, and Voigtlaender [54].

Lemma 3.2.7. Let $\Theta, \Phi : G \to [0,\infty)$ be continuous functions on G and $\Lambda \subseteq G$ be a relatively separated set in G. Then

$$\sup_{y\in G} \sum_{\lambda\in\Lambda} \Theta(y^{-1}\lambda) \leq \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)} \|\Theta\|_{W_Q^L(L^\infty, L^1(G))}, \tag{3.40}$$

and for each $x, y \in G$

$$\sum_{\lambda\in\Lambda} \Phi(y^{-1}\lambda)\Theta(\lambda^{-1}x) \leq \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)} \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Phi * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \Theta\right)(y^{-1}x). \tag{3.41}$$

Proof. If $y \in G$ and $\lambda \in \Lambda$, then for each $z \in \lambda Q$ we have $z^{-1}\lambda \in Q^{-1} = Q$, since Q is a symmetric unit neighbourhood. Therefore, we have $y^{-1}\lambda = y^{-1}zz^{-1}\lambda \in y^{-1}zQ$ for each $z \in \lambda Q$. Now, since Θ is continuous and Q is an open neighbourhood of the identity, we obtain

$$\Theta(y^{-1}\lambda) \leq \sup_{x\in y^{-1}zQ} \Theta(x) = \sup_{x\in Q} \Theta(y^{-1}zx) = \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta(y^{-1}z),$$

for $y \in G$ and $\lambda \in \Lambda$ and each $z \in \lambda Q$. Hence, for $y \in G$ and $\lambda \in \Lambda$ by averaging over λQ and using the left invariance of the Haar measure μ, we obtain

$$\Theta(y^{-1}\lambda) \leq \frac{1}{\mu(\lambda Q)} \int_{\lambda Q} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\mathbf{d}\mu(z) \leq \frac{1}{\mu(Q)} \int_{\lambda Q} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\mathbf{d}\mu(z).$$

For fixed $y \in G$ we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} \Theta(y^{-1}\lambda) &\leq \sum_{\lambda\in\Lambda} \frac{1}{\mu(Q)} \int_{\lambda Q} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\mathbf{d}\mu(z)\\ &= \frac{1}{\mu(Q)} \sum_{\lambda\in\Lambda} \int_{G} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\chi_{\lambda Q}(z)\mathbf{d}\mu(z)\\ &= \frac{1}{\mu(Q)} \int_{G} \sum_{\lambda\in\Lambda} \chi_{\lambda Q}(z)\, \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\mathbf{d}\mu(z),\end{aligned}$$

where we used the monotone convergence theorem for the last equality. Note that for each $x \in G$, we have

$$\mathrm{Rel}_{Q}\,(\Lambda) \geq \#\,(\Lambda \cap xQ) = \sum_{\lambda\in\Lambda} \chi_{xQ}(\lambda) = \sum_{\lambda\in\Lambda} \chi_{\lambda Q}(x).$$

Thus, using the previous and the left invariance of μ, we get

$$\sum_{\lambda\in\Lambda} \Theta(y^{-1}\lambda) \leq \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \int_{G} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(y^{-1}z)\mathbf{d}\mu(z) \leq \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \int_{G} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Theta(z)\mathbf{d}\mu(z) = \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \left\|\Theta\right\|_{W^{L}(L^{\infty},L^{1}(G))}.$$

By taking supremum over all $y \in G$ we arrive at Equation (3.40).

Using a similar technique we prove the second estimate. Initially, we observe that if $y, x \in G$ and $\lambda \in \Lambda$, then for each $z \in \lambda Q$ we have $y^{-1}\lambda \in y^{-1}zQ$ and $\lambda^{-1}x \in Qz^{-1}x$. Hence, since Θ, Φ are continuous, positive and Q is an open neighbourhood of the identity, we obtain $\Phi(y^{-1}\lambda) \leq \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi(y^{-1}z)$ and $\Theta(\lambda^{-1}x) \leq \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta(z^{-1}x)$. Now, averaging over λQ, using the monotone convergence theorem and that $\mathrm{Rel}_{Q}\,(\Lambda) \geq \sum_{\lambda\in\Lambda} \chi_{\lambda Q}(x)$ we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} \Phi(y^{-1}\lambda)\Theta(\lambda^{-1}x) &\leq \sum_{\lambda\in\Lambda} \frac{1}{\mu(Q)} \int_{\lambda Q} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi(y^{-1}z)\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta(z^{-1}x)\mathbf{d}\mu(z)\\ &\leq \frac{1}{\mu(Q)} \int_{G} \sum_{\lambda\in\Lambda} \chi_{\lambda Q}(z)\, \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi(y^{-1}z)\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta(z^{-1}x)\mathbf{d}\mu(z)\\ &\leq \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \int_{G} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi(y^{-1}z)\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta(z^{-1}x)\mathbf{d}\mu(z).\end{aligned}$$

Finally, using the left invariance of the Haar measure and the change of variables $t = y^{-1}z$ we obtain

$$\begin{aligned}\sum_{\lambda\in\Lambda} \Phi(y^{-1}\lambda)\Theta(\lambda^{-1}x) &\leq \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \int_{G} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi(t)\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta(t^{-1}y^{-1}x)\mathbf{d}\mu(z)\\ &\leq \frac{\mathrm{Rel}_{Q}\,(\Lambda)}{\mu(Q)} \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \Phi * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, \Theta\right)(y^{-1}x),\end{aligned}$$

which proves our claim. □

The definition of the left and right Amalgam spaces can be extended to a two-sided version, where both the left and right local maximal functions are being used. A special case of the two-sided Amalgam space, with global component $Y = L^{1}_{w}(G)$, is presented below.

Definition 3.2.8. Let G be a locally compact group and $w : G \to [1, \infty)$ be a measurable, submultiplicative weight on (G, μ). The *two-sided Wiener Amalgam space* with local component L^{∞} and global component $L^{1}_{w}(G)$ is defined by

$$W_{Q}\,(L^{\infty},\, L^{1}_{w}(G)) := \left\{f \in L^{\infty}_{loc}(G) \ :\ \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, f \in L^{1}_{w}(G)\right\} \tag{3.42}$$

with norm

$$\|f\|_{W_{Q}\,(L^{\infty},\, L^{1}_{w}(G))} = \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, f\right\|_{L^{1}_{w}(G)}. \tag{3.43}$$

It can be shown that for discrete groups the Amalgam space is exactly the set of all summable sequences on the discrete group.

Remark 3.2.9. Note that if G is a discrete group, equipped with the counting measure μ_C, then we fix $Q = \{e\}$, where $e \in G$ is the unit element on G. Then Q verifies the assumptions of the above definition and hence we have $\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f(x) = \operatorname{ess\,sup}_{y,y' \in Q=\{e\}} |f(yxy')| = |f|(x)$, for each $x \in G$. Thus, using that the space $L^1_w(G, \mu_C)$ for the counting measure μ_C is the sequence space $\ell^1_w(G)$, it follows that

$$W_Q\left(L^\infty,\, L^1_w(G)\right) = \left\{f \in L^\infty_{loc}(G) \;:\; |f| \in L^1_w(G, \mu_C)\right\} = \left\{f \in L^\infty_{loc}(G) \;:\; f \in \ell^1_w(G)\right\} \tag{3.44}$$

and

$$\|f\|_{W_Q\left(L^\infty,\, L^1_w(G)\right)} = \||f|\|_{L^1_w(G,\mu_C)} = \|f\|_{\ell^1_w(G)}\,. \tag{3.45}$$

We conclude that for discrete groups

$$W_Q\left(L^\infty,\, L^1_w(G)\right) = \ell^1_w(G), \tag{3.46}$$

equipped with the sequence norm $\|\cdot\|_{\ell^1_w(G)}$.

Let $f,\, g \in L^1(G)$ such that $f(x) \geq 0$ and $g(x) \geq 0$ for each $x \in G$. Then, for $x \in G$ we observe that

$$\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}(f * g)(x) = \operatorname*{ess\,sup}_{z\in Q}\left(\int_G f(y) g(y^{-1}xz)\,\mathbf{d}\mu(y)\right).$$

Now, we assume that $g \in W^L_Q\left(L^\infty,\, L^1_w(G)\right)$ we have $f * \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, g \in L^1(G)$. Then by choosing a sequence converging to the essential supremum and by applying the Dominated Convergence Theorem to that sequence, we obtain

$$\operatorname*{ess\,sup}_{z\in Q}\left(\int_G f(y) g(y^{-1}xz)\,\mathbf{d}\mu(y)\right) = \int_G f(y) \operatorname*{ess\,sup}_{z\in Q} g(y^{-1}xz)\,\mathbf{d}\mu(y) = \int_G f(y)\, \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, g(y^{-1}x)\,\mathbf{d}\mu(y).$$

Thus, for each $f \in L^1(G)$ and $g \in W^L_Q\left(L^\infty,\, L^1_w(G)\right)$ such that $f, g \geq 0$

$$\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}(f * g)(x) = \left(f * \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\, g\right)(x), \tag{3.47}$$

for μ-almost every $x \in G$. Similarly, for $g \in L^1(G)$ and $f \in W^R_Q\left(L^\infty,\, L^1_w(G)\right)$ such that $f, g \geq 0$ we have

$$\begin{aligned}\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}(f * g)(x) &= \operatorname*{ess\,sup}_{z\in Q}\left(\int_G f(y) g(y^{-1}zx)\,\mathbf{d}\mu(y)\right) = \operatorname*{ess\,sup}_{z\in Q}\left(\int_G f(zy) g(y^{-1}x)\,\mathbf{d}\mu(y)\right)\\ &= \int_G \operatorname*{ess\,sup}_{z\in Q}(f(zy)) g(y^{-1}x)\,\mathbf{d}\mu(y) = \int_G \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, f(y) g(y^{-1}x)\,\mathbf{d}\mu(y),\end{aligned}$$

where for the second equality we have used the Dominated convergence theorem and for the third the left invariance of the Haar measure μ. Thus, for each $g \in L^1(G)$ and $f \in W^R_Q\left(L^\infty,\, L^1_w(G)\right)$ such that $f, g \geq 0$

$$\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}(f * g)(x) = \left(\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\, f * g\right)(x), \tag{3.48}$$

for μ-almost every $x \in G$.

Using Equations (3.47) and (3.48) above, we deduce that the two-sided Amalgam space is closed under convolution.

Lemma 3.2.10. Let $f \in W^R_Q\left(L^\infty,\, L^1_w(G)\right)$ and $g \in W^L_Q\left(L^\infty,\, L^1_w(G)\right)$. Then

$$\|f * g\|_{W_Q\left(L^\infty,\, L^1_w(G)\right)} \leq \|f\|_{W^R_Q\left(L^\infty,\, L^1_w(G)\right)} \|g\|_{W^L_Q\left(L^\infty,\, L^1_w(G)\right)}\,. \tag{3.49}$$

Moreover, we have the following embedding

$$W_Q\left(L^\infty,\, L^1_w(G)\right) * W_Q\left(L^\infty,\, L^1_w(G)\right) \hookrightarrow W_Q\left(L^\infty,\, L^1_w(G)\right). \tag{3.50}$$

Proof. Note that

$$\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f * g)(x) = \operatorname*{ess\,sup}_{z \in Q} \left| \int_G f(y) g(y^{-1} z x) \mathbf{d}\mu(y) \right| \leq \operatorname*{ess\,sup}_{z \in Q} \int_G |f|(y) \, |g| (y^{-1} z x) \mathbf{d}\mu(y)$$

and hence

$$\mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f * g)(x) \leq \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(|f| * |g|)(x). \tag{3.51}$$

Since $L^1(G)$ is a solid function space we have

$$\left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f * g) \right\|_{L^1_w(G)} \leq \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(|f| * |g|) \right\|_{L^1_w(G)}. \tag{3.52}$$

Combining the previous and Equations (3.47) and (3.48), we have

$$\begin{aligned} \|f * g\|_{W_Q(L^\infty,\, L^1_w(G))} &= \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f * g) \right\|_{L^1_w(G)} \leq \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(|f| * |g|) \right\|_{L^1_w(G)} & (3.53) \\ &= \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}(\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} |f| * |g|) \right\|_{L^1_w(G)} = \left\| (\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} |f| * \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} |g|) \right\|_{L^1_w(G)} \\ &\leq \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f \right\|_{L^1_w(G)} \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} g \right\|_{L^1_w(G)} \\ &= \|f\|_{W_Q^R(L^\infty,\, L^1_w(G))} \|g\|_{W_Q^L(L^\infty,\, L^1_w(G))}, & (3.54) \end{aligned}$$

using Young's inequality, i.e. $\|F * G\|_{L^p(G)} \leq \|F\|_{L^p(G)} \|G\|_{L^1(G)}$, for all $F \in L^p(G)$, $G \in L^1(G)$ and $p \in [1, \infty]$. We conclude that $W_Q^R(L^\infty,\, L^1_w(G)) * W_Q^L(L^\infty,\, L^1_w(G))$ is embedded in $W_Q(L^\infty,\, L^1_w(G))$,

$$W_Q^R(L^\infty,\, L^1_w(G)) * W_Q^L(L^\infty,\, L^1_w(G)) \hookrightarrow W_Q(L^\infty,\, L^1_w(G)). \tag{3.55}$$

By the definition of the two-sided Amalgam space $W_Q(L^\infty,\, L^1_w(G))$ we have that $W_Q(L^\infty,\, L^1_w(G))$ is embedded in the left and right Amalgam spaces. Thus, using Equation (3.55) we obtain

$$W_Q(L^\infty,\, L^1_w(G)) * W_Q(L^\infty,\, L^1_w(G)) \hookrightarrow W_Q(L^\infty,\, L^1_w(G)). \tag{3.56}$$

□

The following result shows that the two-sided Amalgam space $W_Q(L^\infty, L^1_w(G))$ is a Banach space. To prove the completeness of $W_Q(L^\infty, L^1_w(G))$, we use the completeness of the Lebesgue spaces $L^1_w(G)$. Here we only prove the completeness of the Amalgam space with global component $L^1_w(G)$, however the space $W_Q(L^\infty, Y)$ for more general function spaces Y can also be shown to be complete [53, 66].

Theorem 3.2.11. Let G be a locally compact group with Haar measure μ and $w : G \to [1, \infty)$ be a measurable, submultiplicative weight on G. Then $W_Q(L^\infty, L^1_w(G))$ is a Banach space.

Proof. Let $f,\, g \in W_Q(L^\infty, L^1_w(G))$ we obtain

$$\begin{aligned} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f + g)(x) &= \operatorname*{ess\,sup}_{y, y' \in Q} |(f + g)(y x y')| \leq \operatorname*{ess\,sup}_{y, y' \in Q} |f(y x y')| + \operatorname*{ess\,sup}_{y, y' \in Q} |g(y x y')| \\ &\leq \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f(x) + \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} g(x). \end{aligned}$$

Then using the triangle inequality and that $L^1_w(G)$ is solid we have

$$\begin{aligned} \|f + g\|_{W_Q(L^\infty, L^1_w(G))} &= \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f + g) \right\|_{L^1_w(G)} \leq \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f \right\|_{L^1_w(G)} + \left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} g \right\|_{L^1_w(G)} \\ &\leq \|f\|_{W_Q(L^\infty, L^1_w(G))} + \|g\|_{W_Q(L^\infty, L^1_w(G))}. \end{aligned}$$

Moreover, since $\mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}(cf) = |c| \, \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}(f)$ and $\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(cf) = |c| \, \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(f)$ for each $c \in \mathbb{C}$ and each $f \in W_Q(L^\infty, L^1_w(G))$, we deduce the absolute homogeneity of $\|\cdot\|_{W_Q(L^\infty, L^1_w(G))}$ from the absolute homogeneity of $\|\cdot\|_{L^1_w(G)}$. By the definition of the maximal functions we have that $\|f\|_{W_Q(L^\infty, L^1_w(G))} \geq 0$ for each $f \in W_Q(L^\infty, L^1_w(G))$. Now, if $\|f\|_{W_Q(L^\infty, L^1_w(G))} = 0$ then we have $\left\| \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f \right\|_{L^1_w(G)} = 0$ and hence $\mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} f = 0$, $\mu-$almost everywhere in G. Using Lemma 3.2.2 we deduce that $f = 0$ $\mu-$almost everywhere in G. Thus, we conclude that $\left(W_Q(L^\infty, L^1_w(G)),\, \|\cdot\|_{W_Q(L^\infty, L^1_w(G))} \right)$ is a normed space.

To show that $\left(W_Q(L^\infty, L^1_w(G)),\ \|\cdot\|_{W_Q(L^\infty, L^1_w(G))}\right)$ is complete, it suffices to show that if $(f_n)_{n\in\mathbb{N}} \subseteq W_Q(L^\infty, L^1_w(G))$ with $\sum_{n\in\mathbb{N}} \|f_n\|_{W_Q(L^\infty, L^1_w(G))} < \infty$, then

$$\sum_{n\in\mathbb{N}} f_n \in W_Q(L^\infty, L^1_w(G)).$$

Using Lemma 3.2.2, for each $n \in \mathbb{N}$ we have $|f_n| \leq \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n$ and hence, since $L^1_w(G)$ is a solid function space (see the inequalities 3.37), we obtain

$$\sum_{n\in\mathbb{N}} \|f_n\|_{L^1_w(G)} \leq \sum_{n\in\mathbb{N}} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n\right\|_{L^1_w(G)} = \sum_{n\in\mathbb{N}} \|f_n\|_{W_Q(L^\infty, L^1_w(G))} < \infty.$$

By the completeness of $L^1_w(G)$ we define $f := \sum_{n\in\mathbb{N}} f_n \in L^1_w(G)$. For $g_n = \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n$ we have

$$\sum_{n\in\mathbb{N}} \|g_n\|_{L^1_w(G)} = \sum_{n\in\mathbb{N}} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n\right\|_{L^1_w(G)} = \sum_{n\in\mathbb{N}} \|f_n\|_{W_Q(L^\infty, L^1_w(G))} < \infty$$

and using once again the completeness of $L^1_w(G)$ we deduce that $\sum_{n\in\mathbb{N}} g_n \in L^1_w(G)$, or equivalently

$$\left\|\sum_{n\in\mathbb{N}} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n\right\|_{L^1_w(G)} < \infty.$$

For $\mu-$ almost every $x \in G$

$$\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f(x) = \|f\chi_{QxQ}\|_{L^\infty(G)} \leq \sum_{n\in\mathbb{N}} \|f_n\chi_{QxQ}\|_{L^\infty(G)} = \sum_{n\in\mathbb{N}} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f_n(x).$$

Since $L^1_w(G)$ is solid, from the previous we deduce that $\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} f \in L^1_w(G)$, which completes the proof. □

It should be noted that the two-sided Amalgam space $W_Q(L^\infty, L^1_w(G))$ is independent of the choice of open, relatively compact, symmetric unit neighbourhood Q, with equivalent norms [53, 66].

We now define the subspace of continuous functions in the Amalgam space, which will be shown to be closed in the two-sided Amalgam space. Furthermore, this subspace will be used in the next section for the definition of convolution-dominated matrices in locally compact groups. Initially, we show that the subspace of continuous Amalgam functions is closed, by following the proof given by Voigtlaender [66].

Lemma 3.2.12. The subspace

$$C(G) \cap W_Q(L^\infty, L^1_w(G)) \tag{3.57}$$

is a closed subspace of $W_Q(L^\infty, L^1_w(G))$.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset C(G) \cap W_Q(L^\infty, L^1_w(G))$ be a sequence such that $f_n \xrightarrow{n\to\infty} f \in W_Q(L^\infty, L^1_w(G))$ with convergence in $W_Q\,(L^\infty,\, L^1_w(G))$. Now, suppose that $V_0 \subseteq G$ is a compact set, then choosing a symmetric open unit neighbourhood V, such that $VV \subseteq Q$, and from the compactness of V_0 we have that there exist $x_1, \ldots, x_N \in G$ such that

$$V_0 \subseteq \bigcup_{i=1}^{N} x_i V.$$

Then, for each $i = 1, \ldots, N$, by averaging over V (see Equation (3.39) in the proof of Lemma 3.2.6) we obtain

$$\|f\chi_{x_iV}\|_{L^\infty(G)} \leq \frac{1}{\mu(V)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1(G)}$$

and hence

$$\|f\chi_{V_0}\|_{L^\infty(G)} \leq \sum_{i=1}^{N} \|f\chi_{x_iV}\|_{L^\infty(G)} \leq \sum_{i=1}^{N} \frac{1}{\mu(V)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} f\right\|_{L^1(G)}.$$

Thus,

$$\|f\chi_{V_0}\|_{L^\infty(G)} \leq C_{V_0,Q}\, \|f\|_{W_Q\,(L^\infty,\, L^1_w(G))}\,, \tag{3.58}$$

where $C_{V_0,Q} > 0$. For a relatively compact open set $V_1 \subseteq G$ we have

$$\|(f_n - f_m)\chi_{V_1}\|_{L^\infty(G)} \leq \left\|(f_n - f_m)\chi_{\overline{V_1}}\right\|_{L^\infty(G)} \leq C_{\overline{V_1},Q}\, \|f_n - f_m\|_{W_Q\,(L^\infty,\, L^1_w(G))} \xrightarrow{n,m\to\infty} 0$$

using the estimate in Equation (3.58) and the convergence of the sequence $(f_n)_{n\in\mathbb{N}}$. From the completeness of $C(V_1)$ we have that there exists $g_{V_1} \in C(V_1)$ such that

$$\|(f_n - g_{V_1})\chi_{V_1}\|_{L^\infty(G)} \xrightarrow{n\to\infty} 0.$$

Similarly, for a relatively compact open set $V_2 \subseteq G$ we define $g_{V_2} \in C(V_2)$ such that

$$\|(f_n - g_{V_2})\chi_{V_2}\|_{L^\infty(G)} \xrightarrow{n\to\infty} 0.$$

Thus, for two relatively compact open sets $V_1, V_2 \subseteq G$ we have

$$\begin{aligned} \|(g_{V_2} - g_{V_1})\chi_{V_1\cap V_2}\|_{L^\infty(G)} &\leq \|(f_n - g_{V_1})\chi_{V_1\cap V_2}\|_{L^\infty(G)} + \|(f_n - g_{V_2})\chi_{V_1\cap V_2}\|_{L^\infty(G)} \\ &\leq \|(f_n - g_{V_1})\chi_{V_1}\|_{L^\infty(G)} + \|(f_n - g_{V_2})\chi_{V_2}\|_{L^\infty(G)} \xrightarrow{n\to\infty} 0, \end{aligned} \tag{3.59}$$

hence $g_{V_2} = g_{V_1}$ on $V_1 \cap V_2$. Since G is compactly generated, we have $G = \bigcup_{m=0}^{\infty} U^m$ where $U \subseteq G$ is a relatively compact symmetric unit neighbourhood and we define $h_m = g_{U^m} \in C(G)$ (similarly to g_{V_1}) such that

$$\|(f_n - h_m)\chi_{U^m}\|_{L^\infty(G)} \xrightarrow{n\to\infty} 0.$$

Then we define

$$g(x) = h_m(x), \text{ for each } x \in U^m \tag{3.60}$$

and by Equation (3.59) we have that $g \in C(G)$. Let $K \subseteq G$ be a compact set. Then since G is compactly generated there exists $m_0 := m_0(K) > 0$ such that $K \subseteq U^{m_0}$ and hence

$$\begin{aligned} \|(f_n - g\chi_K)\chi_K\|_{L^\infty(G)} &\leq \|(f_n - g\chi_{U^{m_0}})\chi_{U^{m_0}}\|_{L^\infty(G)} \\ &\leq \|(f_n - h_{m_0})\chi_{U^{m_0}}\|_{L^\infty(G)} \xrightarrow{n\to\infty} 0. \end{aligned}$$

Therefore, f_n converges pointwise to $g \in C(G)$. We conclude that $g \in C(G)$ is a continuous representative of $f \in W_Q(L^\infty, L^1_w(G))$ and hence f_n converges to $g \in C(G) \cap W_Q(L^\infty, L^1_w(G))$ in the amalgam norm. Thus, $C(G) \cap W_Q(L^\infty, L^1_w(G))$ is a closed subspace of $W_Q(L^\infty, L^1_w(G))$. ☐

Since from the previous $C(G)\cap W_Q(L^\infty, L^1_w(G))$ is a closed subspace of $W_Q(L^\infty, L^1_w(G))$ and from Theorem 3.2.11 we have that $W_Q(L^\infty, L^1_w(G))$ is a Banach space, we deduce that $C(G)\cap W_Q(L^\infty, L^1_w(G))$ is also a Banach space.

We denote the subspace of continuous Amaglam functions by

$$W_Q\,(C,\, L^1_w(G)) \;= C(G) \cap W_Q(L^\infty, L^1_w(G))$$

and we interpret this space as the two-sided Amalgam space with local component $C(G)$ and and global component $L^1_w(G)$. Moreover, since the Amalgam space is independent of the choice of Q, for a measurable, submultiplicative weight w on G we use the notation

$$W^L_w(G) = C(G) \cap W^L_Q(L^\infty, L^1_w(G)),$$

$$W^R_w(G) = C(G) \cap W^R_Q(L^\infty, L^1_w(G)),$$

and

$$W_w(G) = C(G) \cap W_Q(L^\infty, L^1_w(G))$$

throughout the rest of the paper.

3.3. Convolution-dominated Matrices

With the preparation done over the previous sections, we can now define the convolution-dominated matrices indexed by relatively separated sets in locally compact groups. Throughout this section we make the same assumptions as the previous section. Precisely, we assume that G is a compactly generated with generating neighbourhood $U \subseteq G$ and (left) Haar measure μ, equipped with the word metric d.

Definition 3.3.1. Let $\Lambda, \Gamma \subseteq G$ be two relatively separated sets and let $w : G \to [1, \infty)$ be a measurable, submultiplicative weight on G. We say that a matrix $A = (A(\lambda,\gamma))_{\lambda\in\Lambda,\,\gamma\in\Gamma} \in \mathbb{C}^{\Lambda\times\Gamma}$ is w-enveloped by a non-negative function $\Theta \in W_Q\left(C,\, L^1_w(G)\right)$ and write $A \prec \Theta$, if

$$|A(\lambda,\gamma)| \le \Theta(\lambda^{-1}\gamma), \tag{3.61}$$

for all $\lambda \in \Lambda,\ \gamma \in \Gamma$. We define the space of *convolution-dominated matrices* in $\mathbb{C}^{\Lambda\times\Gamma}$ by

$$CD_w(\Gamma,\Lambda) := \left\{A \in \mathbb{C}^{\Lambda\times\Gamma} \,:\, \exists\,\Theta \in W_Q\left(C,\, L^1_w(G)\right) \text{ such that } A \prec \Theta\right\} \tag{3.62}$$

and the norm

$$\|A\|_{CD_w(\Gamma,\Lambda)} := \inf_{\Theta}\left\{\|\Theta\|_{W_Q\left(L^\infty,\, L^1_w(G)\right)} \,:\, A \prec \Theta \in W_Q\left(C,\, L^1_w(G)\right)\right\}. \tag{3.63}$$

We refer to the norm above as the *convolution-dominated norm.* Furhermore, when $\Lambda = \Gamma$ we denote $CD_w(\Lambda) = CD_w(\Lambda,\Lambda)$.

For a discrete group G, using (3.46), we have that the convolution-dominated matrices on a discrete group are all the matrices $A = (A(\lambda,\gamma))_{\lambda,\gamma\in G} \in \mathbb{C}^{G\times G}$ such that $|A(\lambda,\gamma)| \le \Theta(\lambda^{-1}\gamma)$, for some $\Theta \in \ell^1_w(G)$. This recovers the definition of convolution-dominated matrices in [19, 63].

The following Proposition proves that $\left(CD_w(\Gamma,\Lambda), \|\cdot\|_{CD_w(\Gamma,\Lambda)}\right)$ is complete, but also it is embedded in the algebra of operators $\mathscr{B}(\ell^p(\Gamma),\ell^p(\Lambda))$, for each $p \in [1,\infty]$. The previous embedding follows from the fact that the convolution-dominated matrices $CD_w(\Gamma,\Lambda)$ have also a finite Schur norm as defined in Equation (2.6). For the aforementioned results we include a proof given in [54].

