Unified correspondence and canonicity

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UNIFIED CORRESPONDENCE AND CANONICITY

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Proefschrift

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aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. ir. K.C.A.M. Luyben,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op vrijdag 23 februari 2018 om 10:00 uur

door

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An electronic version of this dissertation is available at
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to my parents

Contents

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Tai'an Zhiguang Zhao
November, 2017.

Chapter 1

Introduction

This dissertation is about the algebraic understanding of correspondence and canonicity theory, as well as its applications in proof theory. The main focus of this thesis is on the methodology, which is based on algebraic and order-theoretic notions and insights. In this chapter, we will review the extant literature on correspondence and canonicity theory, give an overview of its development, and sketch the contributions of each chapter. We assume that the readers are familiar with modal logic and its relational and algebraic semantics.

1.1 The development of correspondence theory

1.1.1 Correspondence and canonicity theory

Originally, correspondence theory arises as an area of the model theory of modal logic which concerns the relation between modal formulas and first-order formulas interpreted over Kripke frames. We say that a modal formula φ and the first-order formula α correspond to each other if they are valid on the same class of Kripke frames. Canonicity theory is also originated from modal logic. A modal formula φ is canonical if it is valid on its canonical frame. This can be reformulated as the validity preservation from a modal algebra to its canonical extension, or from a descriptive general frame to its underlying Kripke frame. Canonicity is closely related to completeness. If a modal formula φ is canonical, then the normal modal logic axiomatized by φ is complete with respect to the class of Kripke frames defined by φ.

Early results concerning correspondence theory are Sahlqvist's [159] and van Benthem's [180], who gave a syntactic characterization of certain modal formulas (later called *Sahlqvist formulas*) which have first-order correspondents and they are canonical. The Sahlqvist-van Benthem algorithm [159, 180] was given to transform a Sahlqvist formula into its first-order correspondent. Later on, this theory has been extended and generalized in various ways:

- Extending correspondence results to larger fragments of formulas, e.g. inductive formulas [104] and complex formulas [190].

- Extending Sahlqvist-type correspondence and canonicity results to different signatures and languages as well as different logics: polyadic modal logics [102], arbitrary similarity types [67], modal logic with difference modality [65], graded modal logics [143, 188], extended modal logics [66, 100], hybrid logics [103, 116, 173, 175], modal fixed-point logics [16, 17, 19, 181, 182, 183, 184, 186], coalgebraic modal logics [63], modal predicate logic [185], generalized quantifiers related logics [2, 4, 3], Boolean logic with a binary relation [11], monotone modal logics [112], intuitionistic logic [158], intuitionistic modal logics [129], distributive modal logics [88, 189], positive modal logics [27], many valued modal logics [113, 130, 174], relevant modal logics [162, 163], substructural logics [70, 135, 165, 166, 167, 169, 170], modal monoidal t-norm logics [149], precontact logics [10], poset expansion based logics [168], possibility semantics based modal logics [118, 187, 196], (modal) compact Hausdorff space based modal logics [14, 19, 161], etc.

- Different technical aspects of correspondence theory are studied, e.g. alternative canonicity proofs [91, 123, 160], undecidability results of correspondence problems [12, 30, 31, 32, 33, 34], subframe preservation [143], pseudo-correspondence [192], reverse correspondence theory [125, 126, 127, 132, 133], the relation between elementarity and canonicity [75, 96, 97, 98, 99], algebraic canonicity-type preservation results [17, 72, 93, 149, 176], etc.

For comprehensive studies in correspondence theory, we refer the readers to [178, 179]. For a comprehensive survey of correspondence theory, we refer the readers to [50].

1.1.2 Algorithmic correspondence theory

In later development of correspondence theory, the algorithmic approach received increasing attention. In this approach, different algorithms are designed to transform modal formulas into their equivalent corresponding first-order formulas. Examples of these algorithms include SCAN [84, 101], DLS [39, 172] and SQEMA [40, 41, 49, 51, 52, 53, 54, 90, 191]. In particular, the SQEMA algorithm operates on modal formulas and is based on the Ackermann lemmas [1]. It rewrites the input modal formula into a pure modal formula in an expanded language, and translate the pure modal formula into the first-order language. It is shown in [51] that SQEMA succeeds on the class of inductive formulas, which is a strictly larger class than Sahlqvist formulas, and all formulas on which SQEMA succeeds are canonical. In this line of research, the focus is on classical normal modal logic and its variations.

1.1.3 Unified correspondence theory

Unified correspondence theory [48] is based on early developments of the algorithm SQEMA. In this stage, the scattered correspondence and canonicity results for different logics and the algorithmic method are unified. Based on duality-theoretic and order-algebraic insights, a very general syntactic definition of Sahlqvist and inductive formulas is given, which applies uniformly to each logical signature and is given purely in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives. In addition, the Ackermann lemma based algorithm ALBA, which is a generalization of SQEMA based on order-theoretic and algebraic insights, is given in [55], which effectively computes first-order correspondents of input formulas/inequalities, and is guaranteed to succeed on the Sahlqvist and inductive classes of formulas/inequalities.

Thanks to these order-theoretic insights [58, 164], a uniform treatment of Sahlqvist-type correspondence and canonicity theory is available for a wide range of logics, including: intuitionistic and distributive lattice-based (normal modal) logics [55], non-normal (regular) modal logics [153], substructural logics [57], hybrid logics [61], many-valued modal logics [24], and mu-calculus [42, 44, 43]. This work has stimulated many applications. Some are closely related to the core concerns of the theory itself, such as the understanding of different methodologies for obtaining canonicity results [152, 56, 59]. Other applications include the dual characterizations of classes of finite lattices [83], computing the first-order correspondence of rules for one-step frames [15, 92, 140], the epistemic logical theory of categorization [45, 46]. In particular, unified correspondence theory makes it possible to identify the syntactic shape of axioms which can be translated into analytic structural rules[1] of a proper display calculus [109]. This line of research has made it possible the development of systematic design principles for proof calculi with excellent properties [81] for logics which were challenging from a proof-theoretic perspective, such as dynamic epistemic logic [79, 80, 106], propositional dynamic logic [78], first-order logic [13], inquisitive logic [82], linear logic [111], lattice logic [110], bilattice logic [108], semi-de Morgan logic [107], the logic of resources and capabilities [20], etc.

1.2 The contributions of this dissertation

The contributions of the dissertation are listed below.

- Chapter 3 applies the unified correspondence methodology to possibility semantics, and gives alternative proofs of Sahlqvist-type correspondence results to the ones of [196], and extends these results from Sahlqvist formulas to the strictly larger class of inductive formulas, and from the full possibility frames to filter-descriptive possibility frames.

[1] Informally, *analytic* rules are those which can be added to a display calculus with cut elimination obtaining again a display calculus with cut elimination.

- Chapter 4 applies the unified correspondence methodology to modal compact Hausdorff spaces, and gives alternative proofs of canonicity-type preservation results to the ones in [14].
- Chapter 5 examines the power and limits of the translation method in obtaining correspondence and canonicity results. The correspondence via translation results generalize [88] and the canonicity via translation results are new.
- Chapter 6 is about an application of unified correspondence theory to the proof theory of strict implication logics, showing the usefulness of unified correspondence theory in the design of analytic Gentzen sequent calculi, especially when it comes to computing the corresponding analytic rules of a given sequent.

1.3 Outline of each chapter

The present dissertation belongs to the unified correspondence line of research. The outline of each chapter is given as follows:

In Chapter 2, we give the preliminaries on unified correspondence theory.

In Chapter 3, we develop a unified correspondence treatment of the Sahlqvist theory for possibility semantics, extending the results in [196] from Sahlqvist formulas to the strictly larger class of inductive formulas, and from the full possibility frames to filter-descriptive possibility frames. Specifically, we define the possibility semantics version of the algorithm ALBA, and an adapted interpretation of the expanded modal language used in the algorithm. We prove the soundness of the algorithm with respect to both (the dual algebras of) full possibility frames and (the dual algebras of) filter-descriptive possibility frames. We make some comparisons among different semantic settings in the design of the algorithms, and fit possibility semantics into this broader picture.

In Chapter 4, we use the algorithm ALBA to reformulate the proof in [14] and [19] that over modal compact Hausdorff spaces, the validity of Sahlqvist sequents are preserved from open assignments to arbitrary assignments. In particular, we prove an adapted version of the topological Ackermann lemma based on the Esakia-type lemmas proved in [14] and [19].

In Chapter 5, we examine whether the alternative route 'via translation' could be effective for obtaining Sahlqvist-type results of comparable strength for nonclassical logics. This route consists in suitably embedding nonclassical logics into classical polyadic modal logics via some Gödel-type translations, and then obtaining Sahlqvist theory for nonclassical logics as a consequence of Sahlqvist theory of classical polyadic modal logic. We analyze the power and limits of this alternative route for logics algebraically captured by normal distributive lattice expansions, and various sub-varieties thereof. Specifically, we provide a new proof, 'via translation' of the correspondence theorem for inductive inequalities of arbitrary signatures of normal distributive lattice expansions. We also show that canonicity-via-translation can be obtained in a similarly straightforward manner, but only for normal modal expansions of *bi-intuitionistic* *logic*. We also provide a detailed explanation of the difficulties involved in obtaining canonicity-via-translation outside this setting.

In Chapter 6, we specialize unified correspondence theory to strict implication logics and apply it to the proof theory of these logics. We conservatively extend a wide range of strict implication logics to Lambek Calculi over the bounded distributive full non-associative Lambek calculus (BDFNL) as a consequence of a general semantic consevativity result. By a suitably modified version of the Ackermann lemma based algorithm ALBA, we transform many strict implication sequents into analytic rules employing one of the main tools of unified correspondence theory, and develop Gentzen-style cut-free sequent calculi for BDFNL and its extensions with analytic rules which are transformed from strict implication sequents.

Chapter 2

Preliminaries on unified correspondence theory

In the present chapter, which is based on the preliminaries of [145] and [198, Section 6 and 8], we collect the preliminaries on unified correspondence theory, in the language of distributive lattice expansions (DLEs). We report on the two basic ingredients of unified correspondence theory, namely the order-algebraic oriented syntactic definition of inductive DLE-inequalities, and the algorithm ALBA (Ackermann Lemma Based Algorithm) (cf. [55, 48]) for DLE-languages.

2.1 Syntax and semantics for DLE-logics

Throughout this chapter we will use a fixed unspecified language $\mathcal{L}_{\mathrm{DLE}}$, the interpretations of which are distributive lattice expansions.

We will make use of the following definitions: an *order-type* over $n \in \mathbb{N}$ (or an n-order-type) is an n-tuple $\varepsilon \in \{1, \partial\}^n$, which is typically associated with variable tuples $\vec{p} := (p_1, \ldots, p_n)$. We say that p_i has order-type 1 (resp. ∂) if $\varepsilon_i = 1$ (resp. $\varepsilon_i = \partial$), and denote $\varepsilon(p_i) = 1$ or $\varepsilon(i) = 1$ (resp. $\varepsilon(p_i) = \partial$ or $\varepsilon(i) = \partial$). For each order-type ε, its *opposite* order-type is denoted by ε^∂, i.e., $\varepsilon^\partial_i = \partial$ iff $\varepsilon_i = 1$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and let $\mathbb{A}^\partial$ be its dual lattice (the lattice with the reverse partial order of $\mathbb{A}$). For any order-type ε, we denote $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ (we omit $(\mathcal{F}, \mathcal{G})$ when it is clear from the context) consists of: 1) an enumerable set AtProp of propositional variables p, q, r, etc.; 2) two disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$, each $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) has arity n_f (resp. n_g) and order-type ε_f over n_f (resp. ε_g over n_g).[1].