Proposition 3.3.2. Let $\Lambda, \Gamma, K \subseteq G$ be relatively separated sets in G and let $w : G \to [1,\infty)$ be a measurable, submultiplicative weight on G. Then

1. $\left(CD_w(\Gamma,\Lambda), \|\cdot\|_{CD_w(\Gamma,\Lambda)}\right)$ is a Banach space,
2. for all $M \in CD_w(\Gamma,K)$ and $N \in CD_w(\Lambda,\Gamma)$, we have

$$\|MN\|_{CD_w(\Lambda,K)} \le \frac{\mathrm{Rel}\,(\Gamma)}{\mu(Q)}\,\|M\|_{CD_w(\Gamma,K)}\,\|N\|_{CD_w(\Lambda,\Gamma)}\,, \tag{3.64}$$

3. for all $p \in [1,\infty]$, $CD_w(\Gamma,\Lambda)$ is embedded in $\mathscr{B}(\ell^p(\Gamma),\ell^p(\Lambda))$, with

$$\|A\|_{\mathscr{B}(\ell^p(\Gamma),\ell^p(\Lambda))} \le \frac{\max\{\mathrm{Rel}\,(\Lambda)\,\mathrm{Rel}\,(\Gamma)\}}{\mu(Q)}\,\|A\|_{CD_w(\Gamma,\Lambda)}\,, \tag{3.65}$$

for all $A \in CD_w(\Gamma,\Lambda)$.

Proof. 1. The triangle inequality and the absolute homogeneity of the norm $\|\cdot\|_{CD_w(\Gamma,\Lambda)}$ can be shown using Theorem 3.2.11 and the properties of the norm $\|\cdot\|_{W_Q(L^\infty,L^1_w(G))}$. Similarly, we have that $\|A\|_{CD_w(\Gamma,\Lambda)} \ge 0$ for each $A \in CD_w(\Gamma,\Lambda)$. Moreover, for $A \in CD_w(\Gamma,\Lambda)$ using Lemma 3.2.7 we have for each envelope Θ of A and each $\lambda\in\Lambda, \gamma\in\Gamma$

$$|A(\lambda,\gamma)| \le \Theta(\lambda^{-1}\gamma) \le \|\Theta\|_{W_Q(L^\infty,L^1_w(G))}\,. \tag{3.66}$$

Taking supremum over all envelopes of A we have

$$|A(\lambda,\gamma)| \le \|A\|_{CD_w(\Gamma,\Lambda)}\,, \tag{3.67}$$

hence if $\|A\|_{CD_w(\Gamma,\Lambda)} = 0$ then $A = 0$. Thus, $CD_w(\Gamma,\Lambda)$ is a normed vector space.

To show that $CD_w(\Gamma,\Lambda)$ is complete, it suffices to prove that every absolute convergence series is convergent in $CD_w(\Gamma,\Lambda)$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $CD_w(\Gamma,\Lambda)$, such that

$$\sum_{n\in\mathbb{N}} \|A_n\|_{CD_w(\Gamma,\Lambda)} < \infty.$$

Applying the same arguments we used for Equation (3.67), it can be shown that

$$\sum_n |A_n(\lambda,\gamma)| \le \sum_n \|A_n\|_{CD_w(\Gamma,\Lambda)} < \infty$$

for each $\lambda\in\Lambda$ and $\gamma\in\Gamma$. By the completeness of $\mathbb{C}$ for each $\lambda\in\Lambda$ and $\gamma\in\Gamma$ we define

$$A(\lambda,\gamma) := \sum_{n\in\mathbb{N}} A_n(\lambda,\gamma)\in\mathbb{C}$$

and $A := (A(\lambda,\gamma))_{\lambda\in\Lambda,\,\gamma\in\Gamma}\in\mathbb{C}^{\Lambda\times\Gamma}$. We choose an envelope Θ_n for each A_n such that

$$\|\Theta_n\|_{W_Q(L^\infty,L^1_w(G))} \le 2\,\|A_n\|_{CD_w(\Gamma,\Lambda)}\,.$$

Using that

$$\sum_{n=m+1}^{\infty} \|\Theta_n\|_{W_Q(L^\infty,L^1_w(G))} \le 2\sum_{n\in\mathbb{N}} \|A_n\|_{CD_w(\Gamma,\Lambda)} < \infty$$

and that $W_Q(C,L^1_w(G))$ is complete, we deduce that $\Phi_m := \sum_{n=m+1}^{\infty}\Theta_n \in W_Q(L^\infty,L^1_w(G))$ with

$$\|\Phi_m\|_{W_Q(L^\infty,L^1_w(G))} \le \sum_{n=m+1}^{\infty} \|\Theta_n\|_{W_Q(L^\infty,L^1_w(G))} < \infty$$

for each $m\in\mathbb{N}$. Then,

$$\begin{aligned}\left|\left(A-\sum_{n=1}^{m}A_n\right)(\lambda,\gamma)\right| &= \left|A(\lambda,\gamma)-\sum_{n=1}^{m}A_n(\lambda,\gamma)\right| \le \sum_{n=m+1}^{\infty}|A_n(\lambda,\gamma)| \\ &\le \sum_{n=m+1}^{\infty}\Theta_n(\lambda^{-1}\gamma) = \Phi_m(\lambda^{-1}\gamma)\end{aligned}$$

for each $\lambda\in\Lambda$ and $\gamma\in\Gamma$. Thus,

$$\left\|A-\sum_{n=1}^{m}A_n\right\|_{CD_w(\Gamma,\Lambda)} \le \|\Phi_m\|_{W_Q(L^\infty,L^1_w(G))} \le \sum_{n=m+1}^{\infty}\|\Theta_n\|_{W_Q(L^\infty,L^1_w(G))} \xrightarrow{m\to\infty} 0$$

and we deduce that $A=\sum_{n\in\mathbb{N}}A_n\in CD_w(\Gamma,\Lambda)$. From the previous, we conclude that $CD_w(\Gamma,\Lambda)$ is a Banach space.

2. Let $M := (M(\kappa,\gamma))_{\kappa\in K,\,\gamma\in\Gamma}\in CD_w(\Gamma,K)$, $N := (N(\gamma,\lambda))_{\gamma\in\Gamma,\,\lambda\in\Lambda}\in CD_w(\Lambda,\Gamma)$ and suppose that $\Theta,\,\Phi\in W_Q(C,L^1_w(G))$ are envelopes of M and N respectively. Then for each $\lambda\in\Lambda,\kappa\in K$ we obtain

$$|(MN)(\kappa,\lambda)| \le \sum_{\gamma\in\Gamma}|M(\kappa,\gamma)|\,|N(\gamma,\lambda)| \le \sum_{\gamma\in\Gamma}\Theta(\kappa^{-1}\gamma)\Phi(\gamma^{-1}\lambda) \le \frac{\mathrm{Rel}_Q(\Gamma)}{\mu(Q)}\left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\,\Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\,\Phi\right)(\kappa^{-1}\lambda),$$

using Equation (3.41). Since from Equation (3.49)

$$\begin{aligned}\left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\,\Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\,\Phi\right\|_{W_Q(L^\infty,L^1_w(G))} &\le \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\,\Theta\right\|_{W^R_Q(L^\infty,L^1_w(G))}\left\|\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\,\Phi\right\|_{W^L_Q(L^\infty,L^1_w(G))} \\ &\le \|\Theta\|_{W_Q(L^\infty,L^1_w(G))}\,\|\Phi\|_{W_Q(L^\infty,L^1_w(G))}\,,\end{aligned}$$

it follows that MN is enveloped by $\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}}\,\Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}}\,\Phi\in W_Q(L^\infty,L^1_w(G))$ and

$$\|MN\|_{CD_w(\Lambda,K)} \le \frac{\mathrm{Rel}_Q(\Gamma)}{\mu(Q)}\,\|\Theta\|_{W_Q(L^\infty,L^1_w(G))}\,\|\Phi\|_{W_Q(L^\infty,L^1_w(G))}\,.$$

Thus, taking infimum over all envelopes Θ of M and Φ of N we deduce

$$\|MN\|_{CD_w(\Lambda,K)} \le \frac{\mathrm{Rel}\,(\Gamma)}{\mu(Q)}\,\|M\|_{CD_w(\Gamma,K)}\,\|N\|_{CD_w(\Lambda,\Gamma)}\,. \tag{3.68}$$

3. Let $A \in CD_w(\Gamma, \Lambda)$ and $\Theta \in W_Q(C, L^1_w(G))$ be an envelope of A. Then, using Lemma 3.2.7 we obtain

$$\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}|A(\lambda,\gamma)| \leq \sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\Theta(\lambda^{-1}\gamma) \leq \frac{\mathrm{Rel}_Q\,(\Gamma)}{\mu(Q)}\,\|\Theta\|_{W^L_Q(L^\infty,L^1_w(G))} \leq \frac{\mathrm{Rel}_Q\,(\Gamma)}{\mu(Q)}\,\|\Theta\|_{W_Q(L^\infty,L^1_w(G))}$$

and

$$\begin{aligned}\sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}|A(\lambda,\gamma)| \leq \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\Theta(\lambda^{-1}\gamma) = \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\Theta^\vee(\gamma^{-1}\lambda) &\leq \frac{\mathrm{Rel}_Q\,(\Lambda)}{\mu(Q)}\,\|\Theta^\vee\|_{W^L_Q(L^\infty,L^1_w(G))}\\ = \frac{\mathrm{Rel}_Q\,(\Lambda)}{\mu(Q)}\,\|\Theta\|_{W^R_Q(L^\infty,L^1_w(G))} &\leq \frac{\mathrm{Rel}_Q\,(\Lambda)}{\mu(Q)}\,\|\Theta\|_{W_Q(L^\infty,L^1_w(G))}\end{aligned}$$

Hence,

$$\begin{aligned}\max\left\{\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}|A(\lambda,\gamma)|\,,\sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}|A(\lambda,\gamma)|\right\} &\leq \max\left\{\frac{\mathrm{Rel}_Q\,(\Lambda)}{\mu(Q)},\frac{\mathrm{Rel}_Q\,(\Gamma)}{\mu(Q)}\right\}\\ &\leq \frac{\max\left\{\mathrm{Rel}_Q\,(\Lambda)\,,\mathrm{Rel}_Q\,(\Gamma)\right\}}{\mu(Q)}\,\|\Theta\|_{W_Q(L^\infty,L^1_w(G))}\end{aligned}$$

and taking infimum over all envelopes $\Theta \in W_Q(L^\infty, L^1_w(G))$ of A we obtain

$$\max\left\{\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}|A(\lambda,\gamma)|\,,\sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}|A(\lambda,\gamma)|\right\} \leq \frac{\max\left\{\mathrm{Rel}_Q\,(\Lambda)\,,\mathrm{Rel}_Q\,(\Gamma)\right\}}{\mu(Q)}\,\|A\|_{CD_w(\Gamma,\Lambda)}\,.$$

Thus, from the previous and Schur's Test, see e.g. [22, Theorem 6.18], we deduce that

$$\|A\|_{\mathscr{B}(\ell^p(\Gamma),\ell^p(\Lambda))} \leq \frac{\max\left\{\mathrm{Rel}_Q\,(\Lambda)\,,\mathrm{Rel}_Q\,(\Gamma)\right\}}{\mu(Q)}\,\|A\|_{CD_w(\Gamma,\Lambda)}\,, \tag{3.69}$$

for each $p \in [1, \infty]$.

□

From Part 2 of Theorem 3.3.2, we deduce that the multiplication in $CD_w(\Lambda)$ is left and right continuous. Then, since $\left(CD_w(\Lambda), \|\cdot\|_{CD_w(\Lambda)}\right)$ is a Banach space, there exists a norm $|||\cdot|||_{CD_w(\Lambda)}$ which makes $\left(CD_w(\Lambda), |||\cdot|||_{CD_w(\Lambda)}\right)$ a Banach algebra, see e.g. [55, Theorem 10.2]. Moreover, $|||\cdot|||_{CD_w(\Lambda)}$ induces the same topology as $\|\cdot\|_{CD_w(\Lambda)}$.

4

Key Lemmas for the Commutator Technique

In this chapter we prove several lemmas that are required in the proof of the result on ℓ^p-stability and then we proceed to prove the result in Section 5.1.

Throughout this chapter, we fix a locally compact group G of polynomial growth generated by the symmetric, relatively compact, unit neighbourhood $U \subset G$ and we assume that the growth rate of the group is given by $C_G^{-1} n^{D_G} \leq \mu(U^n) \leq C_G n^{D_G}$ for each $n \in \mathbb{N}$, where $C_G, D_G > 0$ are constants. Moreover, assume that $\Lambda, \Gamma \subseteq G$ are two relatively separated sets in G. The group G is equipped with the word metric d and the Haar measure μ, which is both left and right invariant (see Lemma 3.1.7). Moreover, we fix an open, symmetric, relatively compact unit neighbourhood Q and without loss of generality we can further assume that $Q \subseteq U$. For each $\alpha \in \mathbb{N}$,

$$\begin{aligned} w_\alpha \ : \ G &\longrightarrow [1, \infty) \\ x &\longmapsto (1 + \mathrm{d}(x, e))^\alpha \end{aligned} \tag{4.1}$$

denotes the measurable, submultiplicative polynomial weight on G.

For the upcoming results, we use the notation $\lesssim$ and $\gtrsim$ for inequalities up to a constant, and $\asymp$ when we have equality up to a constant, that is when $a \lesssim b \lesssim a$ we denote $a \asymp b$. Furthermore, when the previous symbols have a subscript, then the constant of the inequality depends on the subscript, e.g. we denote by $a \lesssim_m b$ the inequality $a \leq Cb$, where $C = C(m) > 0$. Recall that $\mathrm{Rel}_V(\Lambda) = \sup_{x \in G} \#(\Lambda \cap xV)$ is the relatively separation constant of the set $\Lambda \subseteq G$ with respect to $V \subseteq G$ and we denote by $\mathrm{Rel}(\Lambda) = \mathrm{Rel}_Q(\Lambda)$ the relatively separation constant of the set $\Lambda \subseteq G$ with respect to the fixed unit neighbourhood $Q \subseteq \Lambda$.

4.1. Equivalent norm on the sequence space

Initially, we define a sequence of functions on the group that acts as a partition of unity, up to a constant, and then we define an equivalent norm on the sequence space $\ell^q(\Lambda)$ for each $q \in [1, \infty]$, depending on the sequence of functions.

Before stating the next lemma, we recall that for each $N \in \mathbb{N}$ Lemma 3.1.13 defines a relatively separated and countable set X_N and a covering of G. Using an enumeration of the countable set X_N we uniquely determine the elements of X_N by $X_N = \{x_{k,N} \ : \ k \in \mathbb{N}\}$. Using this notation, we have from Lemma 3.1.13 that $\{x_k U^{2N}\}_{x_{k,N} \in X_N}$ is a cover of G.

Lemma 4.1.1. Fix $N \in \mathbb{N}$. Suppose that $X_N \subseteq G$ is the relatively separated set and $\{x_{k,N} U^{2N}\}_{x_{k,N} \in X_N}$ is the cover of G given by Lemma 3.1.13. Then, for each $x_{k,N} \in X_N$ there exists a function $\psi_k^N \ : \ G \longrightarrow [0, \infty)$, such that $0 \leq \psi_k^N \leq 1$, ψ_k^N is supported in $x_{k,N} U^{4N}$, for each $p \in [1, \infty]$ we have

$$1 \leq \left(\sum_{x_{k,N} \in X_N} \left(\psi_k^N\right)^p \right)^{1/p} \leq K C_G^2 5^{D_G}, \tag{4.2}$$

where $K = K(G, U) > 0$ and for each $x, y \in G$ we have

$$\left|\psi_k^N(x) - \psi_k^N(y)\right| \leq C_\psi \min\left\{1, \frac{1}{2N} \mathrm{d}(x, y)\right\}, \tag{4.3}$$

where $C_\psi > 0$ is independent of N. Moreover, for the multiplication operators Ψ_k^N defined for each $x_{k,N} \in X_N$ as follows

$$\begin{aligned}\Psi_k^N \;:\; \ell^p(\Gamma) &\longrightarrow \ell^p(\Gamma)\\ c &\longmapsto \left(\psi_k^N(\gamma)\; c(\gamma)\right)_{\gamma\in\Gamma},\end{aligned}$$

we have for each $p, q \in [1, \infty]$

$$\|c\|_{\ell^q(\Gamma)} \lesssim_N \left\| \left(\|\Psi_k^N c\|_{\ell^p(\Gamma)} \right)_{x_{k,N}\in X_N} \right\|_{\ell^q} \lesssim_N \|c\|_{\ell^q(\Gamma)}, \quad \forall c \in \ell^q(\Gamma), \tag{4.4}$$

with constants that depend on N, but are independent of $p, q \in [1, \infty]$. Moreover, for $p \in [1, \infty]$ we have

$$\|c\|_{\ell^p(\Gamma)} \lesssim \left\| \left(\|\Psi_k^N c\|_{\ell^p(\Gamma)} \right)_{x_{k,N}\in X_N} \right\|_{\ell^p} \lesssim \|c\|_{\ell^p(\Gamma)}, \quad \forall c \in \ell^p(\Gamma), \tag{4.5}$$

with constants independent of N and $p \in [1, \infty]$.

Proof. Fix $\psi \in C_c^\infty([-2, 2])$, such that $0 \le \psi \le 1,\ \psi = 1$ on $[-1, 1]$ and define

$$\begin{aligned}\psi_k^N \;: G &\longrightarrow [0, \infty)\\ x &\mapsto \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right),\end{aligned}$$

for each $x_{k,N} \in X_N$. For each $x \in G$ we have

$$\begin{aligned}\sum_k \psi_k^N(x) &= \sum_k \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right) \le \sum_k \chi_{[-2,2]}\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)\\ &= \sum_k \chi_{[-4N,4N]}(\mathrm{d}(x, x_{k,N})) = \#\{k \;:\; x_{k,N} \in X_N,\, |\,\mathrm{d}(x, x_{k,N})| \le 4N\}\\ &= \#\left\{k \;:\; x_{k,N} \in xU^{4N} \cap X_N\right\} \le \sup_{x\in G} \#\left\{k \;:\; x_{k,N} \in xU^{4N} \cap X_N\right\}\\ &\le K \sup_{x\in G} \#\left\{k \;:\; x_{k,N} \in xU^{2N} \cap X_N\right\},\end{aligned}$$

where $K := K(G, U) > 0$ is the constant given by Lemma 3.1.16. Then, using Lemma 3.1.13 we obtain

$$\sum_k \psi_k^N(x) \le K \sup_{x\in G} \#\left\{k \;:\; x_{k,N} \in xU^{2N} \cap X_N\right\} = K\,\mathrm{Rel}_{U^{2N}}(X_N) \le K C_G^2 5^{D_G}. \tag{4.6}$$

Furthermore, since $\{x_{k,N} U^{2N}\}_{x_{k,N}\in X_N}$ is a cover of G we get

$$\begin{aligned}\sum_k \psi_k^N(x) &= \sum_k \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right) \ge \sum_k \chi_{[-1,1]}\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)\\ &= \#\{k \;:\; |\,\mathrm{d}(x, x_{k,N})| \le 2N\} = \#\left\{k \;:\; x_{k,N} \in xU^{2N} \cap X_N\right\} \ge 1.\end{aligned}$$

Thus for each $x \in G$ we have $1 \le \sum_k \psi_k^N(x) \le K C_G^2 5^{D_G}$. Similarly, it can be shown that for each $x \in G$ we have

$$1 \le \sum_k \chi_{[-1,1]}\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)^p \le \sum_k \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)^p$$

and

$$\left(\sum_k \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)^p\right)^{1/p} \le \sum_k \psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)^p \le \sum_k \chi_{[-2,2]}\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right)^p \le K C_G^2 5^{D_G}.$$

Hence,

$$1 \leq \left(\sum_{x_{k,N} \in X_N} \psi_k^N(x)^p \right)^{1/p} \leq K C_G^2 5^{D_G}. \tag{4.7}$$

By the smoothness of ψ we deduce that ψ is a Lipschitz function, hence for $x, y \in G$ we obtain

$$\begin{aligned}
\left|\psi_k^N(x) - \psi_k^N(y)\right| &= \left|\psi\left(\frac{\mathrm{d}(x, x_{k,N})}{2N}\right) - \psi\left(\frac{\mathrm{d}(y, x_{k,N})}{2N}\right)\right| \\
&\leq Lip(\psi)\left|\frac{1}{2N}\left(d(x, x_{k,N}) - \mathrm{d}(y, x_{k,N})\right)\right| \\
&\leq Lip(\psi)\frac{1}{2N}\left|\mathrm{d}(x, y)\right|,
\end{aligned}$$

where $Lip(\psi) > 0$ is the Lipschitz constant of ψ and

$$|\psi_k^N(x) - \psi_k^N(y)| \leq |\psi_k^N(x)| + |\psi_k^N(y)| \leq 2.$$

Thus

$$\left|\psi_k^N(x) - \psi_k^N(y)\right| \leq \min\left\{2, Lip(\psi)\frac{1}{2N}\,\mathrm{d}(x, y)\right\} \leq C_\psi \min\left\{1, \frac{1}{2N}\,\mathrm{d}(x, y)\right\}, \tag{4.8}$$

where $C_\psi > 0$ depends on the Lipschitz constant of ψ and is independent of N.

Let $\Gamma \subset G$ be a relatively separated subset of G. For each $x_{k,N} \in X_N$ define the multiplication operator Ψ_k^N as follows

$$\begin{aligned}
\Psi_k^N \ &: \ell^p(\Gamma) \to \ell^p(\Gamma) \\
&c \mapsto \left(\psi_k^N(\gamma)c(\gamma)\right)_{\gamma \in \Gamma},
\end{aligned}$$

for $p \in [1, \infty]$. Note that if $q \leq p$, then $\|y\|_{\ell^p} \leq \|y\|_{\ell^q}$. Moreover, in d dimensional spaces we have for each $p, q \in [1, \infty]$

$$\|y\|_{\ell^p} \leq d^{\max(1/p - 1/q, 0)} \|y\|_{\ell^q}.$$

Since

$$\#\,\mathrm{supp}\left(\Psi_k^N|_\Gamma\right) = \#\left(\Gamma \cap x_{k,N}U^{4N}\right) \leq \sup_{x \in G} \#\left(\Gamma \cap xU^{4N}\right) = \mathrm{Rel}_{U^{4N}}(\Gamma) \lesssim_N \mathrm{Rel}(\Gamma) < \infty,$$

then for $p,\ q \in [1, \infty]$

$$\begin{aligned}
\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)} &\leq \left\|\Psi_k^N c\right\|_{\ell^1(\Gamma)} \leq \#\left(\mathrm{supp}\left(\Psi_k^N|_\Gamma\right)\right) \left\|\Psi_k^N c\right\|_{\ell^\infty(\Gamma)} \\
&\leq \sup_k \{\#(\mathrm{supp}\left(\Psi_k^N|_\Gamma\right))\} \left\|\Psi_k^N c\right\|_{\ell^\infty(\Gamma)} \\
&\lesssim_N \mathrm{Rel}(\Gamma) \left\|\Psi_k^N c\right\|_{\ell^\infty(\Gamma)} \leq \mathrm{Rel}(\Gamma) \left\|\Psi_k^N c\right\|_{\ell^q(\Gamma)}.
\end{aligned} \tag{4.9}$$

Similarly

$$\left\|\Psi_k^N c\right\|_{\ell^q(\Gamma)} \lesssim_N \left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}. \tag{4.10}$$

Hence

$$\left\|\Psi_k^N c\right\|_{\ell^q(\Gamma)} \asymp_N \left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}, \tag{4.11}$$

with constant that depends on N, but independent of $p, q \in [1, \infty]$. For each $p \in [1, \infty]$ we have from Equation (4.7)

$$1 \leq \left(\sum_k \psi_k^N(x)^p \right)^{1/p} \leq K C_G^2 5^{D_G},$$

hence, for $c \in \ell^p(\Gamma)$ we obtain

$$\begin{aligned}\|c\|_{\ell^p(\Gamma)} &= \left(\sum_{\gamma\in\Gamma} |c(\gamma)|^p\right)^{1/p} \leq \left(\sum_{\gamma\in\Gamma}\left(\left(\sum_k \psi_k^N(\gamma)^p\right)^{1/p} |c(\gamma)|\right)^p\right)^{1/p} \\ &= \left(\sum_{\gamma\in\Gamma}\sum_k |\psi_k^N(\gamma)c(\gamma)|^p\right)^{1/p} = \left(\sum_k \left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}^p\right)^{1/p} \\ &= \left\|\left(\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}\right)_k\right\|_{\ell^p},\end{aligned}$$

and

$$\begin{aligned}\left\|\left(\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}\right)_k\right\|_{\ell^p} &= \left(\sum_k \left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}^p\right)^{1/p} = \left(\sum_{\gamma\in\Gamma}\sum_k |\psi_k^N(\gamma)c(\gamma)|^p\right)^{1/p} \\ &= \left(\sum_{\gamma\in\Gamma}\left(\left(\sum_k \psi_k^N(\gamma)^p\right)^{1/p} |c(\gamma)|\right)^p\right)^{1/p} \leq \left(\sum_{\gamma\in\Gamma}(KC_G^2 5^{D_G})^p |c(\gamma)|^p\right)^{1/p} \\ &= KC_G^2 5^{D_G} \|c\|_{\ell^p(\Gamma)}. \end{aligned} \tag{4.12}$$

Therefore,

$$\|c\|_{\ell^p(\Gamma)} \asymp \left\|\left(\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}\right)_k\right\|_{\ell^p} \tag{4.13}$$

with constants independent of $p \in [1,\infty]$ and $N \in \mathbb{N}$. This proves the equivalence in (4.5).

Moreover, combining the previous and Equation (4.11) we conclude that

$$\|c\|_{\ell^q(\Gamma)} \asymp_N \left\|\left(\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)}\right)_k\right\|_{\ell^q} \tag{4.14}$$

with constants that depend on N, but are independent of $p, q \in [1,\infty]$. This proves Equation (4.4).

□

4.2. Estimation of the commutator norms

The proof of the ℓ^p-stability result is inspired by the commutator technique used by Sjöstrand in [60]. Variations of this technique were used by Sun [61] and Gröchenig, Romero, Rottensteiner and Van Velthoven [27] for the ℓ^p-stability result for convolution-dominated matrices indexed by a relatively separated set in the Euclidean space $\mathbb{R}^d$ and in homogeneous groups, respectively, by Shin and Sun [59] for the proof of the inverse-closedness of Banach subalgebras, but also in [14, 58] for the proof of norm-controlled inversion.

Recall from Equation (2.6) that for a matrix $B = (B_{\lambda,\gamma})_{\lambda\in\Lambda,\gamma\in\Gamma} \in \mathbb{C}^{\Lambda\times\Gamma}$ the Schur norm is given by

$$\|B\|_{Schur(\Gamma\to\Lambda)} := \max\left\{\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}|B_{\lambda,\gamma}|\,,\ \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}|B_{\lambda,\gamma}|\right\}.$$

Following this commutator technique, we proceed to estimate the Schur norm of the commutators

$$[A, \Psi_k^N]\Psi_j^N \ :\ \ell^p(\Gamma) \longrightarrow \ell^p(\Lambda),$$

where Ψ_k^N and Ψ_j^N are defined by Lemma 4.1.1 for $x_{k,N}, x_{j,N} \in X_N$. For $N \in \mathbb{N}$, X_N given by Lemma 3.1.13 and $\left(\Psi_k^N\right)_{x_{k,N}\in X_N}$ given by Lemma 4.1.1, we define the matrix $V^N = \left(V^N(k,j)\right)_{x_{k,N},x_{j,N}\in X_N} \in \mathbb{C}^{X_N\times X_N}$ with elements given by

$$V^N(k,j) := \left\|[A, \Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)}, \tag{4.15}$$

for each $x_{k,N}, x_{j,N} \in X_N$.

Lemma 4.2.1. Fix $N, \alpha \in \mathbb{N}$. Suppose that $A \in \mathbb{C}^{\Lambda \times \Gamma}$ and there exists $\Theta \in W_{w_\alpha}(G)$ such that for each $\lambda \in \Lambda$ and $\gamma \in \Gamma$

$$|A(\lambda, \gamma)| \leq \Theta(\lambda^{-1}\gamma). \tag{4.16}$$

Let $\Theta_N(x) := \Theta(x) \min\left\{1, \frac{\mathrm{d}(x,e)}{2N}\right\} \in C(G)$. Then for X_N, as defined in Lemma 3.1.13, and for each $x_{k,N}, x_{j,N} \in X_N$ we have

$$V^N(k,j) \lesssim \begin{cases} \|\Theta_N\|_{W(G)}, & \mathrm{d}(x_{k,N}, x_{j,N}) \leq 10N \\ \min\left\{\|\Theta_N\|_{W(G)}, \left\|\Theta \chi_{U^{5N} x_{k,N}^{-1} x_{j,N} U^{5N}}\right\|_{W(G)}\right\}, & \mathrm{d}(x_{k,N}, x_{j,N}) > 10N, \end{cases} \tag{4.17}$$

where the constant in the above inequality depends on the relatively separated sets Λ and Γ, but is independent of N.

Proof. For $\gamma \in \Gamma$ and $\lambda \in \Lambda$, we have

$$\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma} = \left(A_{\lambda,\gamma}\psi_k^N(\gamma) - \psi_k^N(\lambda)A_{\lambda,\gamma}\right)\psi_j^N(\gamma) = -\left(\psi_k^N(\lambda) - \psi_k^N(\gamma)\right)A_{\lambda,\gamma}\psi_j^N(\gamma). \tag{4.18}$$

Hence, Equation (4.3) and the estimates $\left|\psi_j^N(\gamma)\right| \leq 1$ and $|A_{\lambda,\gamma}| \leq \Theta(\lambda^{-1}\gamma)$ give

$$\begin{aligned} \left|\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma}\right| &= \left|\psi_k^N(\lambda) - \psi_k^N(\gamma)\right| |A_{\lambda,\gamma}| \left|\psi_j^N(\gamma)\right| \\ &\leq \left|\psi_k^N(\lambda) - \psi_k^N(\gamma)\right| \Theta(\lambda^{-1}\gamma) \\ &\leq C_\psi \Theta(\lambda^{-1}\gamma) \min\left\{1, \frac{\mathrm{d}(\lambda,\gamma)}{2N}\right\} \\ &\lesssim \Theta(\lambda^{-1}\gamma) \min\left\{1, \frac{\mathrm{d}(\lambda^{-1}\gamma, e)}{2N}\right\} \\ &= \Theta_N(\lambda^{-1}\gamma), \end{aligned}$$

where the symmetry and the left invariance of the metric are used for the last inequality. Then we have

$$\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\left|\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma}\right| \lesssim \sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\Theta_N(\lambda^{-1}\gamma) \leq \frac{\mathrm{Rel}_Q(\Gamma)}{\mu(Q)}\|\Theta_N\|_{W^L(G)} \leq \frac{\mathrm{Rel}_Q(\Gamma)}{\mu(Q)}\|\Theta_N\|_{W(G)},$$

by Lemma 3.2.7. Similarly, we have

$$\begin{aligned} \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\left|\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma}\right| &\lesssim \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\Theta_N(\lambda^{-1}\gamma) \leq \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\Theta_N^\vee(\gamma^{-1}\lambda) \\ &\leq \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)}\|\Theta_N^\vee\|_{W^L(G)} = \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)}\|\Theta_N\|_{W^R(G)} \\ &\leq \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)}\|\Theta_N\|_{W(G)}. \end{aligned}$$

Thus,

$$\begin{aligned} V^N(k,j) &:= \left\|[A, \Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)} = \max\left\{\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\left|\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma}\right|, \sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}\left|\left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma}\right|\right\} \\ &\lesssim \max\left\{\frac{\mathrm{Rel}_Q(\Gamma)}{\mu(Q)}, \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)}\right\}\|\Theta_N\|_{W(G)}. \end{aligned}$$

Now, we prove refined estimates for $k,\ j$ such that $x_{k,N}, x_{j,N} \in X_N$ and $\mathrm{d}(x_{k,N}, x_{j,N}) > 10N$. Let $\gamma \in G$. Then we have

$$10N < \mathrm{d}(x_{k,N}, x_{j,N}) \leq \mathrm{d}(x_{k,N}, \gamma) + \mathrm{d}(\gamma, x_{j,N})$$

and hence we obtain

$$4N < \mathrm{d}(x_{k,N}, \gamma) \quad \text{or} \quad 4N < \mathrm{d}(x_{j,N}, \gamma),$$

using the symmetry of the metric. This implies that

$$\gamma \notin \operatorname{supp} \psi_k^N \quad \text{or} \quad \gamma \notin \operatorname{supp} \psi_j^N .$$

Thus for each $\gamma \in G$ we have

$$\psi_k^N(\gamma)\psi_j^N(\gamma) = 0,$$

for each k, j such that $x_{k,N}, x_{j,N} \in X_N$ and $\mathrm{d}(x_{k,N}, x_{j,N}) > 10N$. Then, using Equation (4.18) and the previous we obtain for each $x_{k,N}, x_{j,N} \in X_N$ such that $\mathrm{d}(x_{k,N}, x_{j,N}) > 10N$.

$$\begin{aligned}
\sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \left| \left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma} \right| &\leq \sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \psi_k^N(\lambda) \left|A_{\lambda,\gamma}\right| \psi_j^N(\gamma) \leq \sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \psi_k^N(\lambda)\Theta(\lambda^{-1}\gamma)\psi_j^N(\gamma) \\
&\leq \sup_{\lambda\in\Lambda \,\cap\, \operatorname{supp}\psi_k^N} \sum_{\gamma\in\Gamma \,\cap\, \operatorname{supp}\psi_j^N} \psi_k^N(\lambda)\Theta(\lambda^{-1}\gamma)\psi_j^N(\gamma) \\
&\leq \sup_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \sum_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \psi_k^N(\lambda)\Theta(\lambda^{-1}\gamma)\psi_j^N(\gamma).
\end{aligned}$$

Note that if $\lambda \in \Lambda \cap x_{k,N}U^{4N}$ and $\gamma \in \Gamma \cap x_{j,N}U^{4N}$, then

$$\lambda^{-1}\gamma \in \left(x_{k,N}U^{4N}\right)^{-1} x_{j,N}U^{4N} = U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N},$$

by the symmetry of U. Thus,

$$\begin{aligned}
\sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \left| \left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma} \right| &\leq \sup_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \sum_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \psi_k^N(\lambda)\Theta(\lambda^{-1}\gamma)\psi_j^N(\gamma) \\
&\leq \sup_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \sum_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \Theta(\lambda^{-1}\gamma)\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}(\lambda^{-1}\gamma).
\end{aligned}$$

We want to apply Equation (3.40) to the previous, however Equation (3.40) was proved for continuous functions. We can apply Equation (3.40) to a sequence of continuous functions that approximates the function $\Theta\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}$, or to a continuous function that estimates the function $\Theta\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}$. Here we estimate $\Theta\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}$ by a continuous function on G. Let $\eta \in C(\mathbb{R})$, such that $0 \leq \eta \leq 1$ and $\eta(z) = 1$, for $|z| \in [\mathrm{d}(x_{k,N}, x_{j,N}) - 8N, \mathrm{d}(x_{k,N}, x_{j,N}) + 8N]$ and $\eta(z) = 0$, for $|z| \notin [d(x_{k,N}, x_{j,N}) - 8N - 1, d(x_{k,N}, x_{j,N}) + 8N + 1]$. Then $\eta\left(\mathrm{d}(\cdot, e)\right) : G \longrightarrow [0, 1]$ is continuous on G and

$$\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}} \leq \eta\left(\mathrm{d}(\cdot, e)\right) \leq \chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}.$$

Thus, from Equation (3.40) and the previous it follows

$$\begin{aligned}
\sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \left| \left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma} \right| &\leq \sup_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \sum_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \Theta(\lambda^{-1}\gamma)\chi_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}(\lambda^{-1}\gamma) && (4.19) \\
&\leq \sup_{\lambda\in\Lambda} \sum_{\gamma\in\Gamma} \Theta(\lambda^{-1}\gamma)\eta\left(\mathrm{d}(\lambda^{-1}\gamma, e)\right) && (4.20) \\
&\lesssim \left\|\Theta\eta\left(\mathrm{d}(\cdot, e)\right)\right\|_{W(G)} \leq \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)}, && (4.21)
\end{aligned}$$

with constants that depend on the relatively separated sets Λ and Γ, but are independent of N.

Similarly, we obtain

$$\begin{aligned}
\sup_{\gamma\in\Gamma} \sum_{\lambda\in\Lambda} \left| \left([A, \Psi_k^N]\Psi_j^N\right)_{\lambda,\gamma} \right| &\leq \sup_{\gamma\in\Gamma} \sum_{\lambda\in\Lambda} \psi_k^N(\lambda) \left|A_{\lambda,\gamma}\right| \psi_j^N(\gamma) \leq \sup_{\gamma\in\Gamma} \sum_{\lambda\in\Lambda} \psi_k^N(\lambda)\Theta(\lambda^{-1}\gamma)\psi_j^N(\gamma) \\
&\leq \sup_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \sum_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \Theta^{\vee}(\gamma^{-1}\lambda)\chi_{U^{4N}x_{j,N}^{-1}x_{k,N}U^{4N}}(\gamma^{-1}\lambda) \\
&\leq \sup_{\gamma\in\Gamma \,\cap\, x_{j,N}U^{4N}} \sum_{\lambda\in\Lambda \,\cap\, x_{k,N}U^{4N}} \Theta^{\vee}(\gamma^{-1}\lambda)\chi^{\vee}_{U^{4N}x_{k,N}^{-1}x_{j,N}U^{4N}}(\gamma^{-1}\lambda) \\
&\lesssim \left\|(\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}})^{\vee}\right\|_{W(G)} = \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)}.
\end{aligned}$$

Thus for $k,\ j$ such that $\mathrm{d}(x_{k,N}, x_{j,N}) > 10N$ we have

$$V^N(k,j) = \left\|[A, \Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)} \lesssim \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)},$$

with constants that depend on the relatively separated sets Λ and Γ, but are independent of N.

□

4.3. Estimation of the Schur norm

In this section, following the commutator technique from [60] we estimate the Schur norm of the matrix V^N defined by the commutators, see Equation (4.15) for the definition of V^N. We show that the norm of the matrix V^N goes to zero as N approaches infinity. In that way, we can define a new matrix as the Neumann series of powers of the matrix V^{N_1} for some N_1.

In the next lemma, we define a Schur matrix in $\mathscr{S}(X_{N_1})$, which provides an estimate of the norm of a sequence in $\ell^p(\Lambda)$ after the application of the multiplication operator $\Psi_k^{N_1}$ (see Lemma 4.1.1). This is the last step before proving the result on the ℓ^p-stability for the class of convolution-dominated matrices.

Lemma 4.3.1. Fix $\alpha \in \mathbb{N}$, such that $\alpha \geq D_G$. Suppose that $A \in \mathbb{C}^{\Lambda\times\Gamma}$ and that there exists $\Theta \in W_{w_\alpha}(G)$, such that for each $\lambda \in \Lambda$ and $\gamma \in \Gamma$

$$|A(\lambda,\gamma)| \leq \Theta(\lambda^{-1}\gamma). \tag{4.22}$$

Furthermore, assume that A is bounded from below for some $p \in [1,\infty]$, i.e. there exists $C_A > 0$ such that

$$\|c\|_{\ell^p(\Gamma)} \leq C_A \|Ac\|_{\ell^p(\Lambda)}, \quad \forall c \in \ell^p(\Gamma). \tag{4.23}$$

Then there exists $N_1 \in \mathbb{N}$ such that for X_{N_1} given by Lemma 3.1.13 and $\left(\Psi_k^{N_1}\right)_{x_{k,N_1}\in X_{N_1}}$ given by Lemma 4.1.1, there exists $W \in \mathscr{S}(X_{N_1})$ such that for each $x_{k,N_1} \in X_{N_1}$

$$\left\|\Psi_k^{N_1}c\right\|_{\ell^p(\Gamma)} \leq C_A \sum_{x_{j,N_1}\in X_{N_1}} W(k,j)\left\|\Psi_j^{N_1}Ac\right\|_{\ell^p(\Lambda)}, \tag{4.24}$$

for each $c \in \ell^\infty(\Gamma)$.

Proof. Step 1: Commutator technique
Since A is bounded from below for $p \in [1,\infty]$, there exists $C_A > 0$ such that

$$\|c\|_{\ell^p(\Gamma)} \leq C_A \|Ac\|_{\ell^p(\Lambda)}, \quad \forall c \in \ell^p(\Gamma).$$

Without loss of generality we can assume that

$$\|c\|_{\ell^p(\Gamma)} \leq \|Ac\|_{\ell^p(\Lambda)}, \quad \forall c \in \ell^p(\Gamma). \tag{4.25}$$

Then for $c \in \ell^\infty(\Gamma)$ we have

$$\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)} \leq \left\|A\Psi_k^N c\right\|_{\ell^p(\Lambda)} \leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + \left\|[A,\Psi_k^N]c\right\|_{\ell^p(\Lambda)},$$

where the second inequality comes from the triangle inequality and $A\Psi_k^N = [A,\Psi_k^N] + \Psi_k^N A$. Using $\sum_j(\psi_j^N)^2 \asymp 1$ (See Lemma 4.1.1), we have that there exists $C_{2,\psi} > 0$ such that $C_{2,\psi}^{-1} \leq \sum_j(\psi_j^N)^2 \leq C_{2,\psi}$, and hence

$$\begin{aligned}
\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)} &\leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}\left\|\sum_{x_{j,N}\in X_N}[A,\Psi_k^N]\left(\Psi_j^N\right)^2 c\right\|_{\ell^p(\Lambda)} \\
&\leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}\left\|\sum_j[A,\Psi_k^N]\Psi_j^N\right\|_{\ell^p(\Gamma)\to\ell^p(\Lambda)}\left\|\Psi_j^N c\right\|_{\ell^p(\Gamma)} \\
&\leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}\left\|\sum_j[A,\Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)}\left\|\Psi_j^N c\right\|_{\ell^p(\Gamma)} \\
&\leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}\sum_j\left\|[A,\Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)}\left\|\Psi_j^N c\right\|_{\ell^p(\Gamma)}.
\end{aligned}$$

Thus, for each $x_{k,N} \in X_N$

$$\left\|\Psi_k^N c\right\|_{\ell^p(\Gamma)} \leq \left\|\Psi_k^N Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_{x_{j,N} \in X_N} V^N(k,j) \left\|\Psi_j^N c\right\|_{\ell^p(\Gamma)}, \tag{4.26}$$

where we recall that $V^N(k,j) = \left\|[A, \Psi_k^N]\Psi_j^N\right\|_{Schur(\Gamma\to\Lambda)}$, for each $x_{k,N}, x_{j,N} \in X_N$.

Step 2: Estimate $V^N(k,j)$
From Lemma 4.2.1 we have for each $x_{k,N}, x_{j,N} \in X_N$

$$\begin{aligned} V^N(k,j) &\lesssim \begin{cases} \|\Theta_N\|_{W(G)}, & \mathrm{d}(x_{k,N}, x_{j,N}) \leq 10N \\ \min\left\{\|\Theta_N\|_{W(G)}, \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)}\right\}, & \mathrm{d}(x_{k,N}, x_{j,N}) > 10N, \end{cases} \\ &\leq \begin{cases} \|\Theta_N\|_{W(G)}, & \mathrm{d}(x_{k,N}, x_{j,N}) \leq 50N \\ \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)}, & \mathrm{d}(x_{k,N}, x_{j,N}) > 50N, \end{cases} \end{aligned} \tag{4.27}$$

for each $x_{k,N},\ x_{j,N} \in X_N$.
Since

$$\begin{aligned} \left\|V^N\right\|_{Schur(X_N\to X_N)} &= \max\left\{\sup_{x_{k,N}\in X_N} \sum_{x_{j,N}\in X_N} \left|V^N(k,j)\right|, \sup_{x_{j,N}\in X_N} \sum_{x_{k,N}\in X_N} \left|V^N(k,j)\right|\right\} \\ &\leq \sup_{x_{k,N}\in X_N} \sum_{x_{j,N}\in X_N} \left|V^N(k,j)\right| + \sup_{x_{j,N}\in X_N} \sum_{x_{k,N}\in X_N} \left|V^N(k,j)\right| \end{aligned}$$

Then, to estimate the Schur norm of V^N it suffices to estimate the following sums

$$S_1 = \sup_{x_{k,N}\in X_N} \sum_{\substack{x_{j,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \left|V^N(k,j)\right| + \sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \left|V^N(k,j)\right| \tag{4.28}$$

and

$$S_2 = \sup_{x_{k,N}\in X_N} \sum_{\substack{x_{j,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})> 50N}} \left|V^N(k,j)\right| + \sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})> 50N}} \left|V^N(k,j)\right|. \tag{4.29}$$

Step 3: Estimate S_1
Using the estimate from Equation (4.27) we obtain

$$\begin{aligned} S_1 &\leq \sup_{x_{k,N}\in X_N} \sum_{\substack{x_{j,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \left|V^N(k,j)\right| + \sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \left|V^N(k,j)\right| \\ &\lesssim \sup_{x_{k,N}\in X_N} \sum_{\substack{x_{j,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \|\Theta_N\|_{W(G)} + \sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})\leq 50N}} \|\Theta_N\|_{W(G)}. \end{aligned}$$

Then

$$\begin{aligned}
S_1 &\lesssim \|\Theta_N\|_{W(G)} \left(\sup_{x_{k,N} \in X_N} \# \{x_{j,N} \in X_N : \mathrm{d}(x_{k,N}, x_{j,N}) \leq 50N\} \right) \\
&+ \|\Theta_N\|_{W(G)} \left(\sup_{x_{j,N} \in X_N} \# \{x_{k,N} \in X_N : \mathrm{d}(x_{k,N}, x_{j,N}) \leq 50N\} \right) \\
&\leq \|\Theta_N\|_{W(G)} \left(\sup_{x_{k,N} \in X_N} \# \left\{x_{j,N} \in X_N : x_{j,N} \in x_{k,N} U^{50N}\right\} \right) \\
&+ \|\Theta_N\|_{W(G)} \left(\sup_{x_{j,N} \in X_N} \# \left\{x_{k,N} \in X_N : x_{k,N} \in x_{j,N} U^{50N}\right\} \right).
\end{aligned}$$

From Lemma 3.1.13 we have that each $x \in G$ belongs to at most $C_G^2 5^{D_G}$ sets $y_m U^{2N}$ with $y_m \in X_N$, or, equivalently, we have $\mathrm{Rel}_{U^{2N}}(X_N) \leq C_G^2 5^{D_G}$. Hence, by applying multiple times Lemma 3.1.16 we obtain

$$\begin{aligned}
\# \left\{x_{j,N} \in X_N : x_{j,N} \in x_{k,N} U^{50N}\right\} &\leq \# \left\{x_{j,N} \in X_N : x_{j,N} \in x_{k,N} U^{64N}\right\} \leq \mathrm{Rel}_{U^{64N}}(X_N) && (4.30) \\
&\leq K \,\mathrm{Rel}_{U^{32N}}(X_N) \leq K^5 \,\mathrm{Rel}_{U^{2N}}(X_N) \leq K^5 C_G^2 5^{D_G}, && (4.31)
\end{aligned}$$

where $K := K(G, U) > 0$ is the constant given by Lemma 3.1.16. Similarly,

$$\# \left\{x_{k,N} \in X_N : x_{k,N} \in x_{j,N} U^{50N}\right\} \leq K^5 \,\mathrm{Rel}_{U^{2N}}(X_N) \leq K^5 C_G^2 5^{D_G},$$

hence

$$\begin{aligned}
S_1 &\lesssim \|\Theta_N\|_{W(G)} \left(\sup_{x_{k,N} \in X_N} \# \left\{x_{j,N} \in X_N : x_{j,N} \in x_{k,N} U^{50N}\right\} \right) \\
&+ \|\Theta_N\|_{W(G)} \left(\sup_{x_{j,N} \in X_N} \# \left\{x_{k,N} \in X_N : x_{k,N} \in x_{j,N} U^{50N}\right\} \right) \\
&\leq \|\Theta_N\|_{W(G)} \left(\sup_{x_{k,N} \in X_N} K^5 C_G^2 5^{D_G} + \sup_{x_{j,N} \in X_N} K^5 C_G^2 5^{D_G} \right) \\
&\leq 2K^5 C_G^2 5^{D_G} \|\Theta_N\|_{W(G)} \\
&\lesssim \|\Theta_N\|_{W(G)}, && (4.32)
\end{aligned}$$

with constants that depend on G, Λ, Γ and are independent of N.

Step 4: Estimate S_2

Using the estimate from Equation (4.27) we obtain

$$\begin{aligned}
\sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N}, x_{j,N}) > 50N}} \left|V^N(k,j)\right| &\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N}, x_{j,N}) > 50N}} \left\| \Theta \chi_{U^{5N} x_{k,N}^{-1} x_{j,N} U^{5N}} \right\|_{W(G)} \\
&\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N}, x_{j,N}) > 50N}} \int_G \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} (\Theta \chi_{U^{5N} x_{k,N}^{-1} x_{j,N} U^{5N}})(x) \,\mathbf{d}\mu(x) \\
&\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N}, x_{j,N}) > 50N}} \int_G \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} (\Theta)(x) \,\mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} (\chi_{U^{5N} x_{k,N}^{-1} x_{j,N} U^{5N}})(x) \,\mathbf{d}\mu(x) \\
&\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N}, x_{j,N}) > 50N}} \int_G \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}} (\Theta)(x) \chi_{U^{6N} x_{k,N}^{-1} x_{j,N} U^{6N}}(x) \,\mathbf{d}\mu(x),
\end{aligned}$$

where for the last inequality we have used that if $x \notin U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}$, then $yxy' \notin U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}$ for each $y, y' \in Q$. Moreover, observe that for fixed $x \in G$ and for each $x_{k,N} \in X_N$

$$\#\left\{x_{j,N} \in X_N \;:\; x \in U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}\right\} = \#\left\{x_{j,N} \in X_N \;:\; x_{j,N} \in x_{k,N}U^{6N}xU^{6N}\right\},$$

by the symmetry of U. For $z \in U^{6N}xU^{6N}$ using the triangle inequality of the metric we have

$$\mathrm{d}(z,e) \leq 6N + \mathrm{d}(x,e) + 6N = \mathrm{d}(x,e) + 12N,$$

or, equivalently, by the definition of the word metric $z \in U^{\mathrm{d}(x,e)+12N}$. Hence from Lemma 3.1.14 for $\rho = 1$ we obtain

$$\begin{aligned}
\#\left\{x_{j,N} \in X_N \;:\; x \in U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}\right\} &= \#\left\{x_{j,N} \in X_N \;:\; x_{j,N} \in x_{k,N}U^{\mathrm{d}(x,e)+12N}\right\} \\
&\leq \#\left(X_N \cap x_{k,N}U^{\mathrm{d}(x,e)+12N}\right) \\
&\lesssim \mu\left(x_{k,N}U^{\mathrm{d}(x,e)+12N+1}\right) \\
&= \mu\left(U^{\mathrm{d}(x,e)+12N+1}\right) \\
&\leq C_G(\mathrm{d}(x,e)+12N+1)^{D_G},
\end{aligned} \tag{4.33}$$

where for the last inequality we have used the polynomial growth. Note that the constant in (4.33) is given by Lemma 3.1.14, hence it depends on $\mathrm{Rel}_U(X_N)$ and $\rho = 1$. Since $\mathrm{Rel}_U(X_N) \leq \mathrm{Rel}_{U^{2N}}(X_N) \leq C_G^2 5^{D_G}$ from Lemma 3.1.12, we obtain that the implicit constant above is independent of N. From the previous we obtain for fixed $x_{k,N} \in X_N$

$$\sum_{x_{j,N} \in X_N} \chi_{U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}}(x) = \#\left\{x_{j,N} \in X_N \;:\; x \in U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}\right\} \lesssim C_G(\mathrm{d}(x,e)+12N+1)^{D_G},$$

for each $x \in G$, with constant independent of N. Moreover, for $x \in U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}$ and $\mathrm{d}(x_{k,N}, x_{j,N}) > 50N$ we have from the inverse triangle inequality of the metric that $\mathrm{d}(x,e) \geq 37N$ and hence $x \in G \backslash U^{36N}$. Combining the previous and using the monotone convergence theorem, it follows

$$\begin{aligned}
\sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left|V^N(k,j)\right| &\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left\|\Theta\chi_{U^{5N}x_{k,N}^{-1}x_{j,N}U^{5N}}\right\|_{W(G)} \\
&\leq \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \int_G \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x)\chi_{U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}}(x)\mathbf{d}\mu(x) \\
&= \sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \int_{G\backslash U^{36N}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x)\chi_{U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}}(x)\mathbf{d}\mu(x) \\
&= \sup_{x_{k,N} \in X_N} \int_{G\backslash U^{36N}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x) \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \chi_{U^{6N}x_{k,N}^{-1}x_{j,N}U^{6N}}(x)\mathbf{d}\mu(x) \\
&\lesssim \sup_{x_{k,N} \in X_N} \int_{G\backslash U^{36N}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x)(\mathrm{d}(x,e)+12N+1)^{D_G}\mathbf{d}\mu(x) \\
&\leq \sup_{x_{k,N} \in X_N} \int_{G\backslash U^{36N}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x)2^{D_G}(\mathrm{d}(x,e)+1)^{D_G}\mathbf{d}\mu(x),
\end{aligned}$$

where for the last inequality we have used that $12N \leq 36N \leq \mathrm{d}(x,e)$. Thus, for the polynomial weight $w_{D_G} : x \mapsto (\mathrm{d}(x,e)+1)^{D_G}$ we obtain

$$\sup_{x_{k,N} \in X_N} \sum_{\substack{x_{j,N} \in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left|V^N(k,j)\right| \lesssim \int_{G\backslash U^{36N}} \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}}\,\mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(\Theta)(x)w_{D_G}(x)\mathbf{d}\mu(x) = \left\|\Theta\chi_{G\backslash U^{36N}}\right\|_{W_{w_{D_G}}(G)}.$$

Similarly, we obtain

$$\sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left|V^N(k,j)\right| \lesssim \left\|\Theta\chi_{G\setminus U^{36N}}\right\|_{W_{w_{D_G}}(G)}$$

Thus,

$$\begin{aligned} S_2 &= \sup_{x_{k,N}\in X_N} \sum_{\substack{x_{j,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left|V^N(k,j)\right| + \sup_{x_{j,N}\in X_N} \sum_{\substack{x_{k,N}\in X_N \\ \mathrm{d}(x_{k,N},x_{j,N})>50N}} \left|V^N(k,j)\right| \\ &\lesssim \left\|\Theta\chi_{G\setminus U^{36N}}\right\|_{W_{w_{D_G}}(G)}, \end{aligned} \tag{4.34}$$

with constant that depends on G, Λ, Γ and is independent of N.

Step 5: Estimate Schur norm of V^N
Combining the estimates given by Equations (4.32) and (4.34) it follows

$$\left\|V^N\right\|_{Schur(X_N\to X_N)} \le S_1 + S_2 \lesssim \|\Theta_N\|_{W(G)} + \left\|\Theta\chi_{G\setminus U^{36N}}\right\|_{W_{w_{D_G}}(G)},$$

with constants independent of N. Using the Dominated Convergence Theorem, since $\Theta \in W_{w_\alpha}(G)$, with $\alpha \ge D_G$, we obtain

$$\left\|V^N\right\|_{Schur(X_N\to X_N)} \lesssim \|\Theta_N\|_{W(G)} + \left\|\Theta\chi_{G\setminus U^{36N}}\right\|_{W_{w_{D_G}}(G)} \xrightarrow{N\to\infty} 0.$$

Thus, there exists $N_1 \in \mathbb{N}$ such that

$$\left\|V^{N_1}\right\|_{Schur(X_{N_1}\to X_{N_1})} \le \frac{1}{2C_{2,\psi}}, \tag{4.35}$$

for $C_{2,\psi} > 0$ from Equation (4.26). Recall from Equation (2.9) that for matrices $A, B \in \mathscr{S}(X_{N_1})$ we have

$$\|AB\|_{Schur(X_{N_1}\to X_{N_1})} \le \|A\|_{Schur(X_{N_1}\to X_{N_1})} \|B\|_{Schur(X_{N_1}\to X_{N_1})}. \tag{4.36}$$

Hence we can define the Neumann series

$$W := \mathbb{I} + \sum_{m=1}^{\infty} (C_{2,\psi}V^{N_1})^m \in \mathbb{C}^{X_{N_1}\times X_{N_1}}. \tag{4.37}$$

From Equation (4.36) we have

$$\left\|(V^{N_1})^m\right\|_{Schur(X_{N_1}\to X_{N_1})} \le \left\|V^{N_1}\right\|^m_{Schur(X_{N_1}\to X_{N_1})} \le \frac{1}{(2C_{2,\psi})^m} \tag{4.38}$$

and it follows that

$$\|W\|_{Schur(X_{N_1}\to X_{N_1})} \le \sum_{m=0}^{\infty} \left\|C_{2,\psi}V^{N_1}\right\|^m_{Schur(X_{N_1}\to X_{N_1})} \le \sum_{m=0}^{\infty} C^m_{2,\psi}\frac{1}{(2C_{2,\psi})^m} = 2, \tag{4.39}$$

hence $W \in \mathscr{S}(X_{N_1})$.