2.1.1. Definition. The *terms* (*formulas*) of $\mathcal{L}_{\mathrm{DLE}}$ are recursively defined as follows:

$$\varphi ::= p \mid \top \mid \bot \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid g(\overline{\varphi}) \mid f(\overline{\varphi})$$

[1]Unary f (resp. g) will be sometimes denoted as $\Diamond$ (resp. $\Box$) if the order-type is 1, and $\lhd$ (resp. $\rhd$) if the order-type is ∂.

where $p \in \mathsf{AtProp}$, $g \in \mathcal{G}$ and $f \in \mathcal{F}$. Terms (formulas) in $\mathcal{L}_{\mathrm{DLE}}$ are denoted by lowercase Greek letters e.g. φ, ψ, γ, or by lower case Latin letters e.g. s, t. An $\mathcal{L}_{\mathrm{DLE}}$-sequent is an expression of the form $\varphi \vdash \psi$.

2.1.2. Definition. For any tuple $(\mathcal{F}, \mathcal{G})$ as defined above, a *normal distributive lattice expansion* (DLE for short, or DLE-algebra, $\mathcal{L}_{\mathrm{DLE}}$-algebra) is a tuple $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ where L is a bounded distributive lattice, $\mathcal{F}^{\mathbb{A}} = \{f^{\mathbb{A}} \mid f \in \mathcal{F}\}$, $\mathcal{G}^{\mathbb{A}} = \{g^{\mathbb{A}} \mid g \in \mathcal{G}\}$, where each $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is an n_f-ary (resp. n_g-ary) function on L, and morover, each $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) preserves finite (therefore also empty) joins (resp. meets) in each coordinate where $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reverses finite (therefore also empty) meets (resp. joins) in each coordinate where $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$).

We will abuse notation and write e.g. f for $f^{\mathbb{A}}$ when no confusion arises. For every DLE $\mathbb{A}$, each $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \to \mathbb{A}$).

For each operator $f \in \mathcal{F}$ (respectively $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (respectively $1 \leq j \leq n_g$), we use the symbol $f_i[a]$ (resp. $g_j[a]$) to represent compactly that a is in the i-th argument of f (j-th argument of g) and omit the other coordinates which are taken as parameters. The class of all normal DLEs, denoted by $\mathbb{DLE}$, is equationally definable by identities of distributive lattices and the following for each $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. $1 \leq j \leq n_g$):

(1) if $\varepsilon_f(i) = 1$, then $f_i[a \vee b] = f_i[a] \vee f_i[b]$ and $f_i[\bot] = \bot$,

(2) if $\varepsilon_f(i) = \partial$, then $f_i[a \wedge b] = f_i[a] \vee f_i[b]$ and $f_i[\top] = \bot$,

(3) if $\varepsilon_g(j) = 1$, then $g_j[a \wedge b] = g_j[a] \wedge g_j[b]$ and $g_j[\top] = \top$,

(4) if $\varepsilon_g(j) = \partial$, then $g_j[a \vee b] = g_j[a] \wedge g_j[b]$ and $g_j[\bot] = \top$.

Every language $\mathcal{L}_{\mathrm{DLE}}$ is interpreted in DLEs with the same similarity type.

The relational semantics of $\mathcal{L}_{\mathrm{DLE}}$ is given below:

2.1.3. Definition. An $\mathcal{L}_{\mathrm{DLE}}$*-frame* is a tuple $\mathbb{F} = (\mathbb{X}, \mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{G}})$ such that $\mathbb{X} = (W, \leq)$ is a (nonempty) poset, $\mathcal{R}_{\mathcal{F}} = \{R_f \mid f \in \mathcal{F}\}$, and $\mathcal{R}_{\mathcal{G}} = \{R_g \mid g \in \mathcal{G}\}$ such that for each $f \in \mathcal{F}$, the symbol R_f denotes an $(n_f + 1)$-ary relation on W such that for all $\overline{w}, \overline{v} \in \mathbb{X}^{\eta_f}$,

$$\text{if } R_f(\overline{w}) \text{ and } \overline{w} \leq^{\eta_f} \overline{v}, \text{ then } R_f(\overline{v}), \tag{2.1}$$

where η_f is the order-type on $n_f + 1$ defined as follows: $\eta_f(1) = 1$ and $\eta_f(i+1) = \varepsilon_f^{\partial}(i)$ for each $1 \leq i \leq n_f$.

Likewise, for each $g \in \mathcal{G}$, the symbol R_g denotes an $(n_g + 1)$-ary relation on W such that for all $\overline{w}, \overline{v} \in \mathbb{X}^{\eta_g}$,

$$\text{if } R_g(\overline{w}) \text{ and } \overline{w} \geq^{\eta_g} \overline{v}, \text{ then } R_g(\overline{v}), \tag{2.2}$$

where η_g is the order-type on $n_g + 1$ defined as follows: $\eta_g(1) = 1$ and $\eta_g(i+1) = \varepsilon_g^\partial(i)$ for each $1 \leq i \leq n_g$.

An $\mathcal{L}_{\mathrm{DLE}}$-*model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is an $\mathcal{L}_{\mathrm{DLE}}$-frame, and $V : \mathsf{AtProp} \to \mathcal{P}^{\uparrow}(W)$ is a persistent valuation.

The defining clauses for the interpretation of each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ on $\mathcal{L}_{\mathrm{DLE}}$-models are given as follows:

$\mathbb{M}, w \Vdash f(\overline{\varphi})$	iff	there exists some $\overline{v} \in W^{n_f}$ s.t. $R_f(w, \overline{v})$ and $\mathbb{M}, v_i \Vdash^{\varepsilon_f(i)} \varphi_i$ for each $1 \leq i \leq n_f$,
$\mathbb{M}, w \Vdash g(\overline{\varphi})$	iff	for any $\overline{v} \in W^{n_g}$, if $R_g(w, \overline{v})$ then $\mathbb{M}, v_i \Vdash^{\varepsilon_g(i)} \varphi_i$ for some $1 \leq i \leq n_g$,

where $\Vdash^1$ is $\Vdash$ and $\Vdash^\partial$ is $\nVdash$.

2.1.4. Definition. For any language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the minimal DLE-logic is the set of $\mathcal{L}_{\mathrm{DLE}}$-sequents $\varphi \vdash \psi$ containing the following sequents:

(1) Lattice axioms:

$$p \vdash p, \qquad \bot \vdash p, \qquad p \vdash \top, \qquad p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r),$$
$$p \vdash p \vee q, \qquad q \vdash p \vee q, \qquad p \wedge q \vdash p, \qquad p \wedge q \vdash q,$$

(2) $\mathcal{F}$ and $\mathcal{G}$-axioms:

$\varepsilon_f(i) = 1$	$\varepsilon_f(i) = \partial$
$f_i[\bot] \vdash \bot$ $f_i[p \vee q] \vdash f_i[p] \vee f_i[q]$	$f_i[\top] \vdash \bot$ $f_i[p \wedge q] \vdash f_i[p] \vee f_i[q]$
$\varepsilon_g(j) = 1$	$\varepsilon_g(j) = \partial$
$\top \vdash g_j[\top]$ $g_j[p] \wedge g_j[q] \vdash g_j[p \wedge q]$	$\top \vdash g_j[\bot]$ $g_j[p] \wedge g_j[q] \vdash g_j[p \vee q]$

and is closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi[\chi/p] \vdash \psi[\chi/p]} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

$$\frac{\varphi \vdash \psi}{f_i[\varphi] \vdash f_i[\psi]} \, (\varepsilon_f(i) = 1) \qquad \frac{\varphi \vdash \psi}{f_i[\psi] \vdash f_i[\varphi]} \, (\varepsilon_f(i) = \partial)$$

$$\frac{\varphi \vdash \psi}{g_j[\varphi] \vdash g_j[\psi]} \, (\varepsilon_g(j) = 1) \qquad \frac{\varphi \vdash \psi}{g_j[\psi] \vdash g_j[\varphi]} \, (\varepsilon_g(j) = \partial).$$

The formula $\varphi[\chi/p]$ is obtained from φ by substituting χ for p uniformly. The minimal DLE-logic is denoted by $\mathbf{L}_{\mathbb{DLE}}$. For any DLE-language $\mathcal{L}_{\mathrm{DLE}}$, by a DLE-*logic* we understand any axiomatic extension of $\mathbf{L}_{\mathbb{DLE}}$.

In DLE-algebras, the turnstile $\vdash$ is interpreted as their order $\leq$. $\varphi \vdash \psi$ is valid in $\mathbb{A}$ if $\mu(\varphi) \leq \mu(\psi)$ for every assignment μ over PROP to $\mathbb{A}$. The notation $\mathbb{DLE} \models \varphi \vdash \psi$ denotes that $\varphi \vdash \psi$ is valid in all DLEs. Then by standard algebraic completeness, $\varphi \vdash \psi$ is provable in $\mathbf{L}_{\mathrm{DLE}}$ iff $\mathbb{DLE} \models \varphi \vdash \psi$.

2.2 The expanded language $\mathcal{L}^*_{\mathrm{DLE}}$

For any $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, it can be extended to the "tense" language $\mathcal{L}^*_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by adding the following connectives into $\mathcal{L}_{\mathrm{DLE}}$:

(1) the Heyting implications $\leftarrow_H$ and $\rightarrow_H \in \mathcal{G}^*$, to be respectively interpreted as the right residuals of $\wedge$ in the 1st and 2nd coordinate, and the Heyting co-implications $\succ\!\!-$ and $-\!\!\prec \in \mathcal{F}^*$, to be respectively interpreted as the left residuals of $\vee$ in the 1st and 2nd coordinate;

(2) for $f \in \mathcal{F}$ and $1 \leq i \leq n_f$, the connective $f_i^\sharp$ of arity n_f, to be interpreted as the right residual of f in its i-th coordinate and $f_i^\sharp \in \mathcal{G}^*$ if $\varepsilon_f(i) = 1$ (resp. the Galois-adjoint of f in its i-th coordinate and $f_i^\sharp \in \mathcal{F}^*$ if $\varepsilon_f(i) = \partial$);

(3) for $g \in \mathcal{G}$ and $1 \leq i \leq n_g$, the connective $g_i^\flat$ of arity n_g, to be interpreted as the left residual of g in its i-th coordinate and $g_i^\flat \in \mathcal{F}^*$ if $\varepsilon_g(i) = 1$ (resp. the Galois-adjoint of g in its i-th coordinate and $g_i^\flat \in \mathcal{G}^*$ if $\varepsilon_g(i) = \partial$);

The order-type of the new connectives are specified as follows:

(1) $\varepsilon_{\leftarrow_H}(1) = 1, \varepsilon_{\leftarrow_H}(2) = \partial, \varepsilon_{\rightarrow_H}(1) = \partial, \varepsilon_{\rightarrow_H}(2) = 1$;

(2) $\varepsilon_{\succ\!-}(1) = 1, \varepsilon_{\succ\!-}(2) = \partial, \varepsilon_{-\!\prec}(1) = \partial, \varepsilon_{-\!\prec}(2) = 1$;

(3) for $f_i^\sharp$, if $\varepsilon_f(i) = 1$, then $\varepsilon_{f_i^\sharp}(i) = \varepsilon_f(i)$ and $\varepsilon_{f_i^\sharp}(j) = (\varepsilon_f(j))^\partial$ for all $j \neq i$; if $\varepsilon_f(i) = \partial$, then $\varepsilon_{f_i^\sharp} = \varepsilon_f$;

(4) for $g_i^\flat$, if $\varepsilon_g(i) = 1$, then $\varepsilon_{g_i^\flat}(i) = \varepsilon_g(i)$ and $\varepsilon_{g_i^\flat}(j) = (\varepsilon_g(j))^\partial$ for all $j \neq i$; if $\varepsilon_g(i) = \partial$, then $\varepsilon_{g_i^\flat} = \varepsilon_g$.

2.2.1. Definition. Given the DLE-language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the *minimal $\mathcal{L}^*_{\mathrm{DLE}}$-logic* is defined by specializing Definition 2.1.4 to the expanded language $\mathcal{L}^*_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}^*, \mathcal{G}^*)$ and adding the following rules to the logic:

(1) Heyting and co-Heyting residuation rules:

$$\frac{\varphi \wedge \psi \vdash \chi}{\psi \vdash \varphi \rightarrow_H \chi} \qquad \frac{\varphi \wedge \psi \vdash \chi}{\varphi \vdash \chi \leftarrow_H \psi} \qquad \frac{\varphi \vdash \psi \vee \chi}{\psi -\!\!\prec \varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \vee \chi}{\varphi \succ\!\!- \chi \vdash \psi}$$

(2) f- and g-residuation rules[2]:

$$\frac{f_i[\varphi] \vdash \psi}{\varphi \vdash f_i^\sharp[\psi]}\ (\varepsilon_f(i) = 1), \qquad \frac{\varphi \vdash g_j[\psi]}{g_j^\flat[\varphi] \vdash \psi}\ (\varepsilon_g(j) = 1),$$

[2]In principle, we should use the notation $(g_j^\flat)_j[a]$ for $g_j^\flat$ with a as its j-th argument, but in the case of the adjoints in the rules here, we will avoid the double subscript since they coincide.

$$\frac{f_i[\varphi] \vdash \psi}{f_i^\sharp[\psi] \vdash \varphi} (\varepsilon_f(i) = \partial), \quad \frac{\varphi \vdash g_j[\psi]}{\psi \vdash g_j^\flat[\varphi]} (\varepsilon_g(j) = \partial).$$

The double line means that the rule are invertible. Let $\mathbf{L}^*_{\mathbb{DLE}}$ be the minimal $\mathcal{L}^*_{\mathrm{DLE}}$-logic.

The algebraic semantics of $\mathbf{L}^*_{\mathbb{DLE}}$ is given by the class of all $\mathcal{L}^*_{\mathrm{DLE}}$-algebras, defined as $(H, \mathcal{F}^*, \mathcal{G}^*)$ where H is a bi-Heyting algebra (because there are right adjoints or residuals of $\wedge$ and $\vee$ in the algebra) and moreover,

(1) for every $f \in \mathcal{F}$, all $a_i, b \in H$ with $1 \leq i \leq n_f$,

- if $\varepsilon_f(i) = 1$, then $f_i[a_i] \leq b$ iff $a_i \leq f_i^\sharp[b]$;
- if $\varepsilon_f(i) = \partial$, then $f_i[a_i] \leq b$ iff $a_i \leq^\partial f_i^\sharp[b]$.

(2) for every $g \in \mathcal{G}$, any $a_j, b \in H$ with $1 \leq j \leq n_g$,

- if $\varepsilon_g(j) = 1$, then $b \leq g_j[a_j]$ iff $g_j^\flat[b] \leq a_j$.
- if $\varepsilon_g(j) = \partial$, then $b \leq g_j[a_j]$ iff $g_i^\flat[b] \leq^\partial a_j$.

The soundness and completeness of $\mathbf{L}^*_{\mathbb{DLE}}$ w.r.t. the class of all $\mathcal{L}^*_{\mathrm{DLE}}$-algebras can be proved by the standard Lindenbaum-Tarski construction.

2.2.2. Theorem. *(cf. [109, Theorem 12]) For every $\mathcal{L}_{\mathrm{DLE}}$-sequent $\varphi \vdash \psi$, $\varphi \vdash \psi$ is derivable in $\mathbf{L}^*_{\mathrm{DLE}}$ iff $\varphi \vdash \psi$ is derivable in $\mathbf{L}_{\mathrm{DLE}}$. Therefore, the logic $\mathbf{L}^*_{\mathrm{DLE}}$ is a conservative extension of $\mathbf{L}_{\mathrm{DLE}}$.*

2.3 Canonical extensions

First of all, let us recall some concepts from [87]. For any bounded lattice $\mathbb{L}$, a *completion* of $\mathbb{L}$ is a complete lattice $\mathbb{C}$ such that $\mathbb{L}$ is a sublattice of $\mathbb{C}$. An element $x \in \mathbb{C}$ is *closed* if $x = \bigwedge_{\mathbb{C}} F$ for some subset $F \subseteq \mathbb{L}$, and *open* if $x = \bigvee_{\mathbb{C}} I$ for some subset $I \subseteq \mathbb{L}$. We denote $\mathsf{K}(\mathbb{C})$ (resp. $\mathsf{O}(\mathbb{C})$) as the set of all closed (resp. open) elements in $\mathbb{C}$. The completion $\mathbb{C}$ of $\mathbb{L}$ is

- *dense* if each element of $\mathbb{C}$ is both a join of meets and a meet of joins of elements from $\mathbb{L}$;
- *compact* if for any $S \subseteq \mathsf{K}(\mathbb{C})$ and $T \subseteq \mathsf{O}(\mathbb{C})$, if $\bigwedge S \leq \bigvee T$, then there are finite subsets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$.

A *canonical extension* of a lattice $\mathbb{L}$ is a completion of $\mathbb{L}$ which is both dense and compact. Every lattice has unique (up to isomorphism) canonical extension $\mathbb{L}^\delta$ [87].

A distributive lattice $\mathbb{L}$ is *perfect* if it is complete, completely distributive and every element is a join of its completely join-irreducible elements (the set of which is denoted by $J^\infty(\mathbb{L})$), and every element is a meet of its completely meet-irreducible elements (the set of which is denoted by $M^\infty(\mathbb{L})$)[3]. A normal DLE is *perfect* if its underlying distributive lattice is perfect, and each f-operation (resp. g-operation) is completely join-preserving (resp. meet-preserving) or completely meet-reversing (resp. join-reversing) in each coordinate. It is well-known that the canonical extension of a bounded distributive lattice is perfect (cf. e.g. [88, Definition 2.14]).

Let $h : \mathbb{L} \to \mathbb{M}$ be any map from a lattice $\mathbb{L}$ to $\mathbb{M}$. Following [87, Definition 4.1], one can define two maps $h^\sigma, h^\pi : \mathbb{L}^\delta \to \mathbb{M}^\delta$ by setting:

$$h^\sigma(u) = \bigvee\{\bigwedge\{h(a) : a \in \mathbb{L} \ \& \ x \leq a \leq y\} : \mathsf{K}(\mathbb{L}^\delta) \ni x \leq u \leq y \in \mathsf{O}(\mathbb{L}^\delta)\}.$$
$$h^\pi(u) = \bigwedge\{\bigvee\{h(a) : a \in \mathbb{L} \ \& \ x \leq a \leq y\} : \mathsf{K}(\mathbb{L}^\delta) \ni x \leq u \leq y \in \mathsf{O}(\mathbb{L}^\delta)\}.$$

Both h^σ and h^π extend h, and $h^\sigma \leq h^\pi$ pointwise. In general, if h is order-preserving, then h^σ and h^π are also order-preserving ([87]). The canonical extension of an $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A} = (A, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}^\delta = (A^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

2.4 An informal introduction of the algorithm ALBA

In the present section, we give an illustration on the algorithm ALBA with an example to show how it works. The presentation is based on a revised version of [109, Section 3.3], following the discussion in [55, 48]. Our semantic setting is Kripke frames and their dual complex algebras–Boolean algebras with operators (BAOs).

We consider a well-known example in modal logic, namely the 4-axiom $\Box p \to \Box\Box p$, which corresponds to the transitivity condition: for any Kripke frame $\mathbb{F} = (W, R)$,

$$\mathbb{F} \Vdash \Box p \to \Box\Box p \quad \text{iff} \quad \mathbb{F} \models \forall xyz\,(Rxy \land Ryz \to Rxz).$$

Our argument goes in a purely algebraic way, namely in the dual complex algebras of Kripke frames–complete atomic Boolean algebras with complete operators, which are also known as *perfect* BAOs [22, Definition 40, Chapter 6].

In the complex algebra $\mathbb{A} = \mathbb{F}^+$ of $\mathbb{F}$, the semantic condition $\mathbb{F} \Vdash \Box p \to \Box\Box p$ is reformulated as $[\![\Box p]\!] \subseteq [\![\Box\Box p]\!]$ for every assignment of p into $\mathbb{A}$. In purely algebraic terms, this is equivalent to

$$\mathbb{A} \models \forall p[\Box p \leq \Box\Box p], \tag{2.3}$$

where $\leq$ is interpreted as set-theoretic inclusion $\subseteq$. As is well-known, in perfect BAOs, every element can be represented both as the join of the completely join-prime elements below it and the meet of the completely meet-prime elements above

[3] An element $a \in A$ is *completely join-irreducible* (resp. completely meet-irreducible) if for any $S \subseteq A$, $a = \bigvee S$ (resp. $a = \bigwedge S$) implies $a \in S$.

it[4]. Therefore, condition (2.3) can be equivalently reformulated as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \mid i \in J^\infty(\mathbb{A}) \text{ and } i \leq \Box p\} \leq \bigwedge\{m \mid m \in M^\infty(\mathbb{A}) \text{ and } \Box\Box p \leq m\}].$$

which can be further reformulated as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box p \ \ \& \ \ \Box\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{2.4}$$

where the *nominal* variable **i** ranges over $J^\infty(\mathbb{A})$ and the *co-nominal* variable **m** ranges over $M^\infty(\mathbb{A})$.

Since in complete atomic Boolean algebras with complete operators, $\Box$ preserves arbitrary meets, this is equivalent to that $\Box$ is a right adjoint (cf. [64, Proposition 7.34]), therefore it has a left adjoint $\blacklozenge$. As a result, the equation (2.4) above can be reformulated as follows:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\blacklozenge \mathbf{i} \leq p \ \ \& \ \ \Box\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.5}$$

Now we are ready to eliminate the variable p and get the following condition:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m}[(\Box\Box\blacklozenge \mathbf{i} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.6}$$

To justify the equivalence, it suffices to justify the following equivalence:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m}[\exists p(\blacklozenge \mathbf{i} \leq p \ \ \& \ \ \Box\Box p \leq \mathbf{m}) \Leftrightarrow \Box\Box\blacklozenge \mathbf{i} \leq \mathbf{m}], \tag{2.7}$$

Let us fix the assignment of **i** and **m**. From left to right, if there exists p such that $\blacklozenge \mathbf{i} \leq p$ and $\Box\Box p \leq \mathbf{m}$, then by monotonicity, $\Box\Box\blacklozenge \mathbf{i} \leq \Box\Box p \leq \mathbf{m}$. From right to left, suppose $\Box\Box\blacklozenge \mathbf{i} \leq \mathbf{m}$, then take $p := \blacklozenge \mathbf{i}$, then there exists p such that $\blacklozenge \mathbf{i} \leq p$ and $\Box\Box p \leq \mathbf{m}$.