Now recall Equation (4.26),

$$\left\|\Psi_k^{N_1}c\right\|_{\ell^p(\Gamma)} \le \left\|\Psi_k^{N_1}Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}\sum_j V^{N_1}(k,j)\left\|\Psi_j^{N_1}c\right\|_{\ell^p(\Gamma)}.$$

Applying Equation (4.26) twice we obtain

$$\begin{aligned}
\left\|\Psi_k^{N_1} c\right\|_{\ell^p(\Gamma)} &\leq \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_j V^{N_1}(k,j) \left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \\
&\leq \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_j V^{N_1}(k,j) \left(\left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_i V^{N_1}(j,i) \left\|\Psi_i^{N_1} c\right\|_{\ell^p(\Gamma)} \right) \\
&\leq \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_j V^{N_1}(k,j) \left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)} \\
&\quad + C_{2,\psi} \sum_j V^{N_1}(k,j) C_{2,\psi} \sum_i V^{N_1}(j,i) \left\|\Psi_i^{N_1} c\right\|_{\ell^p(\Gamma)} \\
&= \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_j V^{N_1}(k,j) \left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)} \\
&\quad + C_{2,\psi}^2 \sum_i \sum_j V^{N_1}(k,j) V^{N_1}(j,i) \left\|\Psi_i^{N_1} c\right\|_{\ell^p(\Gamma)} \\
&= \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi} \sum_j V^{N_1}(k,j) \left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)} + C_{2,\psi}^2 \sum_i \left(V^{N_1}\right)^2(k,i) \left\|\Psi_i^{N_1} c\right\|_{\ell^p(\Gamma)}.
\end{aligned}$$

Similarly, by applying Equation (4.26) n times we obtain

$$\begin{aligned}
\left\|\Psi_k^{N_1} c\right\|_{\ell^p(\Gamma)} \leq \left\|\Psi_k^{N_1} Ac\right\|_{\ell^p(\Lambda)} &+ \sum_{m=1}^{n-1} C_{2,\psi}^m \sum_j (V^{N_1})^m(k,j) \left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)} \\
&+ C_{2,\psi}^n \sum_j (V^{N_1})^n(k,j) \left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)}.
\end{aligned} \tag{4.40}$$

Since $\|\cdot\|_{\ell^\infty(X_{N_1})} \leq \|\cdot\|_{\ell^p(X_{N_1})}$ we obtain

$$\begin{aligned}
C_{2,\psi}^n \sum_j (V^{N_1})^n(k,j) \left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} &= (C_{2,\psi} V^{N_1})^n \left(\left(\left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \right)_j \right)(k) \\
&\leq \left\| (C_{2,\psi} V^{N_1})^n \left(\left(\left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \right)_j \right) \right\|_{\ell^p}.
\end{aligned}$$

From Schur's Test, see e.g. [22, Theorem 6.18] and the proof of Theorem 3.3.2 for an application, we have that

$$\left\|V^{N_1}\right\|_{\mathscr{B}(\ell^p(X_{N_1}))} \leq \left\|V^{N_1}\right\|_{Schur(X_{N_1} \to X_{N_1})}. \tag{4.41}$$

Then, using

$$\left\|(V^{N_1})^m\right\|_{Schur(X_{N_1} \to X_{N_1})} \leq \frac{1}{(2C_{2,\psi})^m},$$

for $m \in \mathbb{N}$, and the equivalence of norms given by Lemma 4.1.1, we obtain

$$\begin{aligned}
C_{2,\psi}^n \sum_j (V^{N_1})^n(k,j) \left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} &\leq \left\| (C_{2,\psi} V^{N_1})^n \left(\left(\left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \right)_j \right) \right\|_{\ell^p} \\
&\leq C_{2,\psi}^n \left\|V^{N_1}\right\|_{\mathscr{B}(\ell^p(X_{N_1}))} \left\| \left(\left(\left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \right)_j \right) \right\|_{\ell^p} \\
&\leq C_{2,\psi}^n \left\|(V^{N_1})^n\right\|_{Schur(X_{N_1} \to X_{N_1})} \left\| \left(\left(\left\|\Psi_j^{N_1} c\right\|_{\ell^p(\Gamma)} \right)_j \right) \right\|_{\ell^p} \\
&\lesssim C_{2,\psi}^n \left\|(V^{N_1})^n\right\|_{Schur(X_{N_1} \to X_{N_1})} \|c\|_{\ell^p(\Gamma)} \\
&\leq \frac{1}{2^n} \|c\|_{\ell^p(\Gamma)} \xrightarrow{n \to \infty} 0.
\end{aligned}$$

Thus, taking $n \to \infty$ in Equation (4.40) and using the Definition (4.37) we have

$$\begin{aligned}
\lim_{n\to\infty} \left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)} \leq & \lim_{n\to\infty} \left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} + \sum_{m=1}^{n-1} C_{2,\psi}^m \sum_j (V^{N_1})^m(k,j) \left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \\
& + C_{2,\psi}^n \sum_j (V^{N_1})^n(k,j) \left\| \Psi_j^{N_1} c \right\|_{\ell^p(\Gamma)} \\
= & \left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} + \lim_{n\to\infty} \sum_{m=1}^{n-1} C_{2,\psi}^m \sum_j (V^{N_1})^m(k,j) \left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \\
& + \lim_{n\to\infty} C_{2,\psi}^n \sum_j (V^{N_1})^n(k,j) \left\| \Psi_j^{N_1} c \right\|_{\ell^p(\Gamma)} \\
= & \left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} + \sum_{m=1}^{\infty} C_{2,\psi}^m \sum_j (V^{N_1})^m(k,j) \left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \\
= & \left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} + \sum_{m=1}^{\infty} (C_{2,\psi} V^{N_1})^m \left(\left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_j (k) \\
= & \left(\mathbb{I} + \sum_{m=1}^{\infty} (C_{2,\psi} V^{N_1})^m \right) \left(\left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_j (k) \\
= & W \left(\left\| \Psi_\cdot^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right) (k)
\end{aligned}$$

Thus, for each $c \in \ell^\infty(\Gamma)$ and $x_{k,N_1} \in X_{N_1}$

$$\left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)} \leq \sum_{x_{j,N_1} \in X_{N_1}} W(k,j) \left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)}, \tag{4.42}$$

where $W \in \mathscr{S}(X_{N_1})$. □

5

Stability and Spectral Invariance of Convolution-dominated Matrices

In this chapter, we prove that ℓ^p-stability is independent of $p \in [1,\infty]$ for convolution-dominated matrices indexed by relatively separated sets in groups of polynomial growth, by using the lemmas proved in the previous chapter. In Section 5.2, we state Wiener type Lemmas for convolution-dominated matrices in groups of polynomial growth. Throughout this chapter we make the same assumptions as in Chapter 4.

5.1. Stability

We now have all the ingredients needed to prove the independence of ℓ^p-stability from $p \in [1,\infty]$ for the class of convolution-dominated matrices indexed by relatively separated sets in a group of polynomial growth.

A matrix $A \in \mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$ is said to have *ℓ^p-stability* if there exists $C_p > 0$ such that

$$C_p^{-1}\|c\|_{\ell^p(\Gamma)} \;\le\; \|Ac\|_{\ell^p(\Lambda)} \;\le\; C_p\|c\|_{\ell^p(\Gamma)} \;\;\forall\, c \in \ell^p(\Gamma).$$

We will show that if a convolution-dominated matrix has ℓ^p-stability for some $p \in [1,\infty]$, then it has ℓ^q-stability for each $q \in [1,\infty]$.

Initially, we show that if a convolution-dominated matrix $A \in CD_{w_{D_G}}(\Lambda,\Gamma)$ is bounded from below for some $p \in [1,\infty]$, i.e. there exists $C_{A,p} > 0$ such that

$$\|c\|_{\ell^p(\Gamma)} \;\le\; C_{A,p}\|Ac\|_{\ell^p(\Lambda)} \;\;\forall\, c \in \ell^p(\Gamma),$$

then A is bounded from below for each $q \in [1,\infty]$. The proof given below was inspired by the method developed in [27, 33, 59] and is based on the commutator technique used in Lemma 4.3.1 and the norm equivalence given in Lemma 4.1.1.

Theorem 5.1.1. Let Λ, $\Gamma \subseteq G$ be two relatively separated sets in a locally compact group of polynomial growth and let $\alpha \in \mathbb{N}$, such that $\alpha \ge D_G$, where D_G is the order of growth of G. Suppose that $A \in CD_{w_\alpha}(\Gamma,\Lambda)$ and A is bounded from below for some $p \in [1,\infty]$, i.e. there exists $C_{A,p} > 0$ such that

$$\|c\|_{\ell^p(\Gamma)} \;\le\; C_{A,p}\|Ac\|_{\ell^p(\Lambda)} \;\;\forall\, c \in \ell^p(\Gamma).$$

Then there exists $C'_A > 0$, such that for all $q \in [1,\infty]$ and $c \in \ell^q(\Gamma)$,

$$\|c\|_{\ell^q(\Gamma)} \;\le\; C'_A\|Ac\|_{\ell^q(\Lambda)}.$$

Proof. From Lemma 4.3.1 and for X_N and $\left(\Psi_k^N\right)_{x_{k,N} \in X_N}$ given by Lemma 3.1.13 and Lemma 4.1.1, respectively, we have that there exist $N_1 \in \mathbb{N}$ and $W \in \mathscr{S}(X_{N_1})$ such that

$$\left\|\Psi_k^{N_1} c\right\|_{\ell^p(\Gamma)} \le C_{A,p} \sum_{x_{j,N_1} \in X_{N_1}} W(k,j) \left\|\Psi_j^{N_1} Ac\right\|_{\ell^p(\Lambda)}, \tag{5.1}$$

for each k with $x_{k,N_1} \in X_{N_1}$. Thus, we obtain

$$\begin{aligned}
\left\| \left(\left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)} \right)_k \right\|_{\ell^q(X_{N_1})} &= \left(\sum_{x_{k,N_1} \in X_{N_1}} \left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)}^q \right)^{1/q} \\
&\leq \left(\sum_{x_{k,N_1} \in X_{N_1}} \left(C_{A,p} \sum_j W(k,j) \left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)^q \right)^{1/q} \\
&= \left(\sum_{x_{k,N_1} \in X_{N_1}} \left(C_{A,p} \left(W \left(\left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_j \right)(k) \right)^q \right)^{1/q} \\
&\leq C_{A,p} \left\| \left(\left(W \left(\left\| \Psi_j^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_{x_{j,N_1} \in X_{N_1}} \right)(k) \right)_{x_{k,N_1} \in X_{N_1}} \right\|_{\ell^q(X_{N_1})} \\
&= C_{A,p} \left\| W \left(\left\| \Psi_\cdot^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right) \right\|_{\ell^q(X_{N_1})} \\
&\leq C_{A,p} \left\| W \right\|_{\mathscr{B}(\ell^q(X_{N_1}))} \left\| \left(\left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_{x_{k,N_1} \in X_{N_1}} \right\|_{\ell^q(X_{N_1})} \\
&\leq C_{A,p} \left\| W \right\|_{Schur(X_{N_1} \to X_{N_1})} \left\| \left(\left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_{x_{k,N_1} \in X_{N_1}} \right\|_{\ell^q(X_{N_1})},
\end{aligned}$$

where for the last inequality we have used that the Schur matrices are embedded in the spaces $\mathscr{B}(\ell^q(X_{N_1}))$ for each $q \in [1,\infty]$ by the Schur's Test, see e.g. [22, Theorem 6.18].

On the other hand, from the equivalence of the norms given by Lemma 4.1.1, we obtain

$$\|c\|_{\ell^q(\Gamma)} \lesssim_{N_1} \left\| \left(\left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)} \right)_{x_{k,N_1} \in X_{N_1}} \right\|_{\ell^q(X_{N_1})}$$

and

$$\left\| \left(\left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_{x_{k,N_1} \in X_{N_1}} \right\|_{\ell^q(X_{N_1})} \lesssim_{N_1} \|Ac\|_{\ell^q(\Lambda)},$$

with constants independent of $p,\ q\ \in [1,\infty]$. Thus, combining the previous

$$\begin{aligned}
\|c\|_{\ell^q(\Gamma)} \lesssim_{N_1} \left\| \left(\left\| \Psi_k^{N_1} c \right\|_{\ell^p(\Gamma)} \right)_k \right\|_{\ell^q(X_{N_1})} &\leq C_{A,p} \|W\|_{Schur(X_{N_1} \to X_{N_1})} \left\| \left(\left\| \Psi_k^{N_1} Ac \right\|_{\ell^p(\Lambda)} \right)_k \right\|_{\ell^q(X_{N_1})} \\
\lesssim_{N_1} C_{A,p} \|W\|_{Schur(X_{N_1} \to X_{N_1})} \|Ac\|_{\ell^q(\Lambda)},
\end{aligned}$$

Thus, there exists $C'_A := C'_A(A, N_1, p) > 0$ such that for each $q \in [1,\infty]$ and $c \in \ell^q(\Gamma)$ we have

$$\|c\|_{\ell^q(\Gamma)} \leq C'_A \|Ac\|_{\ell^q(\Lambda)}. \tag{5.2}$$

□

From the embedding of the class of convolution-dominated matrices $CD_{w_{D_G}}(\Gamma,\Lambda)$ into $\mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$, see Theorem 3.3.2, we have that every matrix $A \in CD_{w_{D_G}}(\Gamma,\Lambda)$ is bounded as an operator in $\mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$. Hence, if A is also bounded from below in $\mathscr{B}(\ell^p(\Gamma), \ell^p(\Lambda))$, then it has ℓ^p-stability. Combining the previous and Theorem 5.1.1 we have that if $A \in CD_{w_{D_G}}(\Gamma,\Lambda)$ has ℓ^p-stability for some $p \in [1,\infty]$, then it has ℓ^q-stability for all $q \in [1,\infty]$. This proves the following result.

Corollary 5.1.2. Let Λ, $\Gamma \subseteq G$ be two relatively separated sets in a locally compact group of polynomial growth and let $\alpha \in \mathbb{N}$, such that $\alpha \geq D_G$, where D_G is the order of growth of G. If $A \in CD_{w_\alpha}(\Gamma,\Lambda)$ has ℓ^p-stability for some $p \in [1,\infty]$, then it has ℓ^q-stability for all $q \in [1,\infty]$.

Remark 5.1.3. Assuming further that G is abelian, by using similar estimates it can be shown that the results of this section are true for matrices in $CD_{w_\alpha}(\Gamma,\Lambda)$, for any polynomial weight ($\alpha \in \mathbb{N}$), but also in the unweighted case ($\alpha = 0$). Precisely, using the commutativity of the group in the proof of Lemma 4.3.1 we can obtain a uniform bound in Equation (4.33), independent of $x \in G$ and $x_{k,N} \in X_N$. This recovers the result by Shin and Sun [59] for relatively separated sets in the Euclidean space $\mathbb{R}^d$.

The following theorem extends Theorem 5.1.1 and proves that if a convolution-dominated operator is bounded from below on a subspace for some $p \in [1,\infty]$, then it is bounded from below on the subspace for each $q \in [1,\infty]$. The proof presented below was inspired by similar results in [29] and [27] for Euclidean spaces and homogeneous groups, respectively.

Theorem 5.1.4. Let Λ, $\Gamma \subseteq G$ be two relatively separated sets in a locally compact group of polynomial growth and let $\alpha \in \mathbb{N}$ be such that $\alpha \geq D_G + 1$, where D_G is the order of growth of G. Suppose that $P \in CD_{w_\alpha}(\Gamma)$ and P is idempotent, i.e. $P^2 = P$. Moreover, suppose that $A \in CD_{w_\alpha}(\Gamma,\Lambda)$ and for $p \in [1,\infty]$, there exists $C_{A,p} > 0$, such that for all $c \in \ell^p(\Gamma)$

$$\|Pc\|_{\ell^p(\Gamma)} \;\leq\; C_{A,p}\|APc\|_{\ell^p(\Lambda)}. \tag{5.3}$$

Then there exists $C' > 0$, such that for all $q \in [1,\infty]$ and $c \in \ell^q(\Gamma)$,

$$\|Pc\|_{\ell^q(\Gamma)} \;\leq\; C'\|APc\|_{\ell^q(\Lambda)}. \tag{5.4}$$

Proof. Since $P \in CD_{w_\alpha}(\Gamma)$ and $A \in CD_{w_\alpha}(\Gamma,\Lambda)$, then by Theorem 3.3.2 we deduce that $I - P \in CD_{w_\alpha}(\Gamma)$ and $AP \in CD_{w_\alpha}(\Gamma,\Lambda)$. Hence, there exist $\Theta_1, \Theta_2 \in W_{w_\alpha}(G)$ such that for each $\lambda \in \Lambda$ and $\gamma \in \Gamma$

$$|(AP)(\lambda,\gamma)| \leq \Theta_1(\lambda^{-1}\gamma) \tag{5.5}$$

and for each $\gamma, \gamma' \in \Gamma$

$$|(I-P)(\gamma,\gamma')| \leq \Theta_2(\gamma^{-1}\gamma'). \tag{5.6}$$

Then the operator

$$\begin{aligned} \widetilde{A} \;:\; & \ell^p(\Gamma) \longrightarrow \ell^p(\Lambda) \oplus \ell^p(\Gamma) \\ & c \quad\longmapsto (APc,\, (I-P)c)\,, \end{aligned} \tag{5.7}$$

is well defined and bounded. By using Equation (5.3) we have for each $c \in \ell^p(\Gamma)$

$$\left\|\widetilde{A}c\right\|_{\ell^p(\Lambda)\oplus\ell^p(\Gamma)} = \|APc\|_{\ell^p(\Lambda)} + \|(I-P)c\|_{\ell^p(\Gamma)} \geq C_{A,p}^{-1}\|Pc\|_{\ell^p(\Gamma)} + \|(I-P)c\|_{\ell^p(\Gamma)}\,. \tag{5.8}$$

From the previous and $\|c\|_{\ell^p(\Gamma)} \leq \|Pc\|_{\ell^p(\Gamma)} + \|(I-P)c\|_{\ell^p(\Gamma)}$, it follows that for each $c \in \ell^p(\Gamma)$

$$\left\|\widetilde{A}c\right\|_{\ell^p(\Lambda)\oplus\ell^p(\Gamma)} \gtrsim \|c\|_{\ell^p(\Gamma)}\,. \tag{5.9}$$

Define the group $\widetilde{G} = G \times \mathbb{R}$, with multiplication $(x,z)\cdot(y,\zeta) = (xy, z+\zeta)$, for each $x,y \in G$ and $z,\zeta \in \mathbb{R}$. $\widetilde{G}$ is a locally compact group with Haar measure μ, such that for each measurable sets $V_1 \subseteq G$ and $V_2 \subseteq \mathbb{R}$ we have $\mu((V_1,V_2)) = \mu_G(V_1)\mu_{\mathbb{R}}(V_2)$, where μ_G is the Haar measure on G and $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$, see e.g. [21]. It follows that $\widetilde{G}$ is a group of polynomial growth generated by $(U, B_{\mathbb{R}}(0,1))$, where U is the generating neighbourhood of G and $B_{\mathbb{R}}(0,1) = \{z \in \mathbb{R} \,:\, |z| < 1\}$ and $\widetilde{G}$ has order of growth equal to $D_{\widetilde{G}} = D_G + 1$. Moreover, we define the following relatively separated sets, $\widetilde{\Lambda} = \Lambda \times \{0\} \subseteq \widetilde{G}$, $\widetilde{\Gamma} = \Gamma \times \{1\} \subseteq \widetilde{G}$ and $\Omega = \widetilde{\Lambda} \cup \widetilde{\Gamma} \subseteq \widetilde{G}$.

We consider the matrix $B \in \mathbb{C}^{\Omega\times\widetilde{\Gamma}}$, such that for each $\lambda \in \Lambda$ and $\gamma' \in \Gamma$

$$B\left((\lambda,0),(\gamma',1)\right) = (AP)(\lambda,\gamma'), \tag{5.10}$$

and for each $\gamma, \gamma' \in \Gamma$

$$B\left((\gamma,1),(\gamma',1)\right) = (I-P)(\gamma,\gamma). \tag{5.11}$$

Note that B can be identified with the operator $\widetilde{A}$, by identifying $\widetilde{\Lambda}$ with Λ and $\widetilde{\Gamma}$ with Γ.

Let $\psi \in C_c^\infty(\mathbb{R})$, such that $\operatorname{supp}(\psi) \subseteq [-2,2]$ and $\psi = 1$ on $[-1,1]$. We define

$$\begin{aligned} \Theta\ : \widetilde{G} &\longrightarrow \mathbb{C} \\ (x,z) &\longmapsto (\Theta_1(x) + \Theta_2(x))\psi(z) \end{aligned} \tag{5.12}$$

and the weight $\widetilde{w_\alpha}$ on $\widetilde{G}$, such that $\widetilde{w_\alpha}(x,z) = (1 + \mathrm{d}(x,e) + |z|)^\alpha$. Note that $\widetilde{w_\alpha}$ is exactly the polynomial weight on $\widetilde{G}$ of order α. Since $\Theta_1, \Theta_2 \in W_{w_\alpha}(G)$ and $\psi \in C_c^\infty(\mathbb{R})$, it follows that $\Theta \in W_{\widetilde{w_\alpha}}\left(\widetilde{G}\right)$. By the definition of the matrix B and Equations (5.5) and (5.6) we obtain for each $\lambda \in \Lambda$ and $\gamma, \gamma' \in \Gamma$

$$|B\left((\lambda,0),(\gamma',1)\right)| \leq \Theta_1(\lambda^{-1}\gamma') \leq (\Theta_1(\lambda^{-1}\gamma') + \Theta_2(\lambda^{-1}\gamma'))\psi(1-0) = \Theta((\lambda,0)^{-1}(\gamma',1)) \tag{5.13}$$

and

$$|B\left((\gamma,1),(\gamma',1)\right)| \leq \Theta_2(\gamma^{-1}\gamma') \leq (\Theta_1(\gamma^{-1}\gamma') + \Theta_2(\gamma^{-1}\gamma'))\psi(1-1) = \Theta((\gamma,1)^{-1}(\gamma',1)). \tag{5.14}$$

Thus, $B \in CD_{\widetilde{w_\alpha}}\left(\Omega, \widetilde{\Gamma}\right)$. Moreover, by identifying B and $\widetilde{A}$ and using Equation (5.9) we deduce that B is bounded from below for p. Then since $B \in CD_{\widetilde{w_\alpha}}\left(\Omega, \widetilde{\Gamma}\right)$ is bounded from below for p and $\alpha \geq D_G + 1 = D_{\widetilde{G}}$ we can apply Theorem 5.1.1. Hence B is bounded from below for each $q \in [1,\infty]$ (with constant independent of q). Thus, using once more the identification of B and $\widetilde{A}$, there exists $C' > 0$ such that for each $q \in [1,\infty]$

$$\|c\|_{\ell^q(\Gamma)} \leq C' \left\|\widetilde{A}c\right\|_{\ell^q(\Lambda)\oplus\ell^q(\Gamma)}, \ \forall c \in \ell^q(\Gamma). \tag{5.15}$$

It follows that for each $q \in [1,\infty]$ for each $c \in \ell^q(\Gamma)$

$$\|Pc\|_{\ell^q(\Gamma)} \leq C' \left\|\widetilde{A}Pc\right\|_{\ell^q(\Lambda)\oplus\ell^q(\Gamma)} = C' \|APPc\|_{\ell^q(\Lambda)} + \|(I-P)Pc\|_{\ell^q(\Gamma)} = C' \|APc\|_{\ell^q(\Lambda)}, \tag{5.16}$$

where for the last equality we have used that $P^2 = P$. □

5.1.1. Discussion

We now discuss the optimality of the weight assumption in Theorem 5.1.1 and we compare the results of this section with similar results in the literature.

Recall that for relatively separated subsets Λ and Γ in a group G of polynomial growth, Theorem 5.1.1 proves that if a convolution-dominated matrix in $CD_w(\Gamma, \Lambda)$ is bounded from below for some $p \in [1,\infty]$, then it is bounded from below for each $q \in [1,\infty]$, under the assumption that w is the polynomial weight given by

$$w = w_\alpha : G \longrightarrow [1,\infty), \ x \longmapsto (1 + \mathrm{d}(x,e))^\alpha,$$

where $\alpha \in \mathbb{N}$ and $\alpha \geq D_G$ with D_G the order of growth of the group. For general relatively separated sets Theorem 5.1.1 yields new results, however we expect that the assumptions of the theorem could be improved. Tessera [63] showed that in the case of matrices indexed by discrete groups of polynomial growth Theorem 5.1.1 holds for each polynomial weight ($\alpha \in \mathbb{N}$). Therefore, if we take a discrete subgroup Λ in a group of polynomial growth that also has polynomial growth, for example a uniform lattice (see Lemma 3.1.18), then by Tessera [63] we have that the aforementioned results hold for the convolution-dominated matrices $CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. On the other hand, Theorem 5.1.1 proves the ℓ^p-stability result only for $\alpha \geq D_G$, where D_G is the order of growth of the group. From the previous, we expect that the assumption on the order α of the weight could be improved, however the estimates in the commutator technique seem not good enough to do so and hence another method should be used to obtain optimal assumptions.

For abelian groups of polynomial growth we have from Remark 5.1.3 the result on ℓ^p-stability for the convolution-dominated matrices $CD_w(\Lambda)$ for every polynomial weight, but also in the unweighted case. Since this proves the spectral invariance in the uweighted case we have the optimal result and, moreover, we recover the result given by Shin and Sun [59].

5.2. Spectral Invariance

In this section, we state an inverse-closedness type result for the intersection of the convolution-dominated matrices indexed by a relatively separated set, $CD_{w_\alpha}(\Lambda)$, for $\alpha \in \mathbb{N}$ and the spectral invariance of convolution-dominated matrices in discrete groups of polynomial growth given by Fendler, Gröchenig and Leinert in [19].

The commutator technique used in Lemma 4.3.1 was first used by Sjöstrand [60] for proving the spectral invariance, however Shin and Sun [59] observed that this technique proves also the result on ℓ^p-stability. For proving the spectral invariance of convolution-dominated matrices using the commutator technique, we should estimate the convolution-dominated norm of the matrix V^N in Lemma 4.3.1, instead of the Schur norm. During the project we have attempted to prove the spectral invariance using the commutator technique, but did not succeed. Furthermore, inspired by the use of auxiliary norms in [58], we have also attempted using the commutator technique with auxiliary norms. The estimates used in the commutator technique seem not good enough in order to prove the spectral invariance, therefore we now present a Wiener type Lemma for the intersection of polynomially weighted convolution-dominated matrices. For this, we follow a different method by using the spectral invariance of the weighted Schur matrices given by Sun [61].

Initially, we define the weighted Schur matrices indexed by a relatively separated set Λ, for each polynomial weight

$$w_\alpha : G \longrightarrow [1,\infty),\ x \longmapsto (1+\mathrm{d}(x,e))^\alpha,$$

$\alpha \in \mathbb{N} \cup \{0\}$ and each $p \in [1,\infty]$ as follows

$$\mathscr{S}_{p,w_\alpha}(\Lambda) := \left\{ A \in \mathbb{C}^{\Lambda\times\Lambda} \ :\ \|A\|_{\mathscr{S}_{p,w_\alpha}(\Lambda)} < \infty \right\}, \tag{5.17}$$

where

$$\|A\|_{\mathscr{S}_{p,w_\alpha}(\Lambda)} = \sup_{\lambda\in\Lambda} \left(\sum_{\lambda'\in\Lambda} \left| A(\lambda,\lambda') w_\alpha(\lambda^{-1}\lambda') \right|^p \right)^{1/p} + \sup_{\lambda'\in\Lambda} \left(\sum_{\lambda\in\Lambda} \left| A(\lambda,\lambda') w_\alpha(\lambda^{-1}\lambda') \right|^p \right)^{1/p}, \tag{5.18}$$

for $p \in [1,\infty)$ and

$$\|A\|_{\mathscr{S}_{p,w_\alpha}(\Lambda)} = \sup_{\lambda,\lambda'\in\Lambda} \left| A(\lambda,\lambda') w_\alpha(\lambda^{-1}\lambda') \right|, \tag{5.19}$$

for $p = \infty$. Note that the norms $\|\cdot\|_{Schur(\Lambda\to\Lambda)}$ and $\|\cdot\|_{\mathscr{S}_{1,w_0}(\Lambda)}$ are equivalent, therefore the definition of the unweighted Schur matrices $\mathscr{S}_{1,w_0}(\Lambda)$ given in (5.17) and $\mathscr{S}(\Lambda)$ given in (2.7) coincide. Moreover, we observe that by definition we have

$$\|A\|_{\mathscr{S}_{\infty,w_\alpha}(\Lambda)} \le \|A\|_{\mathscr{S}_{1,w_\alpha}(\Lambda)},$$

for each $A \in \mathscr{S}_{1,w_\alpha}(\Lambda)$ and each polynomial weight, hence

$$\mathscr{S}_{1,w_\alpha}(\Lambda) \subseteq \mathscr{S}_{\infty,w_\alpha}(\Lambda). \tag{5.20}$$

From Lemma 3.2.7, see also the proof of Proposition 3.3.2, we deduce

$$\|A\|_{\mathscr{S}_{1,w_\alpha}(\Lambda)} \le 2\frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)} \|A\|_{CD_{w_\alpha}(\Lambda)},$$

therefore

$$CD_{w_\alpha}(\Lambda) \subseteq \mathscr{S}_{1,w_\alpha}(\Lambda). \tag{5.21}$$

Sun [61] proved the spectral invariance of the weighted Schur matrices indexed by a discrete set with polynomial growth, under further assumptions on the weight, see [61, Theorem 4.1]. We present below Sun's [61] spectral invariance for $\mathscr{S}_{1,w_\alpha}(\Lambda)$, where w_α is the polynomial weight of order $\alpha \in \mathbb{N}$, and Λ is a relatively separated set in a group of polynomial growth. We do not present the proof given of the result, however we show that the assumptions of the theorem are verified, see [61, Theorem 4.1] for a complete proof.

Theorem 5.2.1 ([61])**.** Let Λ be a relatively separated set in a group, G, of polynomial growth. Moreover, suppose that $\alpha \in \mathbb{N}$ and w_α is the polynomial weight on the group given by $w_\alpha : x \longmapsto (1+\mathrm{d}(x,e))^\alpha$. Then $\mathscr{S}_{1,w_\alpha}(\Lambda)$ is inverse-closed in $\mathscr{B}(\ell^2(\Lambda))$.

Proof. Since $\alpha > 0$ we have from [61, Example A.2] that w_α verifies the conditions of [61, Theorem 4.1].

We equip the relatively separated set Λ with the restriction to Λ of word metric of the group, $\mathrm{d}_\Lambda = \mathrm{d}\,|_\Lambda$, and the counting measure μ_C given by

$$\begin{aligned} \mu_C\ &: \mathscr{P}(\Lambda) \longrightarrow [0,\infty) \\ V\ &\longmapsto \#V := \#\{x \in \Lambda : x \in V\}, \end{aligned} \tag{5.22}$$

where $\mathscr{P}(\Lambda) := \{V \subseteq \Lambda\}$ is the power set of Λ. Then using Lemma 3.1.14 and the polynomial growth of the group we obtain

$$\mu_C(B_{\mathrm{d}_\Lambda}(x,n)) = \#\left(\Lambda \cap B_{\mathrm{d}}(x,n)\right) \leq \#\left(\Lambda \cap xU^n\right) \lesssim \mu(xU^n) = \mu(U^n) \leq C_G n^{D_G},$$

for each $x \in \Lambda$ and $n \in \mathbb{N}$. Thus, the triple $(\Lambda, \mu_C, \mathrm{d}_\Lambda)$ verifies the assumptions of [61, Theorem 4.1].

From the previous we can apply [61, Theorem 4.1] and this proves the claim. □

We now show that for a relatively separated set Λ, if a matrix A belongs in the convolution-dominated matrices $CD_{w_\alpha}(\Lambda)$ for each polynomial weight w_α, $\alpha \in \mathbb{N}$ and A is invertible in $\mathscr{B}(\ell^2(\Lambda))$, then $A^{-1} \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. The proof of the following result is based on the spectral invariance of the Schur matrices and the inclusions

$$CD_{w_\alpha}(\Lambda) \subseteq \mathscr{S}_{1,w_\alpha}(\Lambda) \subseteq \mathscr{S}_{\infty,w_\alpha}(\Lambda),$$

that hold for each $\alpha \in \mathbb{N}$.

Theorem 5.2.2. Let $\Lambda \subseteq G$ be a relatively separated set in a locally compact group of polynomial growth and let w_α be the polynomial weight on the group given by $w_\alpha : x \longmapsto (1 + \mathrm{d}(x,e))^\alpha$. Suppose that $A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$ and A is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Then $A^{-1} \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$.

Proof. Suppose that G is compactly generated by the open, relatively compact, unit neighbourhood U. Moreover, suppose that the growth of the generating neighbourhood is given by

$$\mu(U^n) \leq C_G n^{D_G},$$

where $C_G > 0$ and $D_G > 0$ is the order of growth of the group.

Let $\alpha \in \mathbb{N}$ be such that $\alpha \geq D_G + 2$. Since $A \in CD_{w_\alpha}(\Lambda)$, we have that $A \in \mathscr{S}_{1,w_\alpha}(\Lambda)$, from the inclusion (5.21). From the spectral invariance of $\mathscr{S}_{1,w_\alpha}(\Lambda)$ in $\mathscr{B}(\ell^2(\Lambda))$, given by Theorem 5.2.1, and the invertibility of A in $\mathscr{B}(\ell^2(\Lambda))$, it follows that $A^{-1} \in \mathscr{S}_{1,w_\alpha}(\Lambda)$. Moreover, using the inclusion (5.20) we have that

$$A^{-1} = \left(A^{-1}(\lambda,\lambda')\right)_{\lambda,\lambda' \in \Lambda} \in \mathscr{S}_{\infty,w_\alpha}(\Lambda),$$

hence

$$\left\|A^{-1}\right\|_{\mathscr{S}_{\infty,w_\alpha}(\Lambda)} = \sup_{\lambda,\lambda' \in \Lambda} \left|A^{-1}(\lambda,\lambda') w_\alpha(\lambda^{-1}\lambda')\right| < \infty. \tag{5.23}$$

From the above we obtain that for each $\lambda, \lambda' \in \Lambda$

$$\left|A^{-1}(\lambda,\lambda')\right| \leq \left\|A^{-1}\right\|_{\mathscr{S}_{\infty,w_\alpha}(\Lambda)} w_\alpha(\lambda^{-1}\lambda')^{-1}. \tag{5.24}$$

By the submultiplicativity of the weight w_α we have for each $x \in G$ and $y, y' \in Q$

$$w_\alpha(x) = w_\alpha(y^{-1}yxy'(y')^{-1}) \leq w_\alpha(y^{-1}) w_\alpha(yxy') w_\alpha((y')^{-1})$$

hence

$$w_\alpha(x)^{-1} \geq w_\alpha(y^{-1})^{-1} w_\alpha(yxy')^{-1} w_\alpha((y')^{-1})^{-1}.$$

Since the previous holds for each $y, y' \in Q$, by taking the supremum over $y, y' \in Q$, we obtain

$$w_\alpha(x)^{-1} \geq D_1 \, \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \, \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(w_\alpha^{-1})(x), \tag{5.25}$$

where $D_1 = \sup_{y,y' \in Q} w_\alpha(y^{-1})^{-1} w_\alpha((y')^{-1})^{-1} > 0$. Then, using $G = \bigcup_{n=0}^{\infty} U^n$, we have

$$\begin{aligned}
\left\|w_\alpha^{-1}\right\|_{W_{w_{\alpha-(D_G+2)}}(G)} &= \int_G \mathrm{M}_{\mathrm{Q}}^{\mathrm{L}} \, \mathrm{M}_{\mathrm{Q}}^{\mathrm{R}}(w_\alpha^{-1})(x) w_{\alpha - D_G + 2}(x) \mathbf{d}\mu(x) \\
&\leq D_1 \int_G w_\alpha^{-1}(x) w_{\alpha-(D_G+2)}(x) \mathbf{d}\mu(x) \\
&= D_1 \left(\int_{U^1} w_{D_G+2}^{-1}(x) \mathbf{d}\mu(x) + \sum_{n=1}^{\infty} \int_{U^{n+1} \setminus U^n} w_{D_G+2}^{-1}(x) \mathbf{d}\mu(x) \right).
\end{aligned}$$

We observe that for each $n \in \mathbb{N}$, if $x \in U^{n+1} \backslash U^n$, then $\mathrm{d}(x,e) \geq n$ and

$$w_{D_G+2}^{-1}(x) = \frac{1}{w_{D_G+2}(x)} = \frac{1}{(1+\mathrm{d}(x,e))^{D_G+2}} \leq \frac{1}{(1+n)^{D_G+2}}.$$

Thus,

$$\begin{aligned}\left\|w_\alpha^{-1}\right\|_{W_{w_{\alpha-(D_G+2)}}(G)} &\leq D_1 \left(\int_{U^1} 1 \mathbf{d}\mu(x) + \sum_{n=1}^{\infty} \int_{U^{n+1}\backslash U^n} \frac{1}{(1+n)^{D_G+2}} \mathbf{d}\mu(x) \right) \\ &= D_1 \left(\mu(U^1) + \sum_{n=1}^{\infty} \mu(U^{n+1}\backslash U^n) \frac{1}{(1+n)^{D_G+2}} \right).\end{aligned}$$

Using the polynomial growth we deduce

$$\begin{aligned}\left\|w_\alpha^{-1}\right\|_{W_{w_{\alpha-(D_G+2)}}(G)} &\leq D_1 \left(C_G + \sum_{n=1}^{\infty} C_G (n+1)^{D_G} \frac{1}{(1+n)^{D_G+2}} \right) \\ &= D_1 C_G \left(1 + \sum_{n=1}^{\infty} \frac{1}{(1+n)^2} \right) < \infty.\end{aligned}$$

Thus $w_\alpha^{-1} \in W_{w_{\alpha-(D_G+2)}}(G)$.

From $\left|A^{-1}(\lambda,\lambda')\right| \leq \left\|A^{-1}\right\|_{\mathscr{S}_{\infty,w_\alpha}(\Lambda)} w_\alpha(\lambda^{-1}\lambda')^{-1}$, for each $\lambda, \lambda' \in \Lambda$ and $w_\alpha^{-1} \in W_{w_{\alpha-(D_G+2)}}(G)$ it follows that

$$A^{-1} \in CD_{w_{\alpha-(D_G+2)}}(\Lambda). \tag{5.26}$$

Since $\alpha \in \mathbb{N}$, $\alpha \geq D_G + 2$ was chosen arbitrary we deduce that $A^{-1} \in CD_{w_{\alpha-(D_G+2)}}(\Lambda)$ for each $\alpha \in \mathbb{N}$, $\alpha \geq D_G + 2$. Thus, $A^{-1} \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$, which proves our claim. □

Note that an operator $A \in \mathscr{B}(\ell^2(\Lambda))$ is invertible if and only if A is bounded from below. On the other hand, for $p \in [1,\infty]$, $p \neq 2$ this equivalence is not true. Combining Theorem 5.1.1 and 5.2.2 we deduce the following result on left invertibility, inspired by [63].

Theorem 5.2.3. Let $\Lambda \subseteq G$ be a relatively separated set in a locally compact group of polynomial growth. Suppose that $A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. Then the following are equivalent:

(i) A is bounded from below for some $p \in [1,\infty]$,

(ii) A is bounded from below for each $p \in [1,\infty]$,

(iii) $B = (A^*A)^{-1}A^*$ defines a left inverse for A and $B \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$.

Proof. The equivalence (i) $\iff$ (ii) is true by Theorem 5.1.1. It is left to prove the equivalence (ii) $\iff$ (iii).

(ii) ⇒ (iii): From (ii) we have that A is bounded from below for $p = 2$, hence there exists $C_{2,A} > 0$ such that

$$\|c\|_{\ell^2(\Lambda)} \leq C_{2,A} \|Ac\|_{\ell^2(\Lambda)}, \quad \forall\, c \in \ell^2(\Lambda). \tag{5.27}$$

Then for each $c \in \ell^2(\Lambda)$ we obtain

$$\|A^*Ac\|_{\ell^2(\Lambda)} \|c\|_{\ell^2(\Lambda)} \geq \langle A^*Ac, c\rangle_{\ell^2(\Lambda)} = \|Ac\|_{\ell^2(\Lambda)}^2 \geq C_{2,A}^2 \|c\|_{\ell^2(\Lambda)}^2.$$

Thus A^*A is bounded from below for $p = 2$. It follows that A^*A is injective in $\mathscr{B}(\ell^2(\Lambda))$. Suppose that $(c_n)_{n\in\mathbb{N}} \subseteq \ell^2(\Lambda)$, is a sequence such that $A^*Ac_n \xrightarrow{n\to\infty} b \in \overline{\mathrm{Ran}(A^*A)}$ in $\ell^2(\Lambda)$. Then $(A^*Ac_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and from the boundedness from below we deduce that $(c_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. Thus, there exists $c \in \ell^2(\Lambda)$ such that $c_n \xrightarrow{n\to\infty} c$ in $\ell^2(\Lambda)$ and from the continuity we obtain $A^*Ac = b$. From the previous, it follows that A^*A has closed range. Since A^*A is self adjoint and injective we have that

$$\mathrm{Ran}(A^*A) = \overline{\mathrm{Ran}(A^*A)} = \ker(A^*A)^\perp = \ell^2(\Lambda).$$

Thus, A^*A is bijective and hence invertible. For the inverse $(A^*A)^{-1}$ of A^*A, using the boundedness from below of A^*A we obtain for each $c \in \ell^2(\Lambda)$

$$C_{2,A} \left\|(A^*A)^{-1}c\right\|_{\ell^2(\Lambda)} \leq \left\|A^*A(A^*A)^{-1}c\right\|_{\ell^2(\Lambda)} = \|c\|_{\ell^2(\Lambda)} .$$

Thus, $(A^*A)^{-1} \in \mathscr{B}(\ell^2(\Lambda))$ and $(A^*A)^{-1}$ is the inverse of A^*A in $\mathscr{B}(\ell^2(\Lambda))$.

Since $A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$, we have $A^* \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$ and since $CD_{w_\alpha}(\Lambda)$ is closed under multiplications (see Proposition 3.3.2) we have $A^*A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. Moreover, $A^*A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$ and A^*A is invertible in $\mathscr{B}(\ell^2(\Lambda))$, hence it follows from Theorem 5.2.2 that that $(A^*A)^{-1} \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. Therefore, $B = (A^*A)^{-1}A^* \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$ and since $B = (A^*A)^{-1}A^*A = I$, we have that B defines a left inverse for A in $CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$.

(iii) ⇒ (ii): Let $p \in [1, \infty]$. From the embedding $CD_{w_1}(\Lambda) \hookrightarrow B(\ell^p(\Lambda))$ given by Proposition 3.3.2, it follows that B defines a left inverse for A in $\mathscr{B}(\ell^p(\Lambda))$. Thus

$$\|c\|_{\ell^p(\Lambda)} = \|BAc\|_{\ell^p(\Lambda)} \leq \|B\|_{\mathscr{B}(\ell^p(\Lambda))} \|Ac\|_{\ell^p(\Lambda)} . \tag{5.28}$$

It follows that A is bounded from below for $p \in [1, \infty]$. Since $p \in [1, \infty]$ was chosen arbitrary (ii) follows. □

Fendler, Gröchenig and Leinert in [19] showed that the convolution-dominated matrices over a discrete group of polynomial growth are spectrally invariant for each polynomial weight and in the unweighted case. We state below the aforementioned result by Fendler, Gröchenig and Leinert, see [19, Theorem 1] for a proof.

Theorem 5.2.4 ([19])**.** Let G be a discrete group of polynomial growth. Suppose $\alpha \in \mathbb{N} \cup \{0\}$ and the weight w_α in G is given by $w_\alpha : x \longmapsto (1 + \mathrm{d}(x, e))^\alpha$, where d is the word metric on G. Then $CD_{w_\alpha}(G)$ is inverse-closed in $\mathscr{B}(\ell^2(G))$.

Recall that a uniform lattice in a group of polynomial growth is a group of polynomial growth equipped with the counting measure, see Lemma 3.1.18. Thus, the previous result can also be applied to uniform lattices in groups of polynomial growth.

Part II

Applications

6

Coherent Frames

In this chapter, we introduce the background needed in order to present the applications of convolution-dominated matrices that were studied during the project. Throughout this part we assume that G is a locally compact group of polynomial growth, with Haar measure μ, generating neighbourhood $U \subseteq G$ and order of growth equal to $D_G > 0$.

6.1. Discrete Series Representations

In this section we define the discrete series $\sigma-$representations, which will be used for the applications presented in the upcoming chapters. More details on discrete series $\sigma-$representations can be found in [1, 5, 39, 51].

Before we define $\sigma-$projective representations, we initially introduce the notion of a cocycle on a group.

Definition 6.1.1. Let $\sigma \,:\, G \times G \longrightarrow \mathbb{T}$ be a Borel measurable function. The function σ is called a *cocycle* on G if

(i) $\sigma(x,\, yz)\sigma(y,\, z) = \sigma(xy,\, z)\sigma(x,\, y)$ for each $x,\, y,\, z \in G$,
(ii) $\sigma(x,\, e) = \sigma(e,\, x) = 1$ for each $x \in G$, where e is the identity element in G.

An element $x \in G$ is said to be $\sigma-$*regular*, if for each element $y \in Z_G(x) := \{z \in G \,:\, xz = zx\}$ we have $\sigma(x,\, y) = \sigma(y,\, x)$.

Definition 6.1.2. A mapping $\pi \,:\, G \longrightarrow \mathscr{U}(H_\pi)$, where $\mathscr{U}(H_\pi)$ denotes the unitary operators in $\mathscr{B}(H_\pi)$, is a $\sigma-$*projective unitary representation* $(\pi,\, H_\pi)$ of G on a Hilbert space H_π if the following conditions are satisfied:

(i) the map $G \ni x \longmapsto \langle f, \pi(x)g\rangle \in \mathbb{C}$ is a Borel measurable function for each $f,\, g \in H_\pi$,
(ii) $\sigma \,:\, G \times G \longrightarrow \mathbb{T}$ is a function on G, such that $\pi(x)\pi(y) = \sigma(x,\, y)\pi(xy)$ for each $x,\, y \in G$,
(iii) $\pi(e) = I_{H_\pi}$.

Note that from (i) of the previous definition we deduce that σ is a Borel measurable function and from (ii) we obtain the properties (i) and (ii) of Definition 6.1.1. Thus, if π is a σ-projective representation then we deduce that σ is a cocycle on G. Moreover, if $(\pi,\, H_\pi)$ is a $\sigma-$projective unitary representation, then $\pi(x) \in \mathscr{U}(H_\pi)$ for each $x \in G$ and we obtain

$$\pi(x)\pi(x)^* = I_{H_\pi} = \pi(e) = \pi(xx^{-1}) = \overline{\sigma(x, x^{-1})}\pi(x)\pi(x^{-1}).$$

Thus,

$$\pi(x)^* = \overline{\sigma(x, x^{-1})}\pi(x^{-1}). \tag{6.1}$$

Similarly, for each $x \in G$, we obtain that

$$\pi(x)^* = \overline{\sigma(x^{-1}, x)}\pi(x^{-1}). \tag{6.2}$$

From the previous we deduce that for each $x \in G$,

$$\sigma(x, x^{-1}) = \sigma(x^{-1}, x). \tag{6.3}$$

A subspace V of H_π is called $\pi(G)$-invariant if $\pi(g)V \subseteq V$ for each $g \in G$. The $\sigma-$projective representation $(\pi,\, H_\pi)$ is said to be *irreducible* if the only closed $\pi(G)$-invariant subspaces are $\{0\}$ and H_π. For $f, g \in H_\pi$ we define the *matrix coefficient* associated with the representation $(\pi,\, H_\pi)$, by

$$V_g f(x) = \langle f,\, \pi(x)g \rangle\,, \tag{6.4}$$

for $x \in G$. We call the irreducible $\sigma-$projective representation $(\pi,\, H_\pi)$ a *discrete series $\sigma-$representation*, if there exists $g \in H_\pi \backslash \{0\}$ such that

$$V_g g = \langle g,\, \pi(\cdot)g \rangle \in L^2(G). \tag{6.5}$$

For a discrete series $\sigma-$ representation $(\pi,\, H_\pi)$ it can be shown that there exists a unique $d_\pi > 0$, such that the *orthogonality relations*

$$\int_G \langle f_1,\, \pi(x)g_1 \rangle\, \overline{\langle f_2,\, \pi(x)g_2 \rangle} \mathbf{d}\mu(x) = \frac{1}{d_\pi} \langle f_1,\, f_2 \rangle \overline{\langle g_1,\, g_2 \rangle}, \tag{6.6}$$

hold for each $f_1,\, f_2,\, g_1,\, g_2 \in H_\pi$, see [1, 51]. The constant d_π is called the *formal dimension* of π.

6.2. Frames and Riesz sequences

The definition of Frames and Riesz sequences that are of interest in the application in Chapter 8, are presented in this section. The definitions and properties presented here can be found in [6].

Let

$$\pi \;:\; G \longrightarrow \mathscr{B}(H_\pi) \tag{6.7}$$

be a σ-projective unitary representation of G. Suppose that Λ is a relatively separated subset of G. For $g \in H_\pi$ we consider the set of vectors

$$\pi(\Lambda)g := \{\pi(\lambda)g \;:\; \lambda \in \Lambda\}\,. \tag{6.8}$$

A frame is a set of vectors in a Hilbert space that generalizes the notion of orthonormal basis. We restrict our attention to frames of the form $\pi(\Lambda)g$ for $g \in H_\pi$.

Definition 6.2.1 (Frame). A set $\pi(\Lambda)g$ is called a *frame* for H_π, if there exist constants $A,\, B > 0$ such that the *frame inequalities*

$$A \left\|f\right\|_{H_\pi}^2 \le \sum_{\lambda\in\Lambda} \left|\langle f,\, \pi(\lambda)g \rangle\right|^2 \le B \left\|f\right\|_{H_\pi}^2 \tag{6.9}$$

hold for each $f \in H_\pi$. In that case, the vector g is called a *frame vector* and the constants $A,\, B$ are called the *frame bounds*.

From the first of the frame inequalities (6.9) we deduce that the set $\pi(\Lambda)g$ is complete. On the other hand, by the second frame inequality (6.9), which is known as the *Bessel bound*, we obtain that the *frame operator*, defined as follows

$$\begin{aligned} S_g \;: H_\pi &\longrightarrow H_\pi \\ f \;&\longmapsto S_g f = \sum_{\lambda\in\Lambda} \langle f,\, \pi(\lambda)g \rangle \pi(\lambda)g, \end{aligned} \tag{6.10}$$

is well-defined and bounded. Note that $\pi(\Lambda)g$ being a frame is equivalent to the frame operator S_g being a bounded, positive-definite and invertible operator on H_π. Note that any $f \in H_\pi$ admits the expansions

$$f = S_g S_g^{-1} f = \sum_{\lambda\in\Lambda} \langle f,\, S_g^{-1}\pi(\lambda)g \rangle \pi(\lambda)g \tag{6.11}$$

$$= S_g^{-1} S_g f = \sum_{\lambda\in\Lambda} \langle f,\, \pi(\lambda)g \rangle S_g^{-1}\pi(\lambda)g, \tag{6.12}$$

where the summations converge unconditionally.

A set $\pi(\Lambda)g$ is called a *Bessel sequence* if it verifies the Bessel bound, i.e. there exists $B > 0$ such that

$$\sum_{\lambda\in\Lambda} \left|\langle f,\, \pi(\lambda)g \rangle\right|^2 \le B \left\|f\right\|_{H_\pi}^2\,, \tag{6.13}$$

for each $f \in H_\pi$, and then the vector $g \in H_\pi$ is called a *Bessel vector.* In that case, the *coefficient operator*, given by

$$\begin{aligned} C_g \ : H_\pi &\longrightarrow \ell^2(\Lambda) \\ f \ &\longmapsto C_g f = (\langle f, \, \pi(\lambda) g \rangle)_{\lambda \in \Lambda} \,, \end{aligned} \tag{6.14}$$

is well-defined and bounded. Hence, its adjoint, the *reconstruction operator*

$$\begin{aligned} D_g \ : \ell^2(\Lambda) \qquad &\longrightarrow H_\pi \\ c = (c_\lambda)_{\lambda \in \Lambda} &\longmapsto C_g f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\Lambda) g, \end{aligned} \tag{6.15}$$

is also a well-defined and bounded operator from $\ell^2(\Lambda)$ to H_π. We denote the space of Bessel vectors by B_π.

Definition 6.2.2. A set $\pi(\Lambda)g$ is called a *Riesz sequence* in H_π, if there exist constants $A, \, B > 0$ such that

$$A \, \|c\|^2_{\ell^2(\Lambda)} \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g \right\| \leq B \, \|c\|^2_{\ell^2(\Lambda)} \tag{6.16}$$

holds for each $c = (c_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$. In that case the constants $A, \, B$ are called the *Riesz bounds* and the vector $g \in H_\pi$ is called a *Riesz vector.*

The first inequality in (6.16) provides that a Riesz sequence is linearly independent. Furthermore, if a Riesz sequence is complete, then it is called a *Riesz basis* for H_π.

7

Twisted Group C*-algebras

An application of the spectral invariance of the convolution-dominated matrices in non-commutative geometry is presented in this chapter. The result is applied to prove a Wiener type Lemma for the non-commutative space generated by linear combinations of representations of nilpotent Lie groups restricted to lattices.

Let Λ be a uniform lattice in the locally compact group, G, of polynomial growth with order of growth equal to $D_G > 0$. The lattice Λ is equipped with the counting measure,

$$\begin{aligned} \mu_C \ : \mathscr{P}(\Lambda) &\longrightarrow [0, \infty) \\ V \quad &\longmapsto \#V := \#\{x \in V\}, \end{aligned} \tag{7.1}$$

where $\mathscr{P}(\Lambda) := \{V \subseteq \Lambda\}$ is the power set of Λ. Recall that (Λ, μ_C) is a locally compact group of polynomial growth, with order of growth equal to D_G, by Lemma 3.1.18. For a fixed $\alpha \in \mathbb{N} \cup \{0\}$ we fix the weight function

$$w = w_\alpha : \Lambda \longrightarrow [1, \infty), \ x \mapsto (1 + \mathrm{d}(x, e))^\alpha.$$

Throughout this chapter, we assume that $\pi \ : \ \Lambda \ \longrightarrow \ \mathscr{U}(H_\pi)$ is the restriction, to the lattice Λ, of a discrete series σ-representation of G on the Hilbert space H_π, as defined in Chapter 6.1.

Recall from (6.13) that $\pi(\Lambda)g$ is a Bessel sequence, if there exists $B > 0$ such that

$$\sum_{\lambda \in \Lambda} |\langle f, \, \pi(\lambda) g \rangle|^2 \leq B \, \|f\|^2_{H_\pi},$$

for each $f \in H_\pi$ and in that case $g \in H_\pi$ is called a Bessel vector. Under the assumption that π is the restriction to Λ of a discrete series $\sigma-$representation, it can be shown that the Bessel vectors are dense in the Hilbert space H_π, see [39, Lemma 7.1.].