Indeed, this is a special case of the following *Ackermann lemma* ([1, 51]):

2.4.1. Lemma. *Let $\alpha, \beta(p), \gamma(p)$ be such that α does not contain p, β is positive in p and γ is negative in p. Then for any assignment θ on $\mathbb{A}$, the following are equivalent:*

(1) there exists a p-variant θ' of θ such that $\mathbb{A}, \theta' \models \beta(p) \leq \gamma(p)$ and $\mathbb{A}, \theta' \models \alpha \leq p$;

(2) $\mathbb{A}, \theta \models \beta(\alpha/p) \leq \gamma(\alpha/p)$.

Then by the property of complete meet-primes, condition (2.6) is equivalent to the following:

$$\mathbb{A} \models \forall \mathbf{i}[\{m \mid m \in M^\infty(\mathbb{A}) \text{ and } \Box\Box\blacklozenge \mathbf{i} \leq m\} \subseteq \{m \mid m \in M^\infty(\mathbb{A}) \text{ and } \mathbf{i} \leq m\}], \tag{2.8}$$

[4]We use $J^\infty(\mathbb{A})$ and $M^\infty(\mathbb{A})$ to denote the set of completely join-prime elements and the set of completely meet-prime elements respectively. In perfect BAOs, the completely join-prime elements are the same as the completely join-irreducible elements or the atoms, and the completely meet-prime elements are the same as the completely meet-irreducible elements or the co-atoms.

$$\mathbb{A} \models \forall \mathbf{i}[\bigwedge\{m \mid m \in M^\infty(\mathbb{A}) \text{ and } \mathbf{i} \leq m\} \leq \bigwedge\{m \mid m \in M^\infty(\mathbb{A}) \text{ and } \Box\Box\Diamond\mathbf{i} \leq m\}], \quad (2.9)$$

$$\mathbb{A} \models \forall \mathbf{i}[\mathbf{i} \leq \Box\Box\Diamond\mathbf{i}]. \quad (2.10)$$

Since $\Box$ is the right adjoint of $\Diamond$, the condition above is equivalent to the following:

$$\mathbb{A} \models \forall \mathbf{i}[\Diamond\Diamond\mathbf{i} \leq \Diamond\mathbf{i}]. \quad (2.11)$$

Now we have eliminated all propositional variables which correspond to second-order variables, and all the remaining variables are interpreted as completely join-prime elements or completely meet-prime elements of the perfect BAO $\mathbb{A}$. By discrete Stone duality, they correspond to singletons and complements of singletons of $\mathbb{F}$. On the frame side, the connective $\Diamond$ is interpreted as $R[\_]$, where R is the binary relation used in the interpretation of $\Box$. Therefore, condition (2.11) is transformed into a first-order condition on the frame $\mathbb{F}$ side:

$$\mathbb{F} \models \forall w(R[R[w]] \subseteq R[w]). \quad (2.12)$$

Which is the same as

$$\mathbb{F} \models \forall w \forall x \forall y(Rwx \land Rxy \to Rxy). \quad (2.13)$$

2.5 Inductive inequalities

In this section, we will recall from [109] the definition of *inductive* [104] and *Sahlqvist* [159] $\mathcal{L}_{\mathrm{DLE}}$-inequalities on which the algorithm ALBA is guaranteed to succeed. This definition is based on the order-theoretic properties of the interpretations of the connectives in each algebra of the class of DLEs associated with the given signature.

2.5.1. Definition. [Signed generation tree] The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{DLE}}$-formula φ is defined as follows: First of all, the root node of the generation tree of φ is labelled with sign $+$ (resp. $-$). After this, the children nodes are labelled as follows:

- For any node $\lor$ or $\land$, label the same sign to the children nodes.
- For any node $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$ and any $1 \leq i \leq n_h$, label the same (resp. the opposite) sign to the i-th child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).

Nodes in the signed generation trees are *positive* (resp. *negative*) if signed with $+$ (resp. $-$).

We will use signed generation trees mainly in the context of inequalities $\varphi \leq \psi$, where we will typically consider the positive generation tree $+\varphi$ and the negative signed generation tree $-\psi$. We say that $\varphi \leq \psi$ is *uniform* in p if all occurrences of p have the same sign in both $+\varphi$ and $-\psi$, and that $\varphi \leq \psi$ is *ε-uniform* in $\vec{p}$ if $\varphi \leq \psi$ is uniform in p, occurring with the sign indicated by ε, for each p in $\vec{p}$.

For any formula $\varphi(p_1, \ldots p_n)$, any order-type $\varepsilon \in \{1, \partial\}^n$ and any $1 \leq i \leq n$, an *ε-critical node* in the signed generation tree $+\varphi$ or $-\varphi$ is a leaf node $+p_i$ if $\varepsilon_i = 1$ or $-p_i$ if $\varepsilon_i = \partial$. An *ε-critical branch* in $+\varphi$ or $-\varphi$ is a branch ending with an ε-critical node. We say that $+\varphi$ (resp. $-\varphi$) *agrees with* ε, and denote $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every branch in $+\varphi$ (resp. $-\varphi$) is ε-critical. We also use $+\psi \prec *\varphi$ (resp. $-\psi \prec *\varphi$) to denote that the signed generation tree $+\psi$ (resp. $-\varphi$) is a subtree of $*\varphi$, where $* \in \{+, -\}$. We use $\varepsilon(\gamma) \prec *\varphi$ (resp. $\varepsilon^\partial(\gamma) \prec *\varphi$) to indicate that the signed generation subtree γ of $*\varphi$, agrees with ε (resp. with ε^∂). A propositional variable p is *positive* (resp. *negative*) in φ if $+p \prec +\varphi$ (resp. $-p \prec +\varphi$).

2.5.2. Definition. In any signed generation tree, nodes will be respectively called *syntactically right adjoint (SRA)*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)* and *Δ-adjoints*, according to Table 2.1. In the signed generation tree $*s$ where $* \in \{+, -\}$, a branch is *good* if it is composed of two sub-branches P_1 and P_2, one of which may be empty, where P_2 is a path from the root consisting of Skeleton-nodes and P_1 is a path from the leaf consisting of PIA-nodes.[5] A good branch is *excellent* if P_1 consists only of SRA nodes.

Table 2.1: Skeleton nodes and PIA nodes for DLE.

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \wedge$	$+ \quad \wedge \quad g \quad$ with $n_g = 1$
$- \quad \wedge \quad \vee$	$- \quad \vee \quad f \quad$ with $n_f = 1$
SLR	SRR
$+ \quad \wedge \quad f \quad$ with $n_f \geq 1$	$+ \quad \vee \quad g \quad$ with $n_g \geq 2$
$- \quad \vee \quad g \quad$ with $n_g \geq 1$	$- \quad \wedge \quad f \quad$ with $n_f \geq 2$

2.5.3. Definition. [Inductive and Sahlqvist inequalities] For any transitive and irreflexive relation $<_\Omega$ on variables $p_1, \ldots p_n$ (referred to as the *dependency order*) and any order-type $\varepsilon \in \{1, \partial\}^n$, the signed generation tree $*\varphi(p_1, \ldots p_n)$ (where $* \in \{-, +\}$) is *(Ω, ε)-inductive* if

(1) each ε-critical branch ending with leaf node p_i is good for each $1 \leq i \leq n$;

(2) in each ε-critical branch ending with leaf node p_i, every m-ary SRR-node occurring is of the form $h(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m)$, where $h \in \mathcal{F} \cup \mathcal{G}$ and for any $l \in \{1, \ldots, m\} \setminus j$:

[5] This organization is motivated and discussed in [44] and [48] to establish a connection with analogous terminology in [186].

(a) $\varepsilon^\partial(\gamma_l) \prec *\varphi$, and

(b) $p_k <_\Omega p_i$ for every p_k occurring in γ_l and for every $1 \leq k \leq n$.

Given any order-type ε, $*\varphi(p_1, \ldots p_n)$ is *ε-Sahlqvist* if every ε-critical branch is excellent (cf. Definition 2.5.2).

The inequality $\varphi \leq \psi$ is called *(Ω, ε)-inductive* (resp. *ε-Sahlqvist*) if $+\varphi$ and $-\psi$ are both (Ω, ε)-inductive (resp. *ε-Sahlqvist*). The inequality $\varphi \leq \psi$ is called *inductive* (*Sahlqvist*) if it is (Ω, ε)-inductive (*ε-Sahlqvist*) for some dependency order Ω and order-type ε (resp. order-type ε).

2.5.4. Remark. As we can see from the definitions above, the shape of Sahlqvist and inductive formulas is based on certain syntactic concatenation requirements. These requirements make it possible to transform a given formula into a condition where it is possible to eliminate propositional variables by the Ackermann rules introduced in Section 2.6. The soundness of such transformation rules is guaranteed by the algebraic and order-theoretic properties of the interpretations of the connectives, which are indicated by the names of their classifications. For example, in Section 2.4, the syntactically right adjoint connective $\Box$ is interpreted as the semantic right adjoint of the interpretation of $\blacklozenge$, which guarantees the transformation from (2.4) to (2.5) possible.

2.5.1 Examples

Here we give some examples of Sahlqvist and inductive inequalities. These examples are taken from [60].

2.5.5. Example. [Bi-intuitionistic language] Consider the bi-intuitionistic language $\mathcal{L}_B = (\mathcal{F}, \mathcal{G})$ where $\mathcal{F} = \{\succ\!\!-\}$, $\mathcal{G} = \{\rightarrow\}$, and $\varepsilon_{\succ\!-}(1) = 1$, $\varepsilon_{\succ\!-}(2) = \partial$, $\varepsilon_{\rightarrow}(1) = \partial$, $\varepsilon_{\rightarrow}(2) = 1$.

In [155, Section 4], Rauszer axiomatizes bi-intuitionistic logic considering the following axioms among others, which we present in the form of inequalities:

$$r \succ\!\!- (q \succ\!\!- p) \leq (p \vee q) \succ\!\!- p \qquad (q \succ\!\!- p) \rightarrow \bot \leq p \rightarrow q.$$

The first inequality is not (Ω, ε)-inductive for any Ω and ε; indeed, in the negative generation tree of $(p \vee q) \succ\!\!- p$, the variable p occurs in both subtrees rooted at the children of the root, which is a binary SRR node, making it impossible to satisfy condition 2(b) of Definition 2.5.3 for any order-type ε and strict ordering Ω.

The second inequality is ε-Sahlqvist for $\varepsilon(p) = 1$ and $\varepsilon(q) = \partial$, and is also (Ω, ε)-inductive but not Sahlqvist for $q <_\Omega p$ and $\varepsilon(p) = \varepsilon(q) = \partial$. It is also (Ω, ε)-inductive but not Sahlqvist for $p <_\Omega q$ and $\varepsilon(p) = \varepsilon(q) = 1$.

2.5.6. Example. [Intuitionistic bi-modal language] Consider the intuitionistic bi-modal language $\mathcal{L}_{IBM} = (\mathcal{F}, \mathcal{G})$ where $\mathcal{F} = \{\Diamond\}$, $\mathcal{G} = \{\Box, \rightarrow\}$ and $\varepsilon_\Diamond = \varepsilon_\Box = 1$, $\varepsilon_\rightarrow(1) = \partial$, $\varepsilon_\rightarrow(2) = 1$.