Lemma 7.1 ([39])**.** The Bessel vectors $B_\pi \subseteq H_\pi$ of the restriction π to Λ are norm dense in H_π.

Using the cocycle σ we define the following *σ-twisted convolution* on $\ell^1(\Lambda)$ for each $a, \, b \in \ell^1(\Lambda)$

$$a *_\sigma b(\lambda_1) = \sum_{\lambda_2 \in \Lambda} \sigma(\lambda_2, \, \lambda_2^{-1} \lambda_1) a(\lambda_2) b(\lambda_2^{-1} \lambda_1), \tag{7.2}$$

for $\lambda_1 \in \Lambda$, and the *σ-twisted involution*

$$a^{*_\sigma}(\lambda) = \overline{\sigma(\lambda, \, \lambda^{-1}) a(\lambda^{-1})}, \tag{7.3}$$

for $\lambda \in \Lambda$. The σ-twisted convolution of $a, \, b \in \ell^1(\Lambda)$ is well defined on $\ell^1(\Lambda)$, since for each $\lambda_1 \in \Lambda$ we have

$$|a *_\sigma b(\lambda_1)| \leq \sum_{\lambda_2 \in \Lambda} \left|\sigma(\lambda_2, \, \lambda_2^{-1} \lambda_1) a(\lambda_2) b(\lambda_2^{-1} \lambda_1)\right| = \sum_{\lambda_2 \in \Lambda} |a| \, (\lambda_2) \, |b| \, (\lambda_2^{-1} \lambda_1) = |a| * |b| \, (\lambda_1). \tag{7.4}$$

Thus, for $a, \, b \in \ell^1(\Lambda)$, using Young's inequality we obtain

$$\|a *_\sigma b\|_{\ell^1(\Lambda)} = \sum_{\lambda \in \Lambda} |a *_\sigma b(\lambda)| \leq \sum_{\lambda \in \Lambda} |a| * |b| \, (\lambda) = \| |a| * |b| \|_{\ell^1(\Lambda)} \leq \|a\|_{\ell^1(\Lambda)} \, \|b\|_{\ell^1(\Lambda)}.$$

We deduce that the sequence space $\ell^1(\Lambda)$ is a Banach *-algebra with the σ-twisted convolution and involution, and similarly we have that the weighted sequence space $\ell^1_w(\Lambda)$ is a Banach *-algebra with the σ-twisted convolution and involution.

For each $a \in \ell^1_w(\Lambda)$ we define the *twisted-convolution operator*

$$\begin{aligned} C_a^\sigma \ : \ \ell^2(\Lambda) &\longrightarrow \ell^2(\Lambda) \\ c \quad &\longmapsto c *_\sigma a. \end{aligned} \tag{7.5}$$

Note that for each $c \in \ell^2(\Lambda)$

$$\|C_a^\sigma c\|^2_{\ell^2(\Lambda)} = \|c *_\sigma a\|^2_{\ell^2(\Lambda)} = \sum_{\lambda\in\Lambda} |c *_\sigma a(\lambda)|^2 \leq \sum_{\lambda\in\Lambda} (|c| * |a|(\lambda))^2 = \||c| * |a|\|^2_{\ell^2(\Lambda)} \leq \|c\|^2_{\ell^2(\Lambda)} \|a\|^2_{\ell^1(\Lambda)},$$

by Young's inequality and hence $C_a^\sigma \in \mathscr{B}(\ell^2(\Lambda))$ is well-defined. We denote by $C^*(\ell^1)$ the C*-algebra generated by $\{C_a^\sigma \ : \ a \in \ell^1_w(\Lambda)\} \subseteq \mathscr{B}(\ell^2(\Lambda))$. From the inverse-closedness of the class of convolution-dominated matrices we deduce that the class of twisted-convolution operators is inverse-closed in $\mathscr{B}(\ell^2(\Lambda))$. This was shown by Gröchenig [28] for lattices in the Euclidean spaces $\mathbb{R}^d$ and his proof can be extended for uniform lattices in groups of polynomial growth.

Theorem 7.2. Let $a \in \ell^1_w(\Lambda)$ and

$$\begin{aligned} C_a^\sigma \ : \ \ell^2(\Lambda) &\longrightarrow \ell^2(\Lambda) \\ c \quad &\longmapsto c *_\sigma a. \end{aligned}$$

Suppose that C_a^σ is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Then there exists $b \in \ell^1_w(\Lambda)$, such that $(C_a^\sigma)^{-1} = C_b^\sigma$.

Proof. For each $\lambda \in \Lambda$ and $c \in \ell^2(\Lambda)$ observe that

$$C_a^\sigma c(\lambda) = c *_\sigma a(\lambda) = \sum_{\gamma\in\Lambda} \sigma(\gamma, \gamma^{-1}\lambda) c(\gamma) a(\gamma^{-1}\lambda)$$

Hence, the action of C_a^σ can be interpreted as the matrix action

$$A := (A(\lambda, \gamma))_{\lambda, m\in\Lambda} = \left(\sigma(\gamma, \gamma^{-1}\lambda) a(\gamma^{-1}\lambda)\right)_{\lambda,\gamma\in\Lambda} \in \mathbb{C}^{\Lambda\times\Lambda}. \tag{7.6}$$

Then, we have

$$|A(\lambda, \gamma)| = \left|a(\gamma^{-1}\lambda)\right| = |a^\vee|(\lambda^{-1}\gamma).$$

Recall that the two-sided Amalgam space of continuous functions on the discrete group Λ is the space

$$W_w(\Lambda) = \ell^1_w(\Lambda). \tag{7.7}$$

Thus, since $a \in \ell^1_w(\Lambda)$ and by the definition of the convolution-dominated matrices on Λ we have $A \in CD_w(\Lambda)$ and

$$\|C_a^\sigma\|_{CD_w(\Lambda)} = \|A\|_{CD_w(\Lambda)} = \|a^\vee\|_{W_w(\Lambda)} = \|a^\vee\|_{\ell^1_w(\Lambda)} = \|a\|_{\ell^1_w(\Lambda)}. \tag{7.8}$$

Using the spectral invariance of convolution-dominated matrices indexed by discrete groups of polynomial growth in the operator algebra $\mathscr{B}(\ell^2(\Lambda))$, that is by applying Theorem 5.2.4 for $G = \Lambda$, we deduce that there exists $B \in CD_w(\Lambda)$ such that $(C_a^\sigma)^{-1} = B$.

Moreover, by the invertibility of C_a^σ in $\mathscr{B}(\ell^2(\Lambda))$ there exist $b \in \ell^2(\Lambda)$ such that

$$b *_\sigma a = C_a^\sigma b = \delta_e.$$

We have that the operator C_b^σ is defined on the sequences with finite support, $c_{00}(\Lambda)$, and for each $c \in c_{00}(\Lambda)$ we obtain

$$C_a^\sigma(C_b^\sigma - B)c = C_a^\sigma(c *_\sigma b - Bc) = c *_\sigma b *_\sigma a - C_a^\sigma Bc = c *_\sigma \delta_e - c = 0.$$

Hence, by injectivity of C_a^σ on $\ell^2(\Lambda)$ we obtain that $(C_b^\sigma - B)c = 0$ for each $c \in c_{00}(\Lambda)$ and by the density of $c_{00}(\Lambda)$ in $\ell^2(\Lambda)$ we have that $C_b^\sigma = B$ in $\mathscr{B}(\ell^2(\Lambda))$. Then, by following the same analysis we used to deduce Equation (7.8) we have

$$\|b\|_{\ell^1_w(\Lambda)} = \|C_b^\sigma\|_{CD_w(\Lambda)} = \|B\|_{CD_w(\Lambda)}. \tag{7.9}$$

Thus, $b \in \ell^1_w(\Lambda)$ since $B \in CD_w(\Lambda)$. □

For $a = (a_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ the formal sum $\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda)$ is denoted by

$$\pi(a) = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda). \tag{7.10}$$

For $a \in \ell^1_w(\Lambda)$ we obtain

$$\|\pi(a)\|_{\mathscr{B}(H_\pi)} = \left\| \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) \right\|_{\mathscr{B}(H_\pi)} \le \sum_{\lambda\in\Lambda} |a_\lambda| \, \|\pi(\lambda)\|_{\mathscr{B}(H_\pi)} \le \sum_{\lambda\in\Lambda} |a_\lambda| = \|a\|_{\ell^1(\Lambda)} \le \|a\|_{\ell^1_w(\Lambda)},$$

since $\pi(\lambda) \in \mathscr{U}(H_\pi)$ for each $\lambda \in \Lambda$. Thus, the integrated representation π of $\ell^1_w(\Lambda)$ given by Equation (7.10) is well-defined. Note that for each $a = (a_\lambda)_{\lambda\in\Lambda}$, $b = (b_\lambda)_{\lambda\in\Lambda} \in \ell^1(\Lambda)$ we have

$$\begin{aligned}\pi(a)\pi(b) &= \left(\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda)\right)\left(\sum_{\lambda'\in\Lambda} b_{\lambda'} \pi(\lambda')\right) = \sum_{\lambda,\lambda'\in\Lambda} a_\lambda \pi(\lambda) b_{\lambda'} \pi(\lambda') \\ &= \sum_{\lambda,\lambda'\in\Lambda} a_\lambda b_{\lambda'} \sigma(\lambda,\lambda')\pi(\lambda\lambda').\end{aligned}$$

By using the change of variable $\lambda' \mapsto \lambda^{-1}m$ and by changing the order of summation we obtain

$$\begin{aligned}\pi(a)\pi(b) &= \sum_{\lambda\in\Lambda}\sum_{m\in\Lambda} a_\lambda b_{\lambda^{-1}m} \sigma(\lambda,\lambda^{-1}m)\pi(m) = \sum_{m\in\Lambda}\left(\sum_{\lambda\in\Lambda} \sigma(\lambda,\lambda^{-1}m) a_\lambda b_{\lambda^{-1}m}\right)\pi(m) \\ &= \sum_{m\in\Lambda} (a *_\sigma b)_m \pi(m) = \pi(a *_\sigma b),\end{aligned}$$

for each $a = (a_\lambda)_{\lambda\in\Lambda}$, $b = (b_\lambda)_{\lambda\in\Lambda} \in \ell^1(\Lambda)$. Moreover, for each $a = (a_\lambda)_{\lambda\in\Lambda} \in \ell^1(\Lambda)$ and $f, h \in H_\pi$ we have

$$\begin{aligned}\langle \pi(a)f, h\rangle &= \left\langle \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) f, h \right\rangle = \sum_{\lambda\in\Lambda} a_\lambda \langle \pi(\lambda) f, h\rangle = \sum_{\lambda\in\Lambda} a_\lambda \langle f, \pi(\lambda)^* h\rangle \\ &= \sum_{\lambda\in\Lambda} a_\lambda \langle f, \overline{\sigma(\lambda^{-1},\lambda)} \pi(\lambda^{-1}) h\rangle = \left\langle f, \sum_{\lambda\in\Lambda} \overline{\sigma(\lambda^{-1},\lambda) a_\lambda} \pi(\lambda^{-1}) h \right\rangle \\ &= \left\langle f, \sum_{\lambda\in\Lambda} a^{*_\sigma}_{\lambda^{-1}} \pi(\lambda^{-1}) h \right\rangle = \left\langle f, \sum_{\lambda\in\Lambda} a^{*_\sigma}_{\lambda} \pi(\lambda) h \right\rangle = \langle f, \pi(a^{*_\sigma}) h\rangle,\end{aligned}$$

using that Λ is a discrete group and $a^{*_\sigma}(\lambda) = \overline{\sigma(\lambda,\, \lambda^{-1}) a(\lambda^{-1})}$, for each $\lambda \in \Lambda$. Thus,

$$\pi(a)^* = \pi(a^{*_\sigma}). \tag{7.11}$$

We then define the non-commutative space

$$\mathscr{A}^1_w := \pi(\ell^1_w(\Lambda)) \left\{ A \in \mathscr{B}(H_\pi) \ : \ A = \pi(a) = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda), \ a \in \ell^1_w(\Lambda) \right\}, \tag{7.12}$$

where $w = w_\alpha : \Lambda \longrightarrow [1,\infty), \ x \mapsto (1 + \mathrm{d}(x,e))^\alpha$ and the vector subspace

$$\mathscr{A}^0 := \pi(c_{00}(\Lambda)) = \left\{ A \in \mathscr{B}(H_\pi) \ : \ A = \pi(a) = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda), \ a \in c_{00}(\Lambda) \right\}, \tag{7.13}$$

where $c_{00}(\Lambda)$ is the set of all sequences $a \in \ell^\infty(\Lambda)$ with finitely many non zero $a(\lambda)$, $\lambda \in \Lambda$. We denote with $C^*(\Lambda, \pi)$ the closure of $\mathscr{A}^0$ in the operator norm on H_π, i.e.

$$C^*(\Lambda,\pi) = \overline{\mathscr{A}^0}^{\|\cdot\|_{\mathscr{B}(H_\pi)}}.$$

Since every closed subalgebra of a C*-algebra is again also a C*−algebra, it follows that $C^*(\Lambda, \pi)$ is a C*-algebra. Furthermore, for $S \subseteq \mathscr{B}(H_\pi)$ we denote the *commutant* by

$$S^{'} = \{A \in \mathscr{B}(H_\pi) \ : \ AB = BA, \forall B \in S\},$$

or, equivalently, $S^{'}$ is the set of all operators that commute with the elements of S. By von Neumann's double commutant theorem, see e.g. [9, Theorem 6.4.], we obtain that

$$\left(C^*(\Lambda, \pi)\right)^{''} = \overline{C^*(\Lambda, \pi)}^{\text{sot}} = \overline{C^*(\Lambda, \pi)}^{\text{wot}},$$

where $\overline{\cdot}^{\text{sot}}$ and $\overline{\cdot}^{\text{wot}}$ denote the closure with respect to the strong and weak operator norm respectively. Since by definition $C^*(\Lambda, \pi)$ is the closure of $\mathscr{A}^0$ we deduce that the von Neumann algebra $\pi(\Lambda)^{''}$ is the double commutant of the C*-algebra $C^*(\Lambda, \pi)$,

$$\pi(\Lambda)^{''} = \left(C^*(\Lambda, \pi)\right)^{''}. \tag{7.14}$$

Since the norm topology is stronger than the strong operator topology (sot) and weak operator topology (wot), i.e. if a net is convergent in norm then it is convergent in sot and wot, we deduce that

$$\overline{C^*(\Lambda, \pi)}^{\text{sot}} = \overline{\overline{\mathscr{A}^0}^{\|\cdot\|_{\mathscr{B}(H_\pi)}}}^{\text{sot}} = \overline{\mathscr{A}^0}^{\text{sot}} \tag{7.15}$$

and similarly

$$\overline{C^*(\Lambda, \pi)}^{\text{wot}} = \overline{\mathscr{A}^0}^{\text{wot}}. \tag{7.16}$$

Thus, combining the previous we obtain

$$\pi(\Lambda)^{''} = \left(C^*(\Lambda, \pi)\right)^{''} = \overline{\mathscr{A}^0}^{\text{sot}} = \overline{\mathscr{A}^0}^{\text{wot}}. \tag{7.17}$$

The following result given by Caspers and Van Velthoven [5] provides a Fourier-type expansion of each operator in the von Neumann algebra $\pi(\Lambda)^{''}$ on the subspace of Bessel vectors, under the assumption that (π, H_π) is a discrete series σ−representation of G.

Proposition 7.3 ([5]). If $T \in \pi(\Lambda)^{''}$ then there exists $c = (c_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ such that

$$Tg = \pi(c)g = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g, \tag{7.18}$$

for each $g \in B_\pi$.

In general, the coefficients given in Equation (7.18) do not need to be unique, e.g. when the projective kernel $p\ker \pi := \{x \in G \ : \ \pi(x) \in \mathbb{T}I_{H_\pi}\}$ is not trivial, $p\ker \pi \neq \{e\}$, and $p\ker \pi \cap \Lambda \neq \{e\}$. Motivated by this, we call the pair (π, Λ) a *uniqueness pair* if the expansion given by Proposition 7.3 is unique for each $T \in \pi(\Lambda)^{''}$. The following result explains the importance of (π, Λ) being a uniqueness pair.

Lemma 7.4. Suppose that the pair (π, Λ) is a uniqueness pair. If $T \in \pi(\Lambda)^{''}$, $c \in \ell^2(\Lambda)$ and

$$Tg = \pi(c)g = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g = 0$$

for each $g \in B_\pi \subseteq H_\pi$, then $c = 0$.

Proof. We have $T = \pi(c) \in \pi(\Lambda)$ and $T = \pi(c) = 0$ on the dense subspace B_π. Since (π, Λ) is a uniqueness pair then by the uniqueness of the expansion it follows that $c = 0$. □

We present some examples for which the (π, Λ) is a uniqueness pair.

Remark 7.5. The pair (π, Λ) is a uniqueness pair in the following cases:

(i) (Λ, σ) satisfies Kleppner's condition, see [41]. We say that the pair (Λ, σ) satisfies *Kleppner's condition* if the conjugacy class $C_\Lambda(\lambda) := \left\{\gamma\lambda\gamma^{-1} \ : \ \gamma \in \Lambda\right\}$ of any σ−regular element $\lambda \in \Lambda \backslash \{e\}$ is infinite, see [39].

(ii) if the projective representation $\big(\pi, L^2(\mathbb{R}^d)\big)$ of $\mathbb{R}^{2d}$ is defined by

$$\pi(x,\xi) \;:\; L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),\; f \mapsto e^{2\pi i \xi \cdot} f(\cdot - x)$$

and Λ is a lattice in $\mathbb{R}^{2d}$, see [32, Lemma 3.3.].

(iii) the group G verifies the condition $B(G) = Z(G)$ and π has a trivial projective kernel. We denote by $B(G)$ the set of all $x \in G$ for which the conjugacy class $C_G(x) = \big\{yxy^{-1} \;:\; y \in G\big\}$ has a compact closure and the projective kernel of the representation π is defined by $p\ker\pi = \{x \in G \;:\; \pi(x) \in \mathbb{T} I_{H_\pi}\}$. By combining results from [5] and [13] it can be shown that under these assumptions (π, Λ) is a uniqueness pair. In particular, this is satisfied for simply connected nilpotent Lie groups, see [13].

We will prove a Wiener type Lemma for the non-commutative space $\mathscr{A}^1_w$ in the C*-algebra $C^*(\Lambda,\pi)$, by using an adaptation of the method proposed by Gröchenig [28] and used by Gröchenig and Leinert in [32] in the case of time-frequency shifts.

Theorem 7.6 (Spectral Invariance)**.** Suppose that the pair (π, Λ) is a uniqueness pair. If $a \in \ell^1_w(\Lambda)$ is such that $\pi(a)$ is invertible in $\mathscr{B}(H_\pi)$, then there exists $b \in \ell^1_w(\Lambda)$ such that $\pi(a)^{-1} = \pi(b)$.

In order to prove Theorem 7.6, we first prove some useful norm estimates for $\pi(a)$, where $a \in \ell^1_w(\Lambda)$. The proof of the following estimate is similar to a result given by Gröchenig and Leinert, see [32, Lemma 3.4.].

Lemma 7.7. For each $a \in \ell^2(\Lambda)$ we have

$$\|a\|_{\ell^2(\Lambda)} \leq \|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))}\,. \tag{7.19}$$

Moreover, for each $a \in \ell^1_w(\Lambda)$ we have

$$\|\pi(a)\|_{\mathscr{B}(H_\pi)} \leq \|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))}\,. \tag{7.20}$$

Proof. For the first estimate assume $a \in \ell^2(\Lambda)$. Then, since

$$\|C^\sigma_a \delta_e\|_{\ell^2(\Lambda)} = \|a\|_{\ell^2(\Lambda)}$$

we deduce

$$\|a\|_{\ell^2(\Lambda)} \leq \|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))}\,. \tag{7.21}$$

We will now prove the second estimate. Let $a \in \ell^1_w(\Lambda)$. Let $z \notin \sigma_{\mathscr{B}(\ell^2(\Lambda))}(C^\sigma_a)$. Then by the spectral invariance of twisted convolution operators, see Theorem 7.2, there exists $b \in \ell^1_w(\Lambda)$ such that $(C^\sigma_{z-a})^{-1} = C^\sigma_b$. Then since

$$\|C^\sigma_b\|_{\mathscr{B}(\ell^1(\Lambda))} = \|b\|_{\ell^1(\Lambda)}\,,$$

we deduce $C^\sigma_b \in \mathscr{B}(\ell^1(\Lambda))$ and hence $z \notin \sigma_{\mathscr{B}(\ell^1(\Lambda))}(C^\sigma_a)$. Thus,

$$\sigma_{\mathscr{B}(\ell^1(\Lambda))}(C^\sigma_a) \subseteq \sigma_{\mathscr{B}(\ell^2(\Lambda))}(C^\sigma_a) \tag{7.22}$$

and

$$\mathrm{r}_{\mathscr{B}(\ell^1(\Lambda))}(C^\sigma_a) \leq \mathrm{r}_{\mathscr{B}(\ell^2(\Lambda))}(C^\sigma_a). \tag{7.23}$$

Let $a \in \ell^1_w(\Lambda)$. Then

$$\|\pi(a)\|_{\mathscr{B}(H_\pi)} = \Bigg\|\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda)\Bigg\|_{\mathscr{B}(H_\pi)} \leq \sum_{\lambda\in\Lambda} |a_\lambda|\,\|\pi(\lambda)\|_{\mathscr{B}(H_\pi)} \leq \|a\|_{\ell^1(\Lambda)} = \|C^\sigma_a\|_{\mathscr{B}(\ell^1(\Lambda))}\,,$$

since $\pi(\lambda) \in \mathscr{U}(H_\pi)$ for each $\lambda \in \Lambda$. Hence

$$\begin{aligned}\mathrm{r}_{\mathscr{B}(H_\pi)}(\pi(a)) &= \lim_{n\to\infty} \|\pi(a)^n\|^{1/n}_{\mathscr{B}(H_\pi)} = \lim_{n\to\infty} \|\pi(a *_\sigma \ldots *_\sigma a)\|^{1/n}_{\mathscr{B}(H_\pi)} \\ &\leq \lim_{n\to\infty} \big\|C^\sigma_{a*_\sigma\ldots *_\sigma a}\big\|^{1/n}_{\mathscr{B}(H_\pi)} = \lim_{n\to\infty} \|(C^\sigma_a)^n\|^{1/n}_{\mathscr{B}(\ell^1(\Lambda))} = \mathrm{r}_{\mathscr{B}(\ell^1(\Lambda))}(C^\sigma_a).\end{aligned}$$

Thus,

$$\|\pi(a)\|^2_{\mathscr{B}(H_\pi)} = \|\pi(a)^*\pi(a)\|_{\mathscr{B}(H_\pi)} = \mathrm{r}_{\mathscr{B}(H_\pi)}(\pi(a)^*\pi(a)) \leq \mathrm{r}_{\mathscr{B}(\ell^1(\Lambda))}\left((C^\sigma_a)^* C^\sigma_a\right).$$

Using Equation (7.23) we obtain

$$\|\pi(a)\|^2_{\mathscr{B}(H_\pi)} \leq \mathrm{r}_{\mathscr{B}(\ell^1(\Lambda))}((C^\sigma_a)^* C^\sigma_a) \leq \mathrm{r}_{\mathscr{B}(\ell^2(\Lambda))}((C^\sigma_a)^* C^\sigma_a) = \|(C^\sigma_a)^* C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))} = \|C^\sigma_a\|^2_{\mathscr{B}(\ell^2(\Lambda))}\,.$$

This proves our claim. ☐

The following equality of norms is essential in applying Hulanicki's Lemma (Theorem 2.2.3) for the proof of spectral invariance (Theorem 7.6). We follow the proof given in [32, Theorem 3.1].

Lemma 7.8. Suppose that the pair (π, Λ) is a uniqueness pair. Then, for each $a \in \ell^1_w(\Lambda)$

$$\|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))} = \|\pi(a)\|_{\mathscr{B}(H_\pi)}\,. \tag{7.24}$$

and there exist an isometric *-homomorphism $h : C^*(\ell^1) \longrightarrow C^*(\Lambda)$ such that $h(C^\sigma_a) = \pi(a)^*$, for each $a \in \ell^1_w(\Lambda)$.

Proof. Let $A \in C^*(\ell^1)$. Then, there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \ell^1_w(\Lambda)$ such that

$$\left\|A - C^\sigma_{a_n}\right\|_{\mathscr{B}(\ell^2(\Lambda))} \xrightarrow{n\to\infty} 0.$$

From the previous lemma we have $\|b\|_{\ell^2(\Lambda)} \le \|C^\sigma_b\|_{\mathscr{B}(\ell^2(\Lambda))}$ for each $b \in \ell^2(\Lambda)$, hence we obtain

$$\|a_n - a_m\|_{\ell^2(\Lambda)} \le \left\|C^\sigma_{a_n - a_m}\right\|_{\mathscr{B}(\ell^2(\Lambda))} = \left\|C^\sigma_{a_n} - C^\sigma_{a_m}\right\|_{\mathscr{B}(\ell^2(\Lambda))}.$$

Thus, by convergence of $\left\{C^\sigma_{a_n}\right\}_{n\in\mathbb{N}}$ we deduce that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\ell^2(\Lambda)$. Thus, there exists $a \in \ell^2(\Lambda)$ such that $\|a_n - a\|_{\ell^2(\Lambda)} \xrightarrow{n\to\infty} 0$. Then for each $c \in \ell^1(\Lambda)$ we obtain

$$\begin{aligned}\|(A - C^\sigma_a)c\|_{\ell^2(\Lambda)} &\le \left\|(A - C^\sigma_{a_n})c\right\|_{\ell^2(\Lambda)} + \left\|(C^\sigma_{a_n} - C^\sigma_a)c\right\|_{\ell^2(\Lambda)} \\ &\le \left\|A - C^\sigma_{a_n}\right\|_{\mathscr{B}(\ell^2(\Lambda))} \|c\|_{\ell^1(\Lambda)} + \|a_n - a\|_{\ell^2(\Lambda)} \|c\|_{\ell^1(\Lambda)} \xrightarrow{n\to\infty} 0,\end{aligned}$$

and hence

$$Ac = C^\sigma_a c = c *_\sigma a, \tag{7.25}$$

for each $c \in \ell^1(\Lambda)$. Since $\ell^1(\Lambda)$ is dense in $\ell^2(\Lambda)$ we deduce that $A = C^\sigma_a = \cdot *_\sigma a$ in $\mathscr{B}(\ell^2(\Lambda))$. From Lemma 7.7 we have

$$\|\pi(a_n) - \pi(a_m)\|_{\mathscr{B}(H_\pi)} = \|\pi(a_n - a_m)\|_{\mathscr{B}(H_\pi)} \le \left\|C^\sigma_{a_n - a_m}\right\|_{\mathscr{B}(\ell^2(\Lambda))} \le \left\|C^\sigma_{a_n} - C^\sigma_{a_m}\right\|_{\mathscr{B}(\ell^2(\Lambda))},$$

hence we deduce that $(\pi(a_n))_n \subseteq C^*(\Lambda, \pi)$ is a Cauchy sequence since $\left\|A - C^\sigma_{a_n}\right\|_{\mathscr{B}(\ell^2(\Lambda))} \xrightarrow{n\to\infty} 0$. Thus, there exists a unique $T \in C^*(\Lambda, \pi)$ such that $\|\pi(a_n) - T\|_{\mathscr{B}(H_\pi)} \xrightarrow{n\to\infty} 0$. Moreover, recall that if $g \in B_\pi$ then $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Bessel sequence, hence there exists $B := B(g) > 0$ such that

$$\left\|\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g\right\|^2_{H_\pi} \le B\,\|c\|^2_{\ell^2(\Lambda)}\,,$$

for each $c \in \ell^2(\Lambda)$. Therefore, for each $g \in B_\pi$ we deduce

$$\begin{aligned}\|(\pi(a) - T)g\|_{H_\pi} &\le \|(\pi(a) - \pi(a_n))g\|_{H_\pi} + \|\pi(a_n) - T\|_{\mathscr{B}(H_\pi)} \|g\|_{H_\pi} \\ &\le B^{1/2}\,\|a - a_n\|_{\ell^2(\Lambda)} + \|\pi(a_n) - T\|_{\mathscr{B}(H_\pi)} \|g\|_{H_\pi} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus, $T = \pi(a)$ on B_π.

We have shown that for each $A \in C^*(\ell^1)$ there exists a unique $T \in C^*(\Lambda, \pi)$. Define

$$h \;:\; C^*(\ell^1) \longrightarrow C^*(\Lambda),$$

as the mapping such that $h(A) = T^*$, for A and T as defined above. Note that by construction of h we have $h(C^\sigma_a) = \pi(a)^*$ for $a \in \ell^1_w(\Lambda)$, by choosing the constructing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \ell^1_w(\Lambda)$ as $a_n = a$ for each $n \in \mathbb{N}$.

We will now prove that h is a *-homomorphism. Recall that for each $a,\, b \in \ell^1_w(\Lambda)$ we have

$$\pi(a *_\sigma b) = \pi(a)\pi(b) \;\text{ and }\; \pi(a)^* = \pi(a^*_\sigma). \tag{7.26}$$

Let $A,\, B \in C^*(\ell^1)$. Since for each $A,\, B \in C^*(\ell^1)$, from the analysis above there exist $(a_n)_{n\in\mathbb{N}} \subseteq \ell^1_w(\Lambda)$ and $(b_n)_{n\in\mathbb{N}} \subseteq \ell^1_w(\Lambda)$ such that

$$\|h(A) - \pi(a_n)^*\|_{\mathscr{B}(H_\pi)} \xrightarrow{n\to\infty} 0$$

and

$$\|h(B) - \pi(b_n)^*\|_{\mathscr{B}(H_\pi)} \xrightarrow{n\to\infty} 0.$$

Moreover, there exist $a,\, b \in \ell^2(\Lambda)$ such that $A = C^\sigma_a$ and $B = C^\sigma_b$ in $\mathscr{B}(\ell^2(\Lambda))$ and $h(A) = \pi(a)^*$ and $h(B) = \pi(b)^*$ on B_π. Then, using Equation (7.26) we obtain

$$\begin{aligned} h(AB) &= h(C^\sigma_a C^\sigma_b) = h(C^\sigma_{b *_\sigma a}) = \pi(b *_\sigma a)^* = \lim_{n\to\infty} \pi(b_n *_\sigma a_n)^* \\ &= \lim_{n\to\infty} (\pi(b_n)\pi(a_n))^* = \lim_{n\to\infty} \pi(a_n)^*\pi(b_n)^* = \pi(a)^*\pi(b)^* \\ &= h(C^\sigma_a)h(C^\sigma_b) = h(A)h(B) \end{aligned}$$

on B_π, where the limits are taken in $\mathscr{B}(H_\pi)$. By density of the Bessel vectors B_π (see Lemma 7.1) we deduce that

$$h(AB) = h(A)h(B) \tag{7.27}$$

on H_π and hence h is an homomorphism.

Furthermore, we have

$$\begin{aligned} h(A^*) &= h((C^\sigma_a)^*) = h(C^\sigma_{a^{*\sigma}}) = \pi(a^{*\sigma})^* = \lim_{n\to\infty} \pi(a_n^{*\sigma})^* \\ &= \lim_{n\to\infty} \pi(a_n) = \pi(a) = (\pi(a)^*)^* = h(C^\sigma_a)^* = h(A)^*, \end{aligned}$$

on B_π, where the limits are taken in $\mathscr{B}(H_\pi)$ and using once more the density of the Bessel vectors B_π we deduce that

$$h(A^*) = h(A)^* \tag{7.28}$$

on H_π and hence h is an *-homomorphism.

Suppose that $T^* = h(A) \in C^*(\Lambda, \pi)$ and $T = 0$. Then by definition of h there exists $a \in \ell^2(\Lambda)$ such that $A = C^\sigma_a$ and $T = \pi(a)$ on B_π. Hence, $\pi(a) = 0$ on B_π. Since $T = 0$, we have $T \in \pi(\Lambda)''$ and since $T = \pi(a)$ on B_π then it follows by Lemma 7.4 that $a = 0$. From the previous we deduce that h is injective. Since every injective *-homomorphism between C* algebras is isometric, see e.g. [46, Theorem 3.1.5.], we conclude that h is isometric. Thus, for each $a \in \ell^1_w(\Lambda)$ we obtain

$$\|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))} = \|h(C^\sigma_a)\|_{\mathscr{B}(H_\pi)} = \|\pi(a)^*\|_{\mathscr{B}(H_\pi)} = \|\pi(a)\|_{\mathscr{B}(H_\pi)}\,. \tag{7.29}$$

□

Suppose $a \in \ell^1_w(\Lambda)$ and $(C^\sigma_a)^* = C^\sigma_a$, i.e. $a = a^{*\sigma}$. Then from Lemma 7.8 we have

$$\mathrm{r}_{\mathscr{B}(\ell^2(\Lambda))}(C^\sigma_a) = \|C^\sigma_a\|_{\mathscr{B}(\ell^2(\Lambda))} = \|h(C^\sigma_a)\|_{\mathscr{B}(H_\pi)} = \|\pi(a)\|_{\mathscr{B}(H_\pi)}\,,$$

where $h : C^*(\ell^1) \longrightarrow C^*(\Lambda)$ is the isometric *-homomorphism defined in Lemma 7.8. Then by Theorem 2.2.3 we obtain

$$\sigma_{\mathscr{B}(\ell^2(\Lambda))}(C^\sigma_a) = \sigma_{\mathscr{B}(H_\pi)}(\pi(a)), \tag{7.30}$$

for each $a = a^{*\sigma}$. Now suppose that $a \in \ell^1_w(\Lambda)$ is such that $\pi(a)$ is invertible in $\mathscr{B}(H_\pi)$. Then $\pi(a^{*\sigma} *_\sigma a) = \pi(a)^*\pi(a)$ is invertible in $\mathscr{B}(H_\pi)$. From Equation (7.30) we deduce that $C^\sigma_{a^{*\sigma} *_\sigma a}$ is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Similarly we deduce that $C^\sigma_{a *_\sigma a^{*\sigma}}$ is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Then $C^\sigma_{a^{*\sigma}}(C^\sigma_{a^{*\sigma} *_\sigma a})^{-1}$ is a right inverse and $(C^\sigma_{a *_\sigma a^{*\sigma}})^{-1} C^\sigma_{a^{*\sigma}}$ is a left inverse of C^σ_a in $\mathscr{B}(\ell^2(\Lambda))$. Thus if $\pi(a)$ is invertible in $\mathscr{B}(H_\pi)$ then C^σ_a is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Similarly, we deduce that if C^σ_a is invertible in $\mathscr{B}(\ell^2(\Lambda))$ then $\pi(a)$ is invertible in $\mathscr{B}(H_\pi)$. Thus,

$$\pi(a) = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) \in \mathrm{Inv}(\mathscr{B}(H_\pi)) \iff C^\sigma_a \in \mathrm{Inv}(\mathscr{B}(\ell^2(\Lambda))). \tag{7.31}$$

Using the previous we will now prove Theorem 7.6.

Proof of Theorem 7.6. Let $a \in \ell^1_w(\Lambda)$ and $A = \pi(a) \in \mathscr{A}^1_w$. Assume that $A = \pi(a)$ is invertible in $\mathscr{B}(H_\pi)$. Then from (7.31), it follows that C^σ_a is invertible in $\mathscr{B}(\ell^2(\Lambda))$. Hence, since twisted convolution operators are inverse-closed in $\mathscr{B}(\ell^2(\Lambda))$ (see Theorem 7.2) we deduce that there exists $b \in \ell^1_w$ such that $(C^\sigma_a)^{-1} = C^\sigma_b$ and $a *_\sigma b = \delta_e = b *_\sigma a$. Thus,

$$\pi(a)\pi(b) = \pi(a *_\sigma b) = \pi(\delta_e) = I \tag{7.32}$$

and similarly

$$\pi(b)\pi(a) = \pi(b *_\sigma a) = \pi(\delta_e) = I, \tag{7.33}$$

hence $\pi(a)^{-1} = \pi(b) \in \mathscr{B}(H_\pi)$. This proves the theorem. □

Define the non-commutative space

$$\mathscr{A}^1_\infty := \left\{ A \in \mathscr{B}(H_\pi) \ : \ A = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda),\ a \in \ell^1_{w_\alpha}(\Lambda)\ , \forall \alpha \in \mathbb{N} \right\}, \tag{7.34}$$

where $w_\alpha : \Lambda \longrightarrow [1,\infty),\ x \mapsto (1 + \mathrm{d}(x,e))^\alpha$. Applying Theorem 7.6 for $\mathscr{A}^1_{w_\alpha}$ for each $\alpha \in \mathbb{N}$ we obtain Wiener type Lemma of $\mathscr{A}^1_{w_\alpha}$ in $C^*(\Lambda,\pi)$. Furthermore, taking the intersection of $\mathscr{A}^1_{w_\alpha},\ \alpha \in \mathbb{N}$, we deduce a Wiener type Lemma for the smooth non-commutative space $\mathscr{A}^1_\infty = \bigcap_{\alpha\in\mathbb{N}} \mathscr{A}^1_{w_\alpha}$ in $C^*(\Lambda,\pi)$.

Theorem 7.9. Suppose that the pair (π,Λ) is a uniqueness pair. If $A \in \mathscr{A}^1_\infty$ such that $A^{-1} \in \mathscr{B}(H_\pi)$, then $A^{-1} \in \mathscr{A}^1_\infty$.

The previous recovers the Wiener type Lemma for the smooth non-commutative torus given by Connes [8], where the smooth non-commutative torus is defined by (7.34), where Λ is a lattice in $\mathbb{R}^{2d}$ and $\left(\pi, L^2(\mathbb{R}^d)\right)$ is the projective representation of time-frequency shifts.

8

Frames in Coorbit spaces

In this chapter, we present applications of the ℓ^p-stability and spectral invariance of convolution-dominated matrices in frame theory. We prove that if $\pi(\Lambda)g$ is a p-frame on the coorbit space $Co(L^p)$ for some $p \in [1,\infty]$, then $\pi(\Lambda)g$ is a q-frame on the coorbit space $Co(L^q)$ for each $q \in [1,\infty]$. Moreover, we show that the frame operator of a frame $\pi(\Lambda)g$ for H_π is not only invertible over the Hilbert space H_π, but also as an operator on the coorbit spaces $Co(L^p)$.

Throughout this chapter we assume that (π, H_π) is a discrete series σ-representation of the group G of polynomial growth. For each $\alpha \in \mathbb{N} \cup \{0\}$, we denote by w_α the weight function on G given by

$$w_\alpha : G \longrightarrow [1,\infty),\ x \mapsto (1 + \mathrm{d}(x,e))^\alpha.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space H_π and we define the following subspace

$$A^1_\pi = \left\{ g \in H_\pi \ : \ V_g g \in L^1(G) \right\}. \tag{8.1}$$

Moreover, for a vector $g \in A^1_\pi \backslash \{0\}$ we define

$$H^1 = H^1(g) = \left\{ f \in H_\pi \ : \ V_g f \in L^1(G) \right\} \tag{8.2}$$

and equip H^1 with the norm

$$\|f\|_{H^1} := \|V_g f\|_{L^1(G)}. \tag{8.3}$$

Then, H^1 is a Banach space with respect to the norm $\|\cdot\|_{H^1}$, see [7, 15, 65], and $H^1(g)$ is independent of the choice of g with equivalent norms $\|\cdot\|_{H^1(g)} \asymp \|\cdot\|_{H^1(g')}$ for $g, g' \in A^1_\pi \backslash \{0\}$, see [7, 15, 65]. Moreover, it can be shown that $H^1(g)$ is continuously embedded in H_π and $H^1(g)$ is norm dense in H_π, see e.g. [65, Lemma 4.3].

Let $R^1 = \left(H^1\right)^* = \left(H^1(g)\right)^*$ be the anti-dual of H^1 as a Banach space, i.e. R^1 is the set of all anti-linear functionals on the Banach space $H^1(g)$. We denote the anti-linear pairing for each $f \in R^1$ and $h \in H^1$ by

$$\langle f, h \rangle_{R^1, H^1} = f(h). \tag{8.4}$$

We have that H_π and H^1 are continuously embedded in R^1, $H_\pi \hookrightarrow R^1$ and $H^1 \hookrightarrow R^1$, see e.g [65, Lemma 4.6]. Moreover, the pairing $\langle \cdot, \cdot \rangle_{R^1, H^1}$ extends the inner product $\langle \cdot, \cdot \rangle$, that is for each $f \in H_\pi$ and $h \in H^1$ we have

$$\langle f, h \rangle_{R^1, H^1} = \langle f, h \rangle, \tag{8.5}$$

using the embedding $H_\pi \hookrightarrow R^1$, see e.g [65, Lemma 4.6].

For a vector $g \in A^1_\pi \backslash \{0\}$ and $p \in [1,\infty]$ we define the *coorbit space* of $L^p(G)$,

$$Co(L^p) = Co_g(L^p) = \left\{ f \in R^1 \ : \ V_g f = \langle f, \pi(\cdot) g \rangle_{R^1, H^1} \in L^p(G) \right\}, \tag{8.6}$$

equipped with the norm

$$\|f\|_{Co(L^p)} = \|V_g f\|_{L^p(G)}. \tag{8.7}$$

The coorbit spaces are Banach spaces and are independent of the choice of $g \in A^1_\pi \backslash \{0\}$, with equivalent norms [7, 15, 16, 65]. Moreover, it can be shown that $Co(L^2) = H_\pi$, see e.g. [15, Corollary 4.4]. For more details on coorbit spaces we refer the reader to [7, 15, 16, 65].

For a vector $g \in A^1_\pi \backslash \{0\}$, it can be proved that the coorbit spaces $Co(L^p)$ are embedded in R^1 for each $p \in [1, \infty]$, see e.g. [15] and [65, Proposition 4.8]. Moreover, for $p = 1$ the coorbit space $Co(L^1)$ is equal to $H^1(g)$, see e.g. [65, Proposition 4.10 and Lemma 4.12] for a proof.

Proposition 8.1 ([15, 65])**.** Let $g \in A^1_\pi \backslash \{0\}$. Then:

(i) $Co_g(L^p) \hookrightarrow R^1(g) = \left(H^1(g)\right)^*$, for each $p \in [1, \infty]$,
(ii) $Co_g(L^1) = H^1(g)$.
(iii) $Co_g(L^2) = H_\pi$.

For studying the boundedness of operators on coorbit spaces we need the following subspaces of A^1_π. For $w : G \longrightarrow [1, \infty)$ a measurable, submultiplicative weight on G we define the subspaces

$$A^1_{\pi,w} = \left\{g \in H_\pi \ : \ V_g g \in L^1_w(G)\right\} \tag{8.8}$$

and

$$B^1_w = \left\{g \in H_\pi \ : \ V_g g \in W_w(G)\right\}, \tag{8.9}$$

where $W_w(G)$ is the weighted Amalgam space of continuous functions. For the class of w-integrable representations we have that the subspaces $A^1_{\pi,w}$ and B^1_w are non-trivial. A representation π is said to be a *w-integrable representation* if there exists $g \in H_\pi \backslash \{0\}$ such that $V_g g = \langle g, \pi(\cdot)g\rangle \in L^1_w(G)$. For a $w-$integrable representation we have by definition that $A^1_{\pi,w}$ is non-trivial and we can further show that the vector space B^1_w is also non-trivial [17].

Lemma 8.2. Suppose that $w : G \longrightarrow [1, \infty)$ is a measurable, submultiplicative weight on G. If π is an irreducible, $w-$integrable representation, then there exists $h \in H_\pi$, $h \neq 0$ such that $V_h h \in W_w(G)$.

Proof. Since π is a $w-$integrable representation, there exists $f \in H_\pi$, $f \neq 0$ such that $V_f f \in L^1_w(G)$. Let $\phi \in C_c(G)$. We denote by $\pi(\phi)$ the operator on the Hilbert space H_π given by

$$\langle f_1, \pi(\phi) f_2\rangle = \int_G \langle f_1, \pi(x) f_2\rangle \, \phi(x) \mathbf{d}\mu(x), \tag{8.10}$$

for each $f_1, f_2 \in H_\pi$. Then for each $x \in G$ we obtain

$$\left|V_{\pi(\phi)f}\pi(\phi)f(x)\right| = |\langle \pi(\phi)f, \, \pi(x)\pi(\phi)f\rangle| \leq \int_G \int_G |\langle \pi(z)f, \pi(x)\pi(y)f\rangle| \, |\phi(z)| \, |\phi(y)| \, \mathbf{d}\mu(z)\mathbf{d}\mu(y).$$

From the $\sigma-$projectivity of the representation we have

$$\begin{aligned}
\left|V_{\pi(\phi)f}\pi(\phi)f(x)\right| &\leq \int_G \int_G \left|\langle f, \pi(z^{-1}xy)f\rangle\right| |\phi(y)| \, |\phi(z)| \, \mathbf{d}\mu(z)\mathbf{d}\mu(y) \\
&\leq \int_G \int_G \left|V_f f(z^{-1}xy)\right| |\phi(y)| \, |\phi(z)| \, \mathbf{d}\mu(z)\mathbf{d}\mu(y) \\
&\leq \int_G \left(|\phi| * |V_f f|\right)(xy) \, |\phi(y)| \, \mathbf{d}\mu(y) \\
&= \int_G \left(|\phi| * |V_f f|\right)(xy) \left|\phi^\vee(y^{-1})\right| \mathbf{d}\mu(y) \\
&= \left(|\phi| * |V_f f| * |\phi^\vee|\right)(x).
\end{aligned}$$

Hence, using Equations (3.47) and (3.48) we deduce

$$\begin{aligned}
\left\|V_{\pi(\phi)f}\pi(\phi)f\right\|_{W_w(G)} &= \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \, V_{\pi(\phi)f}\pi(\phi)f\right\|_{L^1_w(G)} = \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \, \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \left(|\phi| * |V_f f| * |\phi^\vee|\right)\right\|_{L^1_w(G)} \\
&= \left\|\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \, |\phi| * |V_f f| * \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \, |\phi^\vee|\right\|_{L^1_w(G)} \\
&\leq \left\|\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \, |\phi|\right\|_{L^1_w(G)} \left\||V_f f|\right\|_{L^1_w(G)} \left\|\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \, |\phi^\vee|\right\|_{L^1_w(G)} \\
&\leq \|\phi\|_{W_w(G)} \left\|V_f f\right\|_{L^1_w(G)} \|\phi^\vee\|_{W_w(G)} \, .
\end{aligned}$$

Since ϕ has compact support and $V_f f \in L^1_w(G)$ we conclude that

$$\left\| V_{\pi(\phi)f}\pi(\phi)f \right\|_{W_w(G)} < \infty, \tag{8.11}$$

and thus $h = \pi(\phi)f \in H_\pi \backslash \{0\}$ is such that $V_h h \in W_w(G)$, or equivalently $h \in B^1_w \backslash \{0\}$. □

Moreover, in order to define and prove the boundedness of the frame operator on coorbit spaces we will use a result for molecules in a Hilbert space. We follow [34, 54] for the definition of molecules.

Definition 8.3. Let $g \in B^1_w \backslash \{0\}$. Let Λ be a relatively separated set in G. A set $(g_\lambda)_{\lambda \in \Lambda} \subseteq H_\pi$ is called a $w-$*molecule*, if there exists $\Theta \in W_w(G)$ such that

$$|V_g g_\lambda|(x) = |\langle g_\lambda, \pi(x)g \rangle| \leq \Theta(\lambda^{-1}x), \tag{8.12}$$

for each $\lambda \in \Lambda$ and $x \in G$.

If $(g_\lambda)_{\lambda \in \Lambda} \subseteq H_\pi$ is a $w-$molecule, then for each $\lambda \in \Lambda$

$$\|V_g g_\lambda\|_{L^1(G)} \leq \int_G \left|\Theta(\lambda^{-1}x)\right| \mathbf{d}\mu(x) = \|\Theta\|_{L^1(G)} < \infty, \tag{8.13}$$

hence $g_\lambda \in H^1$ for each $\lambda \in \Lambda$. Moreover, the functions $V_g g_\lambda$, $\lambda \in \Lambda$ have a common envelope, $\Theta \in W_w(G)$, therefore the assumption in (8.12) is stronger than assuming $g_\lambda \in H^1$ for each $\lambda \in \Lambda$. Note that if $g \in B^1_w \backslash \{0\}$, then for each $\lambda \in \Lambda$ and $x \in G$ we have

$$V_g \pi(\lambda) g(x) = |\langle \pi(\lambda)g, \pi(x)g \rangle| = \left|\langle g, \pi(\lambda^{-1}x)g \rangle\right| = \left|V_g g(\lambda^{-1}x)\right|$$

and $V_g g \in W_w(G)$. Thus, if $g \in B^1_w \backslash \{0\}$, then $\pi(\Lambda)g$ is automatically a $w-$molecule.

For a $w-$molecule $(g_\lambda)_{\lambda \in \Lambda} \subseteq H_\pi$, the *coefficient operator*

$$C : f \longmapsto \left(\langle f, g_\lambda \rangle_{R^1, H^1}\right)_{\lambda \in \Lambda}$$

and the *reconstruction operator*

$$D : c = (c_\lambda)_{\lambda \in \Lambda} \longmapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda$$

can be shown to be bounded as operators from $Co_g(L^p)$ into $\ell^p(\Lambda)$ and from $\ell^p(\Lambda)$ into $Co_g(L^p)$, respectively, see e.g. [65, Proposition 6.11.] and [54]. Hence, we deduce that $S = DC$ is a bounded operator defined on $Co_g(L^p)$.

Lemma 8.4 ([65])**.** Let $g \in B^1_w \backslash \{0\}$. If $(g_\lambda)_{\lambda \in \Lambda} \subseteq H_\pi$ is a $w-$molecule then the operators

$$\begin{aligned} C : Co_g(L^p) &\longrightarrow \ell^p(\Lambda) \\ f &\longmapsto \left(\langle f, g_\lambda \rangle_{R^1, H^1}\right)_{\lambda \in \Lambda} \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} D : \ell^p(\Lambda) &\longrightarrow Co_g(L^p) \\ c = (c_\lambda)_{\lambda \in \Lambda} &\longmapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda \end{aligned} \tag{8.15}$$

are well-defined and bounded. Moreover,

$$\begin{aligned} S : Co_g(L^p) &\longrightarrow Co_g(L^p) \\ f &\longmapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle_{R^1, H^1} g_\lambda \end{aligned} \tag{8.16}$$

is a well-defined and bounded operator.

We now present an application of the ℓ^p-stability result in p-frames. From Lemma 8.4 we have that for $g \in B^1_{w_\alpha} \backslash \{0\}$ the coefficient operator of $\pi(\Lambda)g$, defined by

$$\begin{aligned} C_{g,\Lambda} : Co_g(L^p) &\longrightarrow \ell^p(\Lambda) \\ f &\longmapsto \left(\langle f, \pi(\lambda)g\rangle_{R^1,H^1}\right)_{\lambda\in\Lambda} \end{aligned} \tag{8.17}$$

is well-defined and bounded. The system $\pi(\Lambda)g$ is said to be a *p-frame* for $Co_g(L^p)$ if there exist $A, B > 0$ such that for each $f \in Co_g(L^p)$

$$A \|f\|_{Co_g(L^p)} \leq \|C_{g,\Lambda} f\|_{\ell^p(\Lambda)} \leq B \|f\|_{Co_g(L^p)}. \tag{8.18}$$

Note that the above definition of a 2-frame for $Co_g(L^2)$ coincides with the definition of a frame on the Hilbert space H_π given by Definition 6.2.1, since $Co_g(L^2) = H_\pi$ by Proposition 8.1.

Using Theorem 5.1.1 we will prove that for $g \in B^1_{w_\alpha}$, if $\pi(\Lambda)g$ is a p-frame for $Co_g(L^p)$ for some $p \in [1, \infty]$, then $\pi(\Lambda)g$ is a q-frame for $Co_g(L^q)$ for each $q \in [1, \infty]$.

Theorem 8.5. Let G be a group of polynomial growth, with order of growth equal to $D_G > 0$. Suppose that (π, H_π) is a discrete series σ-representation of the group G and $\Lambda \subseteq G$ is a relatively separated set. Fix $g \in B^1_{w_\alpha} \backslash \{0\}$, where $\alpha \geq D_G + 1$. If $\pi(\Lambda)g$ is a p-frame for $Co_g(L^p)$ for some $p \in [1, \infty]$, then $\pi(\Lambda)g$ is a q-frame for $Co_g(L^q)$ for each $q \in [1, \infty]$.

Before proving the previous theorem, we state a result on the existence of canonical dual frames which are also molecules, see [54, Theorem 5.3].

Proposition 8.6 ([54])**.** Let (π, H_π) be a discrete series σ-representation of the group G of polynomial growth. Fix $\alpha \in \mathbb{N}$ and $h \in B^1_{w_\alpha} \backslash \{0\}$. Then there exists a relatively separated set $\Gamma \subseteq G$, such that $\pi(\Gamma)h$ is a p-frame for each $p \in [1, \infty]$, the canonical dual frame $\left(\widetilde{h}_\gamma\right)_{\gamma\in\Gamma}$ of $\pi(\Gamma)h$ in H_π is a w_α-molecule and there exists $\Theta \in W_{w_\alpha}(G)$ such that for each $\gamma \in \Gamma$ and $x \in G$

$$\left|V_h \widetilde{h}_\gamma(x)\right| \leq \Theta(\gamma^{-1}x). \tag{8.19}$$

Moreover, for each $p \in [1, \infty]$ we have that $(\widetilde{h}_\gamma)_{\gamma\in\Gamma}$ is a p-frame and for each $f \in Co(L^p)$ we have the following expansions

$$f = \sum_{\gamma\in\Gamma} \langle f, \pi(\gamma)h\rangle_{R^1,H^1} \widetilde{h}_\gamma = \sum_{\gamma\in\Gamma} \langle f, \widetilde{h}_\gamma\rangle_{R^1,H^1} \pi(\gamma)h. \tag{8.20}$$

The proof of Theorem 8.5 presented below is inspired by [27, Theorem 2.2], which proves a similar result in the setting of homogeneous groups.

Proof of Theorem 8.5. Fix $p \in [1, \infty]$ and suppose that $\pi(\Lambda)g$ is a p-frame for $Co_g(L^p)$. Let

$$\begin{aligned} C_{g,\Lambda} : Co_g(L^p) &\longrightarrow \ell^p(\Lambda) \\ f &\longmapsto \left(\langle f, \pi(\lambda)g\rangle_{R^1,H^1}\right)_{\lambda\in\Lambda} \end{aligned} \tag{8.21}$$

and

$$\begin{aligned} D_{g,\Lambda} : \ell^p(\Lambda) &\longrightarrow Co_g(L^p) \\ c = (c_\lambda)_{\lambda\in\Lambda} &\longmapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g \end{aligned} \tag{8.22}$$

be the coefficient and reconstruction operators of $\pi(\Lambda)g$, respectively.

From Proposition 8.6 and for $h \in B^1_{w_\alpha} \backslash \{0\}$, there exists a relatively separated set $\Gamma \subseteq G$ such that $\pi(\Gamma)h$ and its dual frame $(\widetilde{h}_\gamma)_{\gamma\in\Gamma}$ are a q-frames for each $q \in [1, \infty]$ and there exists $\Theta \in W_{w_\alpha}(G)$ such that for each $\gamma \in \Gamma$ and $x \in G$

$$\left|V_h \widetilde{h}_\gamma(x)\right| \leq \Theta(\gamma^{-1}x). \tag{8.23}$$

We denote by $C_{h,\Gamma}$ and $D_{h,\Gamma}$ the coefficient and reconstruction operators of $\pi(\Gamma)h$ and by $C_{\widetilde{h}}$ and $D_{\widetilde{h}}$ the coefficient and reconstruction operators of $(\widetilde{h}_\gamma)_{\gamma\in\Gamma}$.