The following Fischer Servi inequalities (cf. [76])

$$\Diamond(q \to p) \leq \Box q \to \Diamond p \qquad \Diamond q \to \Box p \leq \Box(q \to p),$$

are both ε-Sahlqvist for $\varepsilon(p) = \partial$ and $\varepsilon(q) = 1$, and are also both (Ω, ε)-inductive but not Sahlqvist for $p <_\Omega q$ and $\varepsilon(p) = \partial$ and $\varepsilon(q) = \partial$.

2.5.7. Example. [Distributive modal language] Consider the distributive modal language $\mathcal{L}_D = (\mathcal{F}, \mathcal{G})$ where $\mathcal{F} = \{\Diamond\}$, $\mathcal{G} = \{\Box\}$, and $\varepsilon_\Diamond = \varepsilon_\Box = 1$.

The following inequalities are key to Dunn's positive modal logic [69], the language of which is the $\{\lhd, \rhd\}$-free fragment of the language of distributive modal logic [88]:

$$\Box q \wedge \Diamond p \leq \Diamond(q \wedge p) \qquad \Box(q \vee p) \leq \Diamond q \vee \Box p.$$

The inequality on the left (resp. right) is ε-Sahlqvist for $\varepsilon(p) = \varepsilon(q) = 1$ (resp. $\varepsilon(p) = \varepsilon(q) = \partial$), and is (Ω, ε)-inductive but not Sahlqvist for $p <_\Omega q$ and $\varepsilon(p) = 1$ and $\varepsilon(q) = \partial$ (resp. $p <_\Omega q$ and $\varepsilon(p) = \partial$ and $\varepsilon(q) = 1$).

2.6 The algorithm ALBA for $\mathcal{L}_{\text{DLE}}$-inequalities

In what follows we will specify the algorithm ALBA for a fixed but arbitrary language $\mathcal{L}_{\text{DLE}}$. The language of the algorithm $\mathcal{L}^{*+}_{\text{DLE}}$ is defined as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \mathbf{i} \mid \mathbf{m} \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid g(\overline{\varphi}) \mid f(\overline{\varphi})$$

where $p \in \mathsf{AtProp}$, $\mathbf{i} \in \mathsf{NOM}$ is called *nominal*, $\mathbf{m} \in \mathsf{CONOM}$ is called *conomimal*, $f \in \mathcal{F}^*$, $g \in \mathcal{G}^*$. This language is interpreted in perfect $\mathcal{L}_{\text{DLE}}$-algebras A, where nominals (resp. conominals) are interpreted as completely join-irreducibles (resp. completely meet-irreducibles) of $\mathbb{A}$ (cf. page 12).

An $\mathcal{L}^{*+}_{\text{DLE}}$-inequality is $\varphi \leq \psi$ such that φ and ψ are $\mathcal{L}^{*+}_{\text{DLE}}$-formulas. An $\mathcal{L}^{*+}_{\text{DLE}}$-quasi-inequality is $\varphi_1 \leq \psi_1$ & ... & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ where each $\varphi_i \leq \psi_i$ for $1 \leq i \leq n$ and $\varphi \leq \psi$ are $\mathcal{L}^{*+}_{\text{DLE}}$-inequalities.

The algorithm ALBA in the language $\mathcal{L}_{\text{DLE}}$ is defined in [55, 109]. The algorithm transforms the input $\mathcal{L}_{\text{DLE}}$-inequalities into equivalent $\mathcal{L}^{*+}_{\text{DLE}}$ quasi-inequalities with nominals and conominals only, where propositional variables are eliminated by the Ackermann rules. The proof of the soundness of ALBA rules in the language $\mathcal{L}_{\text{DLE}}$ is similar to [55, 48] and hence omitted. ALBA receives the input inequality $\varphi \leq \psi$ and runs in three stages:

First stage: preprocessing and first approximation stage. ALBA preprocesses $\varphi \leq \psi$ by applying the following rules exhaustively in $+\varphi$ and $-\psi$:

(1) (a) Push down $+\wedge$ towards variables by distributing over children node labelled with $+\vee$ which are Skeleton nodes;

(b) Push down $-\vee$ towards variables by distributing over children node labelled with $-\wedge$ which are Skeleton nodes;

(c) For any $f \in \mathcal{F}$, push down $+f$ towards variables by distributing over its i-th child node labelled with $+\vee$ (resp. $-\wedge$) which are Skeleton nodes if $\varepsilon_f(i) = 1$ (resp. $\varepsilon_f(i) = \partial$);

(d) For any $g \in \mathcal{G}$, push down $-g$ towards variables by distributing over its i-th child node labelled with $-\wedge$ (resp. $+\vee$) which are Skeleton nodes if $\varepsilon_g(i) = 1$ (resp. $\varepsilon_g(i) = \partial$).

(2) Splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

(3) Monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

where $\beta(p)$ is positive in p and $\alpha(p)$ is negative in p.

Let $\mathsf{Preprocess}(\varphi \leq \psi) := \{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ be the set of inequalities obtained by applying the above rules exhaustively. Then the following rule (which is called the *first approximation rule*) is applied to each $\varphi_i \leq \psi_i$ in $\mathsf{Preprocess}(\varphi \leq \psi)$:

$$\frac{\varphi \leq \psi}{\mathbf{i}_0 \leq \varphi \quad \psi \leq \mathbf{m}_0}$$

where $\mathbf{i}_0$ is a nominal and $\mathbf{m}_0$ is a conominal. After the first approximation rule, for each inequality $\varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)$, the algorithm produces a system of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$.

Second stage: reduction and elimination stage. The present stage aims at eliminating all propositional variables from each system obtained in the previous stage. The variables are eliminated by the so called Ackermann rules, and the other rules in this stage are applied in order to reach the shape to apply the Ackermann rule.

Splitting rules.

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

Residuation rules. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, every $1 \leq j \leq n_f$ and every $1 \leq k \leq n_g$, we have the following residuation rules:

$$\frac{f_j[\psi_j] \leq \chi}{\psi_j \leq f_j^\sharp[\chi]}\ (\varepsilon_f(j) = 1), \quad \frac{f_j[\psi_j] \leq \chi}{f_j^\sharp[\chi] \leq \psi_j}\ (\varepsilon_f(j) = \partial),$$

$$\frac{\chi \leq g_k[\psi_k]}{\psi_k \leq g_k^\flat[\chi]}\ (\varepsilon_g(k) = \partial), \quad \frac{\chi \leq g_k[\psi_k]}{g_k^\flat[\chi] \leq \psi_k}\ (\varepsilon_g(k) = 1).$$

Approximation rules. For every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, every $1 \leq j \leq n_f$ and every $1 \leq k \leq n_g$, we have the following residuation rules:

$$\frac{\mathbf{i} \leq f_j[\psi_j]}{\mathbf{i} \leq f_j[\mathbf{j}] \quad \mathbf{j} \leq \psi_j}\ (\varepsilon_f(j) = 1), \qquad \frac{g_k[\psi_k] \leq \mathbf{m}}{g_k[\mathbf{n}] \leq \mathbf{m} \quad \psi_k \leq \mathbf{n}}\ (\varepsilon_g(k) = 1),$$

$$\frac{\mathbf{i} \leq f_j[\psi_j]}{\mathbf{i} \leq f_j[\mathbf{n}] \quad \psi_k \leq \mathbf{n}}\ (\varepsilon_f(j) = \partial), \qquad \frac{g_k[\psi_k] \leq \mathbf{m}}{g_k[\mathbf{j}] \leq \mathbf{m} \quad \mathbf{j} \leq \psi_h}\ (\varepsilon_g(k) = \partial),$$

where the variables $\mathbf{i}, \mathbf{j}$ (resp. $\mathbf{m}, \mathbf{n}$) are nominals (resp. conominals). The nominal $\mathbf{j}$ and conominal $\mathbf{n}$ must be *fresh*, i.e. not occur in the system before applying the approximation rule.

Ackermann rules. These two rules aim at eliminating propositional variables and operate on the whole system rather than on a single inequality.

$$\frac{\&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \& \&\{\alpha_i \leq p \mid 1 \leq i \leq n\} \ \Rightarrow\ \mathbf{i}_0 \leq \mathbf{m}_0}{\&\{\beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\} \ \Rightarrow\ \mathbf{i}_0 \leq \mathbf{m}_0}\ \text{(RAR)}$$

where $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p, $\beta_1(p), \ldots, \beta_m(p)$ are positive in p and p does not occur in $\alpha_1, \ldots, \alpha_n$.

$$\frac{\&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \& \&\{p \leq \alpha_i \mid 1 \leq i \leq n\} \ \Rightarrow\ \mathbf{i}_0 \leq \mathbf{m}_0}{\&\{\beta_j(\bigwedge_{i=1}^n \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\} \ \Rightarrow\ \mathbf{i}_0 \leq \mathbf{m}_0}\ \text{(LAR)}$$

where $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p, $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and p does not occur in $\alpha_1, \ldots, \alpha_n$.

Third stage: output stage. If for some systems, some variables cannot be eliminated, then ALBA halts and reports failure. Otherwise, every system $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ has been reduced to a system $\mathsf{Reduce}(\varphi_i \leq \psi_i)$ with no propositional variables. Let $\mathsf{ALBA}(\varphi \leq \psi) := \{\&[\mathsf{Reduce}(\varphi_i \leq \psi_i)] \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0 \mid \varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)\}$, which contains no propositional variables. ALBA outputs $\mathsf{ALBA}(\varphi \leq \psi)$ and terminates.

2.7 Soundness, success and canonicity

In the present section, which is based on a revised version of [198, Section 6 and 8], we recall the sketch of the proof of the soundness of the algorithm ALBA, the statement that ALBA succeeds on all inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities, and sketch of the proof that all inequalities that ALBA succeeds are canonical. The proofs are similar to [55].

2.7.1 Soundness and success

2.7.1. Theorem (Soundness). *If* ALBA *runs successfully on* $\varphi \leq \psi$ *and outputs* $\mathsf{ALBA}(\varphi \leq \psi)$*, then for any perfect* $\mathcal{L}_{\mathrm{DLE}}$*-algebra* $\mathbb{A}$,

$$\mathbb{A} \Vdash \varphi \leq \psi \quad \textit{iff} \quad \mathbb{A} \models \mathsf{ALBA}(\varphi \leq \psi).$$

Proof:
(Sketch.) The proof goes similarly to [55, Theorem 8.1]. Let $\mathsf{Preprocess}(\varphi \leq \psi) := \{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ denote the set of inequalities produced by preprocessing $\varphi \leq \psi$ after Stage 1, $\mathsf{Reduce}(\varphi_i \leq \psi_i)$ denote the set of inequalities after processing the inequality $\varphi_i \leq \psi_i$ in Stage 2. It suffices to show the equivalence from (2.14) to (2.18) given below:

(2.14) $\mathbb{A} \models \varphi \leq \psi$

(2.15) $\mathbb{A} \models \mathsf{Preprocess}(\varphi \leq \psi)$

(2.16) $\mathbb{A} \models (\mathbf{j}_0 \leq \varphi_i \ \&\ \psi_i \leq \mathbf{m}_0) \ \Rightarrow\ \mathbf{j}_0 \leq \mathbf{m}_0$ for all $1 \leq i \leq n$

(2.17) $\mathbb{A} \models \mathsf{Reduce}(\varphi_i \leq \psi_i) \ \Rightarrow\ \mathbf{j}_0 \leq \mathbf{m}_0$ for all $1 \leq i \leq n$

(2.18) $\mathbb{A} \models \mathsf{ALBA}(\varphi \leq \psi)$

- the equivalence of (2.14) and (2.15) follows from the soundness of the preprocessing rules in Stage 1, which follows from Lemma 2.7.2;
- the equivalence between (2.15) and (2.16) follows from that in perfect $\mathcal{L}_{DLE}$-algebras, every element can be represented both as the join of the completely join-prime elements below it and the meet of the completely meet-prime elements above it;
- the equivalence between (2.16) and (2.17) follows from the soundness of the reduction rules in Stage 2, which follows from Lemma 2.7.3;
- the equivalence between (2.17) and (2.18) is immediate.

□

2.7.2. Lemma. *(cf. [55, Lemma 8.3]) Suppose that the set S of inequalities is obtained from S by applying preprocessing rules in Stage 1. Then $\mathbb{A} \models S'$ iff $\mathbb{A} \models S$.*

2.7.3. Lemma. *(cf. [55, Lemma 8.4]) Suppose that the system S' of inequalities is obtained from S by applying reduction rules in Stage 2. Then for any assignment θ on $\mathbb{A}$,*

(1) if $\mathbb{A}, \theta \models S$, then $\mathbb{A}, \theta' \models S'$ for some θ' such that $\theta'(\mathbf{i}_0) = \theta(\mathbf{i}_0)$ and $\theta'(\mathbf{m}_0) = \theta(\mathbf{m}_0)$;

(2) if $\mathbb{A}, \theta \models S'$, then $\mathbb{A}, \theta' \models S$ for some θ' such that $\theta'(\mathbf{i}_0) = \theta(\mathbf{i}_0)$ and $\theta'(\mathbf{m}_0) = \theta(\mathbf{m}_0)$.

In the proof of Lemma 2.7.3, for the case of the Ackermann rules, it is justified by the following Ackermann lemmas. We use the notation $\vec{q}$ (resp. $\vec{\mathbf{j}}$, $\vec{\mathbf{m}}$) to denote an array of propositional variables (resp. nominals, co-nominals) and the notation $\vec{a}$ (resp. $\vec{x}$, $\vec{y}$) to denote an array of elements in $\mathbb{A}$ (resp. $J^{\infty}(\mathbb{A})$, $M^{\infty}(\mathbb{A})$).

2.7.4. Lemma (Right-handed Ackermann lemma). *(cf. [55, Lemma 4.2]) Let α be a formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be positive (resp. negative) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$, $\vec{\mathbf{m}}$) be all the propositional variables (resp. nominals, co-nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}), \vec{y} \in M^\infty(\mathbb{A})$ (cf. page 12), the following are equivalent:*

(1) $\beta_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{A}$ *such that* $\alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ *and* $\beta_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

2.7.5. Lemma (Left-handed Ackermann lemma). *(cf. [55, Lemma 4.3]) Let α be a formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be negative (resp. positive) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$, $\vec{\mathbf{m}}$) be all the propositional variables (resp. nominals, co-nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}), \vec{y} \in M^\infty(\mathbb{A})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{A}$ *such that* $a_0 \leq \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

The following theorem is a generalization of [55, Theorem 10.11], and its proof is omitted here.

2.7.6. Theorem. *(cf. [55, Theorem 10.11])* ALBA *succeeds on all inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities.*

2.7.2 Canonicity

As we recall from [55, Section 9], in the proof of the canonicity of inequalities on which ALBA succeeds, we typically use the following "U-shaped" argument[6]:

$$\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \mathsf{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \mathsf{ALBA}(\varphi \leq \psi)
\end{array}$$

Assume that the inequality $\varphi \leq \psi$ is valid on the $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$. This is equivalent to the validity of $\varphi \leq \psi$ on the canonical extension $\mathbb{A}^\delta$ over assignments sending propositional variables into $\mathbb{A}$ rather than $\mathbb{A}^\delta$. Then the algorithm ALBA can equivalently transform the input inequality into the output $\mathsf{ALBA}(\varphi \leq \psi)$ which contain no

[6] We use the notation $\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi$ to indicate that $\varphi \leq \psi$ is valid in $\mathbb{A}^\delta$ with respect to all assignments sending propositional variables into $\mathbb{A}$ rather than $\mathbb{A}^\delta$ (such assignments are called *admissible assignments*).

propositional variables, therefore their validity is invariant under replacing assignments into $\mathbb{A}$ by assignments into $\mathbb{A}^\delta$. Then by the soundness of ALBA on perfect DLEs, the validity of ALBA($\varphi \leq \psi$) is equivalent to the validity of $\varphi \leq \psi$.

In the argument above, the right arm of equivalence is justified by Theorem 2.7.1, and the bottom equivalence is immediate. For the left arm of equivalence, the proof is similar to the right arm. Indeed, except for the soundness of the Ackermann rules, the rest of the proof goes the same (cf. [55, Section 9]).

When it comes to the Ackermann rules, as is similar to what is discussed in the existing literature (e.g. [55, Section 9]), the soundness proof of the Ackermann rules, namely the Ackermann lemmas, cannot be straightforwardly adapted to the setting of admissible assignments, since formulas in the $\mathcal{L}^*_{\mathrm{DLE}}$ might be interpreted as elements in $\mathbb{A}^\delta \setminus \mathbb{A}$ even if all the propositional variables are interpreted in $\mathbb{A}$, thus we cannot just take $a_0 = \alpha^{\mathbb{A}}(\vec{a}, \vec{x}, \vec{y})$ to be the element in $\mathbb{A}$ in the setting of admissible assignments. Therefore, some adaptations are necessary based on the syntactic shape of the formulas, the definitions of which are analogous to [152, Definition B.3]:

2.7.7. Definition. [Syntactically closed and open formulas]

(1) A formula in $\mathcal{L}^*_{\mathrm{DLE}}$ is *syntactically closed* if all occurrences of nominals, $\succ\!\!-$, $-\!\!\prec$, $f_i^\sharp$ (when $\varepsilon_f(i) = \partial$), $g_i^\flat$ (when $\varepsilon_g(i) = 1$) are positive, and all occurrences of co-nominals, $\leftarrow_H$, $\rightarrow_H$,$f_i^\sharp$ (when $\varepsilon_f(i) = 1$), $g_i^\flat$ (when $\varepsilon_g(i) = \partial$) are negative;

(2) A formula in $\mathcal{L}^*_{\mathrm{DLE}}$ is *syntactically open* if all occurrences of nominals, $\succ\!\!-$, $-\!\!\prec$, $f_i^\sharp$ (when $\varepsilon_f(i) = \partial$), $g_i^\flat$ (when $\varepsilon_g(i) = 1$) are negative, and all occurrences of co-nominals, $\leftarrow_H$, $\rightarrow_H$,$f_i^\sharp$ (when $\varepsilon_f(i) = 1$), $g_i^\flat$ (when $\varepsilon_g(i) = \partial$) are positive.

As is discussed in [55, Section 9], the underlying idea of the definitions above is that given an admissible assignment, the value of a syntactically closed (resp. open) formula is always an closed (resp. open) element in $\mathbb{A}^\delta$ (cf. page 11), i.e., in $\mathsf{K}(\mathbb{A}^\delta)$ (resp. $\mathsf{O}(\mathbb{A}^\delta)$), therefore by compactness, we can get an admissible a_0 as required by the topological Ackermann lemmas stated below, which are analogous to [152, Lemma B.4, B.5]:

2.7.8. Lemma (Right-handed topological Ackermann lemma). *Let α be a syntactically closed formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be syntactically closed (resp. open) and positive (resp. negative) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$, $\vec{\mathbf{m}}$) be all the propositional variables (resp. nominals, co-nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ for $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{A}$ such that $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ and $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ for $1 \leq i \leq n$.

2.7.9. Lemma (Left-handed topological Ackermann lemma). *Let α be a syntactically open formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be syntactically closed (resp. open) and negative (resp. positive) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$, $\vec{\mathbf{m}}$) be all the propositional variables (resp. nominals, co-nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{A}$ *such that* $a_0 \leq \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

The main theorem is summarized as follows:

2.7.10. Theorem. *For any language $\mathcal{L}_{\mathrm{DLE}}$, any inequality on which* ALBA *succeeds is canonical.*

Chapter 3

Algorithmic correspondence and canonicity for possibility semantics

In the present chapter, which is a revised version of the paper [198], we apply unified correspondence theory to possibility semantics, extending the Sahlqvist-type results in [196].

3.1 Introduction

Possibility semantics was proposed by Humberstone [119] as an alternative semantics for modal logic, which is based on possibilities rather than possible worlds in Kripke semantics, where every possibility does not provide truth values of all propositions, but only some of them. Different possibilities are ordered by a refinement relation where some possibilities provide more information about the truth value of propositions than others.

In recent years, possibility semantics has been intensely investigated: [117] gives a construction of canonical possibility models with a finitary flavor, [18] studies the intuitionistic generalization of possibility models, [114] investigates the first-order counterpart of possibility semantics, [115] focuses on the relation between Kripke models and possibility models, [187] provides a bimodal perspective on possibility semantics. A comprehensive study of possibility semantics can be found in [118]. In [196], Yamamoto investigates the correspondence theory for possibility semantics and proves a correspondence theorem for Sahlqvist formulas over full possibility frames—which are the counterparts of (full) Kripke frames in the possibility semantics setting—using insights from algebraic correspondence theory developed in [58]. In [118, Theorem 7.20], Holliday shows that all *inductive* formulas are filter-canonical (i.e. their validity is preserved from the canonical possibility models to their underlying canonical full possibility frames), hence every normal modal logic axiomatized by inductive formulas is sound and complete with respect to its canonical full possibility frame. Inductive formulas [104] form a syntactically defined class of formulas which is strictly larger than the class of Sahlqvist formulas, while they still have first-order correspondence and are canonical [51]. Holliday's result provides the canonicity half of the Sahlqvist-type result for inductive formulas relative to possibility semantics. The present chapter provides the remaining half. Namely, we show that inductive formulas have first-order correspondents in full possibility frames as well as in filter-descriptive possibility frames[1].

We analyze the correspondence phenomenon in possibility semantics using the dual algebraic structures of (full) possibility frames, namely complete (not necessarily atomic) Boolean algebras with complete operator, where atoms are not always available. For correspondence over full possibility frames, we identify two different Boolean algebras with operator as the dual algebraic structures of a given full possibility frame by viewing the full possibility frame in two different ways, namely the Boolean algebra of regular open subsets $\mathbb{B}_{\mathsf{RO}}$ (when viewing the possibility frame as a possibility frame itself) and the Boolean algebra of arbitrary subsets $\mathbb{B}_{\mathsf{K}}$ (when viewing the possibility frame as a bimodal Kripke frame, see [187]), where a canonical order-embedding map $e : \mathbb{B}_{\mathsf{RO}} \to \mathbb{B}_{\mathsf{K}}$ can be defined. The embedding e preserves arbitrary meets, therefore a left adjoint $c : \mathbb{B}_{\mathsf{K}} \to \mathbb{B}_{\mathsf{RO}}$ of e can be defined, which sends a subset X of the domain W of possibilities to the smallest regular open subset containing X. This left adjoint c plays an important role in the dual characterization of the interpretations of nominals and the black connectives (which are going to be interpreted as the adjoints of the interpretations of the box and diamond), which form the ground of the regular open translation of the expanded modal language. In particular, we give an algebraic counterpart of Lemma 3.7 in [196] that every regular open element can be represented as the join of regular open closures of singletons below it, therefore the regular open closures of singletons form the join-generators. When it comes to canonicity, we prove a topological Ackermann lemma similar to [55, Lemma 9.3 and 9.4], which forms the basis of the correspondence result with respect to filter-descriptive frames as well as the canonicity result.