We define the operators

$$A := C_{g,\Lambda}D_{h,\Gamma} : \ell^p(\Gamma) \longrightarrow \ell^p(\Lambda) \tag{8.24}$$

and

$$P := C_{\widetilde{h}}D_{h,\Gamma} : \ell^p(\Gamma) \longrightarrow \ell^p(\Gamma). \tag{8.25}$$

By Lemma 8.4 we have that A and P are well-defined and bounded. For each $\lambda \in \Lambda$, $\gamma \in \Gamma$

$$A(\lambda, \gamma) = \langle \pi(\gamma)h, \pi(\lambda)g\rangle_{R^1,H^1} = \langle \pi(\gamma)h, \pi(\lambda)g\rangle \tag{8.26}$$

and hence

$$|A(\lambda, \gamma)| = |\langle \pi(\gamma)h, \pi(\lambda)g\rangle| = |\langle \pi(\lambda)g, \pi(\gamma)h\rangle| = \left|\langle g, \pi(\lambda^{-1}\gamma)h\rangle\right| = |V_h g|\,(\lambda^{-1}\gamma). \tag{8.27}$$

From $h, g \in B^1_{w_\alpha}$ we have $V_h h, V_g g \in W_{w_\alpha}(G)$, hence $V_h g \in W_{w_\alpha}(G)$ from e.g. [65, Lemma 4.5]. Thus, $A \in CD_{w_\alpha}(\Gamma, \Lambda)$. Moreover, for each $\gamma, \gamma' \in \Gamma$

$$P(\gamma, \gamma') = \langle \pi(\gamma')h, \widetilde{h}_\gamma\rangle_{R^1,H^1} = \langle \pi(\gamma')h, \widetilde{h}_\gamma\rangle \tag{8.28}$$

and hence from Equation (8.23) we obtain

$$|P(\gamma, \gamma')| = \left|\langle \pi(\gamma')h, \widetilde{h}_\gamma\rangle\right| = \left|\langle \widetilde{h}_\gamma, \pi(\gamma')h\rangle\right| = \left|V_h \widetilde{h}_\gamma\right|(\gamma') \leq \Theta(\gamma^{-1}\gamma'). \tag{8.29}$$

From $\Theta \in W_{w_\alpha}(G)$, it follows that $P \in CD_{w_\alpha}(\Gamma)$. From the expansion (8.20) in Proposition 8.6 we have

$$f = \sum_{\gamma\in\Gamma} \langle f, \widetilde{h}_\gamma\rangle_{R^1,H^1}\pi(\gamma)h = D_{h,\Gamma}C_{\widetilde{h}}f,$$

for each $f \in Co(L^p)$. Thus,

$$P^2 = C_{\widetilde{h}}D_{h,\Gamma}C_{\widetilde{h}}D_{h,\Gamma} = C_{\widetilde{h}}D_{h,\Gamma} = P \tag{8.30}$$

on $\ell^p(\Gamma)$, i.e. P is idempotent.

For each $c \in \ell^p(\Gamma)$ we have that $f = D_{h,\Gamma}c \in Co(L^p)$ and then

$$\|APc\|_{\ell^p(\Lambda)} = \left\|C_{g,\Lambda}D_{h,\Gamma}C_{\widetilde{h}}D_{h,\Gamma}c\right\|_{\ell^p(\Lambda)} = \left\|C_{g,\Lambda}D_{h,\Gamma}C_{\widetilde{h}}f\right\|_{\ell^p(\Lambda)} = \|C_{g,\Lambda}f\|_{\ell^p(\Lambda)}, \tag{8.31}$$

where we have used that $f = D_{h,\Gamma}C_{\widetilde{h}}f$, for each $f \in Co(L^p)$. Since $\pi(\Lambda)g$ is a p-frame by assumption and $(\widetilde{h}_\gamma)_{\gamma\in\Gamma}$ is a p-frame by Proposition 8.6, we obtain

$$\|APc\|_{\ell^p(\Lambda)} = \|C_{g,\Lambda}f\|_{\ell^p(\Lambda)} \gtrsim \|f\|_{Co(L^p)} \gtrsim \left\|C_{\widetilde{h}}f\right\|_{\ell^p(\Gamma)} = \left\|C_{\widetilde{h}}D_{h,\Gamma}c\right\|_{\ell^p(\Gamma)}. \tag{8.32}$$

Thus, for each $c \in \ell^p(\Gamma)$

$$\|APc\|_{\ell^p(\Lambda)} \gtrsim \left\|C_{\widetilde{h}}D_{h,\Gamma}c\right\|_{\ell^p(\Gamma)} = \|Pc\|_{\ell^p(\Gamma)}. \tag{8.33}$$

Since $\alpha \geq D_G + 1$, $A \in CD_{w_\alpha}(\Gamma, \Lambda)$, $P \in CD_{w_\alpha}(\Gamma)$, $P^2 = P$ and Equation (8.33) holds, the assumptions of Theorem 5.1.4 are satisfied. Hence, by applying Theorem 5.1.4 there exists $C' > 0$ such that for each $q \in [1, \infty]$

$$C'\|APc\|_{\ell^q(\Lambda)} \geq \|Pc\|_{\ell^q(\Gamma)}. \tag{8.34}$$

Fix $q \in [1, \infty]$. For each $f \in Co(L^q)$ from the Expansion (8.20) there exists $c_f := C_{\widetilde{h}}f \in \ell^q(\Gamma)$ such that $f = D_{h,\Gamma}c_f$. Then for each $f \in Co(L^q)$ we have

$$\begin{aligned} C'\|C_{g,\Lambda}f\|_{\ell^q(\Lambda)} &= C'\left\|C_{g,\Lambda}D_{h,\Gamma}C_{\widetilde{h}}f\right\|_{\ell^q(\Lambda)} = C'\left\|C_{g,\Lambda}D_{h,\Gamma}C_{\widetilde{h}}D_{h,\Gamma}c_f\right\|_{\ell^q(\Lambda)} \\ &= C'\|APc_f\|_{\ell^q(\Lambda)} \geq \|Pc_f\|_{\ell^q(\Gamma)} = \left\|C_{\widetilde{h}}D_{h,\Gamma}c_f\right\|_{\ell^q(\Gamma)} = \left\|C_{\widetilde{h}}f\right\|_{\ell^q(\Gamma)}, \end{aligned}$$

using Equation (8.34). Then using that $(\widetilde{h}_\gamma)_{\gamma\in\Gamma}$ is a q-frame (Proposition 8.6) we obtain for each $f \in Co(L^q)$

$$C'\|C_{g,\Lambda}f\|_{\ell^q(\Lambda)} \geq \left\|C_{\widetilde{h}}f\right\|_{\ell^q(\Gamma)} \gtrsim \|f\|_{Co(L^q)}. \tag{8.35}$$

On the other hand, from $g \in B_{w_\alpha}\backslash\{0\}$ and Lemma 8.4 we have that $C_{g,\Lambda}$ is bounded on $Co(L^q)$. Hence by combining the previous

$$\|f\|_{Co(L^q)} \lesssim \|C_{g,\Lambda}f\|_{\ell^q(\Lambda)} \lesssim \|f\|_{Co(L^q)}. \tag{8.36}$$

for each $f \in Co(L^q)$. Thus, $\pi(\Lambda)g$ is a q-frame. Since $q \in [1, \infty]$ was chosen arbitrary, the previous proves our claim. □

For the next results, we define the following subspace of the Hilbert space H_π

$$B_\pi^\infty := \bigcap_{\alpha\in\mathbb{N}} B^1_{w_\alpha} = \{f \in H_\pi \ : \ V_f f \in W_{w_\alpha}(G),\, \forall \alpha \in \mathbb{N}\}\,, \tag{8.37}$$

where w_α is the submultiplicative weight on G given by

$$w_\alpha : G \longrightarrow [1,\infty),\ x \mapsto (1 + \mathrm{d}(x,e))^\alpha,$$

for each $\alpha \in \mathbb{N}$.

Remark 8.7. For classes of projective representations π of simply connected nilpotent Lie groups, the space B_π^∞ is non-trivial, i.e. $B_\pi^\infty \neq \{0\}$. Assuming that G is a simply connected nilpotent Lie group, we define the smooth vectors

$$H_\pi^\infty = \{f \in H_\pi \ : \ [x \longmapsto \pi(x)f] \in C^\infty(G\,; H_\pi)\}\,. \tag{8.38}$$

It can be shown that H_π^∞ is norm dense in H_π, see e.g. [11]. Let **g** be the Lie algebra of G, with basis $\{Y_1, Y_2, ..., Y_d\}$. We define the algebra $\mathscr{D}(G)$ of all differential operators $D : C^\infty(G) \longrightarrow C^\infty(G)$, such that $D = \sum_{\beta\in(\mathbb{N}\cup\{0\})^d} c_\beta Y_1^{\beta_1} Y_2^{\beta_1} \cdots Y_d^{\beta_d}$, with finitely many non-zero $c_\beta \in \mathbb{C}$. Then, a function $F \in C^\infty(G)$ belongs in the Schwartz space, i.e. $F \in S(G)$, if for all $D \in \mathscr{D}(G)$ and $\alpha \in \mathbb{N}\cup\{0\}$

$$\|F\|_{D,\alpha} := \|DF\|_{L^\infty_{w_\alpha}(G)} < \infty. \tag{8.39}$$

See [44, 57] for more on the Schwartz space on Lie groups. For $g \in H_\pi^\infty$, it can be shown that

$$V_g g = \langle g,\, \pi(\cdot)g\rangle \in S(G),$$

see [11, 12, 37, 50]. In particular, this shows that $V_g g \in \bigcap_{\alpha\in\mathbb{N}} W_{w_\alpha}(G)$. Therefore, it follows that B_π^∞ is non-trivial for nilpotent Lie groups.

Using the Wiener type Lemma for the convolution-dominated matrices $\bigcap_{\alpha\in\mathbb{N}} CD_{w_\alpha}(\Lambda)$, given by Theorem 5.2.2, we prove that if $\pi(\Lambda)g$ is a frame for the Hilbert space H_π, then the dual frame $\left(S_g^{-1}\pi(\lambda)g\right)_{\lambda\in\Lambda}$ is a $w_\alpha-$molecule for each $\alpha \in \mathbb{N}$, under the assumption $g \in B_\pi^\infty$.

Lemma 8.8. Let $g \in B_\pi^\infty \backslash \{0\}$. Suppose that $\pi(\Lambda)g$ is a frame for H_π with frame operator S_g. Then the dual frame $(h_\lambda)_{\lambda\in\Lambda} = \left(S_g^{-1}\pi(\lambda)g\right)_{\lambda\in\Lambda}$ of $\pi(\Lambda)g$ is a w_α-molecule for each $\alpha \in \mathbb{N}$.

Proof. If $\pi(\Lambda)g$ is a frame, then the frame operator S_g is invertible in $\mathscr{B}(H_\pi)$. Denote by

$$A = (\langle \pi(\lambda)g, \pi(\lambda')g\rangle)_{\lambda,\lambda'\in\Lambda}$$

and

$$B = \left(\langle S_g^{-1}\pi(\lambda)g, S_g^{-1}\pi(\lambda')g\rangle\right)_{\lambda,\lambda'\in\Lambda}$$

the Gramian matrix of $\pi(\Lambda)g$ and its dual frame $(h_\lambda)_{\lambda\in\Lambda}$, respectively. Then, by [23, Lemma 3.1.], we obtain

$$B = (A^\dagger)^2 A, \tag{8.40}$$

where $A^\dagger$ is the pseudoinverse of A in $\mathscr{B}(\ell^2(\Lambda))$.

Since $g \in B_\pi^\infty$ we have $V_g g \in W_{w_\alpha}(G)$ for each $\alpha \in \mathbb{N}$ and for each $\lambda, \lambda' \in \Lambda$ we have

$$|A(\lambda,\lambda')| = |\langle \pi(\lambda)g, \pi(\lambda')g\rangle| = \left|\langle g,\, \pi(\lambda^{-1}\lambda')g\rangle\right| = \left|V_g g(\lambda^{-1}\lambda')\right|,$$

hence $A \in CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. From the previous and the inclusion (5.21) it follows that $A \in \mathscr{S}_{1,w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. From Corollary 2.2.5 we deduce that $A^\dagger \in \mathscr{S}_{1,w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$, since $A^\dagger$ is the pseudoinverse of $A = A^*$ in $\mathscr{B}(\ell^2(\Lambda))$ and $\mathscr{S}_{1,w_\alpha}(\Lambda)$ is inverse-closed in $\mathscr{B}(\ell^2(\Lambda))$, see Theorem 5.2.1. Moreover, $B = (A^\dagger)^2 A \in \mathscr{S}_{1,w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$, since $\mathscr{S}_{1,w_\alpha}(\Lambda)$ is closed under composition. Then, following the proof of Theorem 5.2.2 we deduce that $B \in \bigcap_{\alpha\in\mathbb{N}} CD_{w_\alpha}(\Lambda)$.

For $\lambda \in \Lambda$ we obtain using the frame decomposition (6.11) for $h_\lambda = S_g^{-1}\pi(\lambda)g$

$$h_\lambda = \sum_{\lambda'\in\Lambda} \langle h_\lambda, S_g^{-1}\pi(\lambda')g\rangle \pi(\lambda')g = \sum_{\lambda'\in\Lambda} \langle h_\lambda, h_{\lambda'}\rangle \pi(\lambda')g. \tag{8.41}$$

Then, for each $\lambda \in \Lambda$ and each $x \in G$

$$
\begin{aligned}
|V_g h_\lambda(x)| &= \left| V_g \left(\sum_{\lambda' \in \Lambda} \langle h_\lambda, h_{\lambda'} \rangle \pi(\lambda') g \right)(x) \right| = \left| \sum_{\lambda' \in \Lambda} \langle h_\lambda, h_{\lambda'} \rangle V_g \left(\pi(\lambda') g \right)(x) \right| \\
&\leq \sum_{\lambda' \in \Lambda} |\langle h_\lambda, h_{\lambda'} \rangle| \left| V_g \left(\pi(\lambda') g \right)(x) \right| = \sum_{\lambda' \in \Lambda} |B(\lambda, \lambda')| \left| V_g g((\lambda')^{-1} x) \right|.
\end{aligned}
$$

Fix $\alpha \in \mathbb{N}$. Since $B \in \bigcap_{\beta \in \mathbb{N}} CD_{w_\beta}(\Lambda)$, there exists $\Theta \in W_{w_\alpha}(G)$ such that $|B(\lambda, \lambda')| \leq \Theta(\lambda^{-1}\lambda')$. Thus,

$$
|V_g h_\lambda(x)| = \sum_{\lambda' \in \Lambda} \Theta(\lambda^{-1}\lambda') \left| V_g g((\lambda')^{-1} x) \right|
$$

and from Lemma 3.2.7 we obtain

$$
|V_g h_\lambda(x)| \leq \frac{\mathrm{Rel}_Q(\Lambda)}{\mu(Q)} \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} |V_g g| \right)(\lambda^{-1} x)
$$

Then using Equations (3.47) and (3.48) we have

$$
\begin{aligned}
\left\| \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} |V_g g| \right) \right\|_{W_{w_\alpha}(G)} &= \left\| \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} V_g g \right) \right\|_{L^1_{w_\alpha}(G)} \\
&= \left\| \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \left(\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta \right) * \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \left(\mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} |V_g g| \right) \right\|_{L^1_{w_\alpha}(G)} \\
&\leq \left\| \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta \right\|_{L^1_{w_\alpha}(G)} \left\| \mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} (|V_g g|) \right\|_{L^1_{w_\alpha}(G)} \\
&\leq \|\Theta\|_{W_{w_\alpha}(G)} \|V_g g\|_{W_{w_\alpha}(G)}.
\end{aligned}
$$

From $\Theta \in W_{w_\alpha}(G)$ and $V_g g \in W_{w_\alpha}(G)$ we deduce that $\mathrm{M}^{\mathrm{L}}_{\mathrm{Q}} \Theta * \mathrm{M}^{\mathrm{R}}_{\mathrm{Q}} |V_g g| \in W_{w_\alpha}(G)$ and then $(h_\lambda)_{\lambda \in \Lambda}$ is a w_α-molecule. Since $\alpha \in \mathbb{N}$ was chosen arbitrary, it follows that $(h_\lambda)_{\lambda \in \Lambda}$ is a w_α-molecule for each $\alpha \in \mathbb{N}$. ☐

The analysis above is sufficient for proving the following result. We fix a vector $g \in B^\infty_\pi$. Moreover, we assume that $\pi(\Lambda)g$ defines a frame for H_π, or equivalently the frame operator

$$
\begin{aligned}
S_g : H_\pi &\longrightarrow H_\pi \\
f &\longmapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g
\end{aligned}
\tag{8.42}
$$

is invertible in $\mathscr{B}(H_\pi)$. Then, the frame operator S_g is well-defined and invertible as an operator on the coorbit spaces $Co_g(L^p)$ for $p \in [1, \infty]$.

Theorem 8.9. Fix $p \in [1, \infty]$. Suppose that (π, H_π) is a discrete series σ-representation of the group G of polynomial growth and $\Lambda \subseteq G$ is a relatively separated set. Fix $g \in B^\infty_\pi \backslash \{0\}$ and assume that $\pi(\Lambda)g$ is a frame for H_π. Then the frame operator

$$
\begin{aligned}
S : Co_g(L^p) &\longrightarrow Co_g(L^p) \\
f &\longmapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle_{R^1, H^1} \pi(\lambda) g
\end{aligned}
\tag{8.43}
$$

is well-defined and invertible.

Proof. Fix $\alpha \in \mathbb{N}$. Note that for each $\lambda \in \Lambda$ and $x \in G$ we obtain

$$
|V_g \pi(\lambda) g|(x) = |\langle \pi(\lambda) g, \pi(x) g \rangle| = \left| \left\langle g, \pi(\lambda^{-1}) \pi(x) g \right\rangle \right| = \left| \left\langle g, \pi(\lambda^{-1} x) g \right\rangle \right| = |V_g g|(\lambda^{-1} x).
$$

Moreover, from the assumption $g \in B^\infty_\pi \backslash \{0\}$ we have $V_g g \in W_{w_\alpha}(G)$. Thus, by combining the previous we deduce that $\pi(\Lambda)g$ is a $w_\alpha-$molecule and hence by applying Lemma 8.4 we obtain that

$$
\begin{aligned}
S : Co_g(L^p) &\longrightarrow Co_g(L^p) \\
f &\longmapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle_{R^1, H^1} \pi(\lambda) g
\end{aligned}
\tag{8.44}
$$

is a well-defined and bounded operator, i.e. $S \in \mathscr{B}(Co_g(L^p))$.

By Lemma 8.8 we deduce that the dual frame $(h_\lambda)_{\lambda\in\Lambda} = \left(S^{-1}\pi(\lambda)g\right)_{\lambda\in\Lambda}$ of $\pi(\Lambda)g$ is a w_β-molecule for each $\beta \in \mathbb{N}$, since $g \in B_\pi^\infty\backslash\{0\}$ and $\pi(\Lambda)g$ is a frame. In particular, $(h_\lambda)_{\lambda\in\Lambda}$ is a w_α-molecule. Thus, from Lemma 8.4 we obtain that the operator

$$\begin{aligned} S_h : Co_g(L^p) &\longrightarrow Co_g(L^p) \\ f \quad &\longmapsto \sum_{\lambda\in\Lambda} \langle f, h_\lambda\rangle_{R^1,H^1} h_\lambda \end{aligned} \tag{8.45}$$

is well-defined and bounded. On the other hand, from the duality of the frames $(h_\lambda)_{\lambda\in\Lambda}$ and $\pi(\Lambda)g$ and since $\langle\cdot,\cdot\rangle_{R^1,H^1}$ extends $\langle\cdot,\cdot\rangle$ we obtain for each $f \in H_\pi$

$$S_h^{-1} f = (S_h|_{H_\pi})^{-1} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_{R^1,H^1} \pi(\lambda)g = Sf. \tag{8.46}$$

Hence $S_h = S^{-1}$ in $\mathscr{B}(H_\pi)$. In particular $S_h = S^{-1}$ on $H^1 \subseteq H_\pi$, and hence from Proposition 8.1 we have $S_h = S^{-1}$ on $Co_g(L^1) = H^1$.

From the previous we have for each $f \in H^1$

$$f = SS_h f = \sum_{\lambda\in\Lambda} \langle S_h f, \pi(\lambda)g\rangle\, \pi(\lambda)g = \sum_{\lambda\in\Lambda} \langle f, S_h\pi(\lambda)g\rangle\, \pi(\lambda)g = \sum_{\lambda\in\Lambda} \langle f, h_\lambda\rangle\, \pi(\lambda)g.$$

Then, from [65, Corollary 6.13] it follows that $f = SS_h f$ for each $f \in R^1$. Moreover, since

$$f = S_h S f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, S_h\pi(\lambda)g = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, h_\lambda,$$

for each $f \in H^1$, by applying [65, Corollary 6.13] it follows that $f = S_h S f$ for each $f \in R^1$. In particular, using the embedding given by Proposition 8.1, we have $f = SS_h f = S_h S f$ for each $f \in Co_g(L^p)$. We conclude that $S_h = S^{-1}$ on $Co_g(L^p)$. Thus, from $S_h \in \mathscr{B}(Co_g(L^p))$ it follows that S has an inverse in $\mathscr{B}(Co_g(L^p))$. This proves our claim. ☐

If for $g \in B_\pi^\infty\backslash\{0\} \subseteq B^1_{w_{D_G+1}}$ we assume that $\pi(\Lambda)g$ is a q-frame for some $q \in [1,\infty]$, then from Theorem 8.5 it follows that $\pi(\Lambda)g$ is a 2-frame, or, equivalently, $\pi(\Lambda)g$ is a frame for H_π. Therefore, for $g \in B_\pi^\infty\backslash\{0\}$, such that $\pi(\Lambda)g$ is a q-frame for some $q \in [1,\infty]$, the assumptions of Theorem 8.9 are satisfied. This proves the following result.

Corollary 8.10. Fix $p \in [1,\infty]$. Suppose that (π, H_π) is a discrete series σ-representation of the group G of polynomial growth and $\Lambda \subseteq G$ is a relatively separated set. Fix $g \in B_\pi^\infty\backslash\{0\}$ and assume that $\pi(\Lambda)g$ is a q-frame for some $q \in [1,\infty]$. Then the frame operator

$$\begin{aligned} S : Co_g(L^p) &\longrightarrow Co_g(L^p) \\ f \quad &\longmapsto \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle_{R^1,H^1} \pi(\lambda)g \end{aligned} \tag{8.47}$$

is well-defined and invertible.

9

Conclusion

In this thesis, we intended to study convolution-dominated matrices on locally compact groups of polynomial growth and investigate the ℓ^p-stability and the spectral invariance of this algebra of operators. Using a commutator technique by Sjöstrand [60] we have proved that if a convolution-dominated matrix is bounded from below for some $p \in [1, \infty]$, then it is bounded from below for each $q \in [1, \infty]$. The previous was proved for the weighted class of convolution-dominated matrices, $CD_w(\Lambda)$, under the assumption that w_α is a polynomial weight of order $\alpha \geq D$, where D is the order of growth of the group. This result is new for matrices indexed by general relatively separated sets in a group of polynomial growth. In the case of convolution-dominated matrices indexed by a uniform lattice in a group of polynomial growth, we recover the result on boundedness from below given by Tessera [63], however Tessera [63] proved the result for each polynomial weight.

Regarding the spectral invariance, using a result given by Sun [61] we have deduced that if a matrix, indexed by a relatively separated set Λ in a group G of polynomial growth, belongs in $CD_{w_\alpha}(\Lambda)$ for each polynomial weight w_α, $\alpha \in \mathbb{N}$ and is invertible in $\mathscr{B}(\ell^2(\Lambda))$, then its inverse belongs in $CD_{w_\alpha}(\Lambda)$ for each $\alpha \in \mathbb{N}$. Fendler, Gröchenig and Leinert in [19] showed that the weighted class of convolution-dominated matrices over a discrete group of polynomial growth is spectrally invariant for each polynomial weight, but also in the unweighted case. We expect a similar result for convolution-dominated matrices indexed by relatively separated sets in groups of polynomial growth, however our estimates in the commutator technique do not seem to be good enough to obtain this spectral invariance.

Additionally, during the project we have studied applications of convolution-dominated matrices and applications of the ℓ^p-stability and spectral invariance of such matrices. In general, spectral invariance of algebras of matrices is useful for studying invertibility and offers a tool to check whether the inverse of a matrix preserves its decay. The spectral invariance of convolution-dominated matrices can be applied to a smooth non-commutative torus that was studied by Connes [8]. For a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and the representation $\big(\pi, L^2(\mathbb{R}^d)\big)$ of $\mathbb{R}^{2d}$ given by

$$\pi(x,\xi) \;:\; L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),\; f \mapsto e^{2\pi i \xi \cdot} f(\cdot - x)$$

for each $x, \xi \in \mathbb{R}^d$, we define the smooth non-commutative torus as follows

$$\mathscr{A}^1_\infty := \left\{ A \in \mathscr{B}(L^2(\mathbb{R}^d)) \;:\; A = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda),\; a \in \ell^1_{w_\alpha}(\Lambda)\;, \forall \alpha \in \mathbb{N} \right\}. \tag{9.1}$$

Connes [8] showed that if $A \in \mathscr{A}^1_\infty$, such that A is invertible in $\mathscr{B}(L^2(\mathbb{R}^d))$, then $A^{-1} \in \mathscr{A}^1_\infty$ and we have extended this in a more general setting by applying the spectral invariance of convolution-dominated matrices. For a uniform lattice Λ in a nilpotent Lie group and a discrete series $\sigma-$representation (π, H_π), we have shown a Wiener type Lemma for the non-commutative space

$$\mathscr{A}^1_{w_\alpha} = \left\{ A \in \mathscr{B}(H_\pi) \;:\; A = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda),\; a \in \ell^1_{w_\alpha}(\Lambda) \right\} \tag{9.2}$$

in $\mathscr{B}(H_\pi)$, where $\alpha \in \mathbb{N} \cup \{0\}$. This recovers the Wiener type Lemma given by Gröchenig and Leinert [32] in the case of $\big(\pi, L^2(\mathbb{R}^d)\big)$ given by $\pi(x,\xi) : f \mapsto e^{2\pi i \xi \cdot} f(\cdot - x)$ for $(x,\xi) \in \mathbb{R}^{2d}$, but also it recovers the result given by

Connes [8], by taking the intersection $\bigcap_{a\in\mathbb{N}} \mathscr{A}^1_{w_\alpha}$. In the aforementioned application several results from frame theory were used in the proof and, therefore, we decided to investigate further applications in frame theory. For this we assume that G is a group of polynomial growth, (π, H_π) is a discrete series $\sigma-$representation and Λ is a general relatively separated set in the group, in contrast to the previous application where Λ was a discrete group. During the project, we have studied frames in the coorbit spaces $Co(L^p)$ and using the result on ℓ^p-stability we have proved that for $w = w_\alpha$, where $\alpha \in \mathbb{N}$, $\alpha \geq D_G + 1$, if $g \in H_\pi \backslash \{0\}$ is such that $V_g g = \langle g, \pi(\cdot)g\rangle \in W_w(G)$ and $\pi(\Lambda)g$ is a p-frame for $Co(L^p)$ for some $p \in [1, \infty]$, then $\pi(\Lambda)g$ is a q-frame for $Co(L^q)$ for each $q \in [1, \infty]$. Moreover, it was proved that if $g \in H_\pi \backslash \{0\}$ is such that $V_g g = \langle g, \pi(\cdot)g\rangle \in \bigcap_{a\in\mathbb{N}} W_{w_\alpha}(G)$ and $\pi(\Lambda)g$ is a frame for H_π, then the frame operator is invertible on the coorbit spaces $Co(L^p)$, for $p \in [1, \infty]$.

There are some interesting directions for future research on convolution-dominated matrices. First of all, it would be interesting to check the optimality of the assumption on the weight in the ℓ^p-stability result. The estimates used in the commutator technique followed in the proof seem not good enough to improve the result, therefore another method should be used. Moreover, an option for future research would be to investigate the spectral invariance of convolution-dominated matrices indexed by relatively separated sets. A variation of the method followed by Sun [61] for the spectral invariance of the Schur matrices could be used for this result. Another interesting direction is investigating whether spectral invariance holds for convolution-dominated matrices in the algebra $\mathscr{B}(\ell^p)$ on the space of p-summable sequences ℓ^p, for $p \in (0, 1)$. Recall that for abelian groups, we have shown that boundedness from below is independent of $p \in [1, \infty]$ for convolution-dominated matrices, in the unweighted case and for polynomial weights. An interesting question is whether for such matrices the result holds in the case of a logarithmic weight [63]. Moreover, based on the previous the interplay between the growth of the group and the weight can be investigated, since currently, this interplay is not well-understood.

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