The chapter is structured as follows. Section 3.2 presents preliminaries on possibility semantics, both frame-theoretically and algebraically, as well as the duality theory background of possibility semantics. Section 3.3 gives an algebraic analysis of the semantic environment of possibility semantics for the interpretation of the expanded modal language, the details of which will be given in Section 3.4 together with the regular open translation and the syntactic definition of Sahlqvist and inductive formulas. The Ackermann Lemma Based Algorithm (ALBA) for possibility semantics is given in Section 3.5 as well as some examples, with its soundness proof with respect to full possibility frames in Section 3.6. The soundness proof with respect to filter-descriptive possibility frames and the canonicity-via-correspondence proof are given in Section 3.7. Section 3.8 provides some discussions, and gives some further directions.

[1] As remarked in [118, page 103], correspondence results might be lost when moving from full possibility frames to filter-descriptive possibility frames.

3.2 Preliminaries on possibility semantics

In the present section we collect the preliminaries on possibility semantics. For more details, see e.g. [118, Section 1 and 2] and [196].

3.2.1 Language

Given a set Prop of propositional variables, the basic modal language $\mathcal{L}$ is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$. We define $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\bot := p \wedge \neg p$, $\top := \neg\bot$ and $\Diamond\varphi := \neg\Box\neg\varphi$, respectively. We also use $\mathsf{Prop}(\alpha)$ to denote the propositional variables occuring in α. In the present chapter we will consider only the modal language with only one unary modality.

We will find it convenient to work on *inequalities* (cf. [55]), i.e. expressions of the form $\varphi \leq \psi$, the interpretation of which is equivalent to the implicative formula $\varphi \to \psi$ being true at any point in a model. Throughout the chapter, we will also make substantial use of *quasi-inequalities*, i.e. expressions of the form $\varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$, where & is the meta-level conjunction and $\Rightarrow$ is the meta-level implication.

3.2.2 Downset topology

In order to define possibility frames and possibility models, we will make use of the following auxiliary notions (see e.g. [118, 196]). For every partial order $(W, \sqsubseteq)$, a subset $Y \subseteq W$ is *downward closed* (or a *down-set*) if for all $x, y \in W$, if $x \in Y$ and $y \sqsubseteq x$, then $y \in Y$. For every $X \subseteq W$, the set $\Downarrow X := \{x \in W \mid (\exists y \sqsupseteq x)(y \in X)\}$ is the smallest down-set containing X. The set of all down-sets of $(W, \sqsubseteq)$ forms a topology on W, denoted by $\tau_{\sqsubseteq}$, which we call *the downset topology*.

For any $X \subseteq W$, we let $\mathsf{cl}(X) := \{x \in W \mid (\exists y \sqsubseteq x)(y \in X)\}$ (resp. $\mathsf{int}(X) := \{x \in W \mid (\forall y \sqsubseteq x)(y \in X)\}$) denote the *closure* (resp. *interior*) of X. We also let

$$\mathsf{RO}(W, \tau_{\sqsubseteq}) := \{X \subseteq W \mid \mathsf{int}(\mathsf{cl}(X)) = X\}$$

denote the collection of *regular open* subsets of W. We say a set $Y \subseteq W$ the *regular open closure* of X if Y is the least regular open subset of W containing X, and denote $Y = \mathsf{ro}(X)$.

We collect some useful facts about the downset topology:

3.2.1. Proposition. *(cf. [118, page 16-18]) For every partial order* $(W, \sqsubseteq)$*,*

(1) every regular open subset of $(W, \tau_{\sqsubseteq})$ *is a down-set, and hence a* $\tau_{\sqsubseteq}$*-open subset.*

(2) $\mathsf{ro}(X) = \mathsf{int}(\mathsf{cl}(\Downarrow X))$ *for any subset* $X \subseteq W$.

(3) $\mathsf{ro}(X) = \mathsf{int}(\mathsf{cl}(X))$ *for any* $X \in \tau_{\sqsubseteq}$.

(4) $\varnothing, W \in \mathsf{RO}(W, \tau_{\sqsubseteq})$.

(5) $X \cap Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$ *if* $X, Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$.

(6) $\mathsf{int}(\mathsf{cl}(X \cup Y)) \in \mathsf{RO}(W, \tau_{\sqsubseteq})$ *if* $X, Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$.

(7) $\mathsf{int}(W \setminus X) \in \mathsf{RO}(W, \tau_{\sqsubseteq})$ *if* $X \in \mathsf{RO}(W, \tau_{\sqsubseteq})$.

(8) $\mathsf{RO}(W, \tau_{\sqsubseteq})$ *is closed under arbitrary intersection (cf. [118, footnote 13 on page 17]).*

(9) $(\mathsf{RO}(W, \tau_{\sqsubseteq}), \varnothing, W, \wedge, \vee, -)$ *is a Boolean algebra such that for all* $X, Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$,

$$X \wedge Y = X \cap Y \qquad X \vee Y = \mathsf{int}(\mathsf{cl}(X \cup Y)) \qquad -X = \mathsf{int}(W \setminus X).$$

(10) *for all* $X, Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$,

$$X \supset Y := -X \vee Y = \mathsf{int}((W \setminus X) \cup Y)$$

3.2.3 Relational semantics

In the present subsection, we will collect basic definitions about possibility frames and models, as well as the relational semantics. For more details, we refer the readers to e.g. [55, 118, 196].

For every binary relation R on a set W, we denote $R[X] = \{w \in W \mid (\exists x \in X)Rxw\}$ and $R^{-1}[X] = \{w \in W \mid (\exists x \in X)Rwx\}$, and denote $R[w] := R[\{w\}]$ and $R^{-1}[w] := R^{-1}[\{w\}]$, respectively. Below we give a slightly different but equivalent definition of possibility frames than the one given in [118, Definition 2.21].

3.2.2. Definition. [Possibility frames and models]

A *possibility frame* is a tuple $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, where $W \neq \varnothing$ is the *domain* of $\mathbb{F}$, the *refinement relation*[2] $\sqsubseteq$ is a partial order on W, the *accessibility relation* R is a binary relation[3] on W, and the collection $\mathsf{P} \subseteq \mathsf{RO}(W, \tau_{\sqsubseteq})$ of *admissible subsets* forms a sub-Boolean algebra of $\mathsf{RO}(W, \tau_{\sqsubseteq})$ such that $\Box_{\mathsf{P}}(X) = \{w \in W \mid R[w] \subseteq X\} \in \mathsf{P}$ for any $X \in \mathsf{P}$. A *pointed possibility frame* is a pair $(\mathbb{F}, w)$ where $w \in W$. A *possibility model* is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $V : \mathsf{Prop} \to \mathsf{P}$ is a *valuation* on $\mathbb{F}$. A possibility frame is *full* if $\mathsf{P} = \mathsf{RO}(W, \tau_{\sqsubseteq})$.

[2]We adopt the order of the refinement relation as in [118, 196], which is used in the theory of weak forcing [151], while in the literature of intuitionistic logic, the order is typically the reverse order.

[3]In the literature, some interaction conditions are imposed between the accessibility relation and the refinement relation (cf. [187]). Since these conditions are not needed for our treatment, we do not impose them and we do not discuss them any further in the present chapter.

It follows straightforwardly from the conditions above that P is endowed with the algebraic structure of a BAO. In general, for any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, the *underlying full possibility frame* $\mathbb{F}^\sharp = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_\sqsubseteq))$ might not be well-defined, since $\mathsf{RO}(W, \tau_\sqsubseteq)$ might not be closed under the box operation arising from the relation R. However, in certain situations which we will discuss in Remark 3.7.3, the underlying full possibility frame is well-defined.

Given any possibility model $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$ and any $w \in W$, the *satisfaction relation* is defined as follows (see e.g. [118, Definition 2.3]):

$$
\begin{array}{lll}
\mathbb{M}, w \Vdash p & \text{iff} & w \in V(p) \\
\mathbb{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathbb{M}, w \Vdash \varphi \text{ and } \mathbb{M}, w \Vdash \psi \\
\mathbb{M}, w \Vdash \neg\varphi & \text{iff} & (\forall v \sqsubseteq w)(\mathbb{M}, v \nVdash \varphi) \\
\mathbb{M}, w \Vdash \Box\varphi & \text{iff} & \forall v(Rwv \ \Rightarrow\ \mathbb{M}, v \Vdash \varphi).
\end{array}
$$

For any formula φ, we let $[\![\varphi]\!]^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \Vdash \varphi\}$ denote the *truth set* of φ in $\mathbb{M}$. The formula φ is *globally true* on a possibility model $\mathbb{M}$ (notation: $\mathbb{M} \Vdash \varphi$) if $\mathbb{M}, w \Vdash \varphi$ for every $w \in W$. Moreover, φ is *valid* on a pointed possibility frame $(\mathbb{F}, w)$ (notation: $\mathbb{F}, w \Vdash \varphi$) if $\mathbb{F}, V, w \Vdash \varphi$ for every valuation V. We say that φ is *valid* on a possibility frame $\mathbb{F}$ (notation: $\mathbb{F} \Vdash \varphi$) if φ is valid on $(\mathbb{F}, w)$ for every $w \in W$. The definition of global truth for inequalities and quasi-inequalities is given as follows (see e.g. [55, page 344]):

$$
\begin{array}{lll}
\mathbb{M} \Vdash \varphi \leq \psi & \text{iff} & \text{for all } w \in W, \text{ if } \mathbb{M}, w \Vdash \varphi \text{ then } \mathbb{M}, w \Vdash \psi \\
\mathbb{M} \Vdash \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi & \text{iff} & \text{if } \mathbb{M} \Vdash \varphi_i \leq \psi_i \text{ for all } i \text{ then } \mathbb{M} \Vdash \varphi \leq \psi.
\end{array}
$$

An inequality (resp. quasi-inequality) is valid on $\mathbb{F}$ if it is globally true on $(\mathbb{F}, V)$ for every valuation V.

It is easy to check that the following truth conditions hold for defined connectives (see e.g. [118, 196]):

3.2.3. Proposition. *The following equivalences hold for any possibility model* $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$ *and any* $w \in W$*:*

$$
\begin{array}{lll}
\mathbb{M}, w \Vdash \top & : & \textit{always} \\
\mathbb{M}, w \Vdash \bot & : & \textit{never} \\
\mathbb{M}, w \Vdash \varphi \vee \psi & \textit{iff} & (\forall v \sqsubseteq w)(\exists u \sqsubseteq v)(\mathbb{M}, u \Vdash \varphi \textit{ or } \mathbb{M}, u \Vdash \psi) \\
\mathbb{M}, w \Vdash \varphi \rightarrow \psi & \textit{iff} & (\forall v \sqsubseteq w)(\mathbb{M}, v \Vdash \varphi \ \Rightarrow\ \mathbb{M}, v \Vdash \psi) \\
\mathbb{M}, w \Vdash \Diamond\varphi & \textit{iff} & (\forall v \sqsubseteq w)\exists u(Rvu \wedge (\exists t \sqsubseteq u)(\mathbb{M}, t \Vdash \varphi)).
\end{array}
$$

The following proposition can be understood as stating that inequalities are equivalent to implicative formulas at the level of global truth and validity:

3.2.4. Proposition. *The following equivalences hold for any possibility model* $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$ *and any* $w \in W$*:*

$$
\begin{array}{lll}
\mathbb{M} \Vdash \varphi \rightarrow \psi & \textit{iff} & [\![\varphi]\!]^{\mathbb{M}} \subseteq [\![\psi]\!]^{\mathbb{M}} \textit{ iff } \mathbb{M} \Vdash \varphi \leq \psi \\
\mathbb{F} \Vdash \varphi \rightarrow \psi & \textit{iff} & \mathbb{F} \Vdash \varphi \leq \psi.
\end{array}
$$

The following proposition can be understood as stating that the interpretation of modal formulas on possibility frames and models can be obtained from the standard algebraic semantics for modal logic via the duality established in [118]. Indeed, since P is a sub-Boolean algebra of RO, all Boolean operations in P are defined in the same way as in RO.

3.2.5. Proposition. *(cf. [118, Fact 2.5]) For any possibility model* $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$ *and any* $w \in W$,

$$
\begin{array}{rcll}
[\![\top]\!]^{\mathbb{M}} & = & W & \\
[\![\bot]\!]^{\mathbb{M}} & = & \varnothing & \\
[\![\varphi \wedge \psi]\!]^{\mathbb{M}} & = & [\![\varphi]\!]^{\mathbb{M}} \wedge [\![\psi]\!]^{\mathbb{M}} & \\
[\![\varphi \vee \psi]\!]^{\mathbb{M}} & = & [\![\varphi]\!]^{\mathbb{M}} \vee [\![\psi]\!]^{\mathbb{M}} & = \mathsf{ro}([\![\varphi]\!]^{\mathbb{M}} \cup [\![\psi]\!]^{\mathbb{M}}) \\
[\![\neg\varphi]\!]^{\mathbb{M}} & = & -[\![\varphi]\!]^{\mathbb{M}} & \\
[\![\varphi \to \psi]\!]^{\mathbb{M}} & = & [\![\varphi]\!]^{\mathbb{M}} \supset [\![\psi]\!]^{\mathbb{M}} & \\
[\![\Box\varphi]\!]^{\mathbb{M}} & = & \Box_{\mathsf{P}}([\![\varphi]\!]^{\mathbb{M}}) & \\
[\![\Diamond\varphi]\!]^{\mathbb{M}} & = & \mathsf{int}(R^{-1}[\mathsf{cl}([\![\varphi]\!]^{\mathbb{M}})]). &
\end{array}
$$

3.2.4 Algebraic semantics

Thanks to the duality in [118], we will be able to work throughout the chapter in the environment of the dual algebras of the possibility frames, namely the Boolean algebras with operators. In the present subsection, we collect basic definitions on the algebraic semantics dual to possibility semantics. For more details, see e.g. [21, Chapter 5], [22, Chapter 6], [88] and [118].

3.2.6. Definition. [Boolean algebra with operator](cf. e.g. [21, Definition 5.19]) A Boolean algebra with operator (BAO) is a tuple $\mathbb{B} = (B, \bot, \top, \wedge, \vee, -, \Box)$, where $(B, \bot, \top, \wedge, \vee, -)$ is a Boolean algebra and moreover, $\Box\top = \top$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $a, b \in B$. The order on $\mathbb{B}$ is defined as $a \leq b$ iff $a \wedge b = a$. We will sometimes abuse notation and use $\mathbb{B}$ to denote B.

For any BAO $\mathbb{B}$, let $\mathbb{B}^{\partial}$ denote its order-dual BAO, and let $\mathbb{B}^1 = \mathbb{B}$. An *order-type* ε over $n \in N$ (or an *n-order-type*) is an element of $\{1, \partial\}^n$, and we use ε_i to denote its i-th coordinate. We omit n if no confusion arises. The *dual order-type* of ε is denoted by ε^{∂}, where $\varepsilon^{\partial}_i = 1$ (resp. ∂) iff $\varepsilon_i = \partial$ (resp. 1). For any n-order-type ε, we let $\mathbb{B}^{\varepsilon}$ be the product algebra $\mathbb{B}^{\varepsilon_1} \times \ldots \times \mathbb{B}^{\varepsilon_n}$.

An *assignment* on $\mathbb{B}$ is a map $\theta : \mathsf{Prop} \to \mathbb{B}$, which can be extended to all formulas as usual. We use $\varphi^{(\mathbb{B},\theta)}$ or $\theta(\varphi)$ to denote the value of φ in $\mathbb{B}$ under θ. We say that a formula φ (resp. an inequality $\varphi \leq \psi$) is *true* on $\mathbb{B}$ under θ (notation: $(\mathbb{B}, \theta) \models \varphi$, $(\mathbb{B}, \theta) \models \varphi \leq \psi$) if $\varphi^{(\mathbb{B},\theta)} = \top$ (resp. $\varphi^{(\mathbb{B},\theta)} \leq \psi^{(\mathbb{B},\theta)}$), and φ (resp. $\varphi \leq \psi$) is *valid* on $\mathbb{B}$ (notation: $\mathbb{B} \models \varphi$, $\mathbb{B} \models \varphi \leq \psi$) if $(\mathbb{B}, \theta) \models \varphi$ (resp. $(\mathbb{B}, \theta) \models \varphi \leq \psi$) for every θ.

A quasi-inequality $\&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ is true on $\mathbb{B}$ under θ (notation: $(\mathbb{B}, \theta) \models \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$) if $\varphi^{(\mathbb{B},\theta)} \leq \psi^{(\mathbb{B},\theta)}$ holds whenever $\varphi_i^{(\mathbb{B},\theta)} \leq \psi_i^{(\mathbb{B},\theta)}$ holds for every

$1 \leq i \leq n$, and $\&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ is valid on $\mathbb{B}$ (notation: $\mathbb{B} \models \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$) if $(\mathbb{B}, \theta) \models \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ for every θ.

Another useful way to look at a formula $\varphi(p_1, \ldots, p_n)$ is to interpret it as an n-ary function $\varphi^{\mathbb{B}} : \mathbb{B}^n \to \mathbb{B}$ such that $\varphi^{\mathbb{B}}(a_1, \ldots, a_n) = \theta(\varphi)$ where $\theta : \mathsf{Prop} \to \mathbb{B}$ satisfies $\theta(p_i) = a_i$, $i = 1, \ldots, n$.

Recall that an element $b \in \mathbb{B}$ is an *atom* if $b \neq \bot$ and for any $c \in B$ s.t. $c \leq b$, either $c = \bot$ or $c = b$. Moreover, a modal operator $\Box$ on $\mathbb{B}$ is *completely multiplicative* if $\bigwedge\{\Box a \mid a \in A\}$ exists for any $A \subseteq B$ such that $\bigwedge A$ exists, and $\bigwedge\{\Box a \mid a \in A\} = \Box(\bigwedge A)$. A BAO is (see e.g. [118, Definition 5.1]):

($\mathcal{C}$) *complete* if the greatest lower bound $\bigwedge A$ and least upper bound $\bigvee A$ exist for any $A \subseteq B$;

($\mathcal{A}$) *atomic* if for any $a \neq \bot$ there exists some atom $b \in B$ such that $\bot \neq b \leq a$;

($\mathcal{V}$) *completely multiplicative*[4] if $\Box$ is completely multiplicative.

A BAO is a *$\mathcal{CV}$-BAO* if it is complete and completely multiplicative, and is a *$\mathcal{CAV}$-BAO* if it is complete, atomic and completely multiplicative, and other abbreviations are given in a similar way. Notice that the definitions of completeness and atomicity also apply to Boolean algebras.

As to the correspondence between BAOs and possibility frames, the following definition provides the frame-to-algebra direction:

3.2.7. Definition. (cf. [118, Definition 5.4]) For any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, let the BAO P of Definition 3.2.2 be the BAO *dual* to $\mathbb{F}$, denoted by $\mathbb{B}_{\mathsf{P}}$. If $\mathbb{F}$ is a full possibility frame, then $\mathsf{P} = \mathsf{RO}(W, \tau_{\sqsubseteq})$ and we refer to $\mathbb{B}_{\mathsf{P}}$ as $\mathbb{B}_{\mathsf{RO}}$ (the *regular open dual BAO* of $\mathbb{F}$).

3.2.8. Proposition. *(cf. [118, Theorem 5.6(2)]) For any full possibility frame $\mathbb{F}$, $\mathbb{B}_{\mathsf{RO}}$ is a $\mathcal{CV}$-BAO.*

The existence of the frame-to-algebra direction of the duality defined above induces a bijection between valuations on $\mathbb{F}$ and interpretations of propositional variables into $\mathbb{B}_{\mathsf{P}}$.

3.2.9. Definition. For any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, any valuation V on $\mathbb{F}$ can be associated with the assignment $\theta_V : \mathsf{Prop} \to \mathbb{B}_{\mathsf{P}}$ defined by $\theta_V(p) := V(p)$ for every $p \in \mathsf{Prop}$. Conversely, any assignment $\theta : \mathsf{Prop} \to \mathbb{B}_{\mathsf{P}}$ can be associated with the valuation $V_\theta : \mathsf{Prop} \to \mathsf{P}$ defined by $V_\theta(p) = \theta(p)$ for every $p \in \mathsf{Prop}$.

It is easy to check the following equivalences hold:

[4]In [118], algebras satisfying condition ($\mathcal{V}$) are called completely additive.

3.2.10. Proposition. *(Folklore.) For any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, for any valuation $V : \mathsf{Prop} \to \mathsf{P}$ on $\mathbb{F}$, any assignment $\theta : \mathsf{Prop} \to \mathsf{P}$ on $\mathbb{B}_\mathsf{P}$, any $w \in W$, any formula φ, any inequality $\varphi \leq \psi$, any quasi-inequality $\&_{i=1}^n(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$,*

(1) $\mathbb{F}, V, w \Vdash \varphi$ *iff* $w \in \theta_V(\varphi)$;

(2) $\mathbb{F}, V_\theta, w \Vdash \varphi$ *iff* $w \in \theta(\varphi)$;

(3) $\mathbb{F} \Vdash \varphi$ *iff* $\mathbb{B}_\mathsf{P} \models \varphi$;

(4) $\mathbb{F} \Vdash \varphi \leq \psi$ *iff* $\mathbb{B}_\mathsf{P} \models \varphi \leq \psi$;

(5) $\mathbb{F} \Vdash \&_{i=1}^n(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ *iff* $\mathbb{B}_\mathsf{P} \models \&_{i=1}^n(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$.

The duality theoretic facts outlined above make it possible to transfer the development of correspondence theory from frames to algebras, similarly to the way in which algebraic correspondence is developed for Kripke frames. In particular, $\mathbb{B}_{\mathsf{RO}}$ will be the algebra where correspondence theory over full possibility frames is developed. However, the essential difference between the correspondence for Kripke semantics and the present setting is that the algebra $\mathbb{B}_{\mathsf{RO}}$ is not atomic in general (see [118, Example 2.40]). This implies that some of the rules of the original ALBA-type algorithm (cf. [51, 55]) for complex algebras of Kripke frames (namely, the so-called approximation rules which relied on atomicity) are not going to be sound in this setting. The analysis of the semantic environment of the regular open dual BAOs, developed in the next section, will give insights on how to design the algorithm in this semantic setting.

3.3 Semantic environment for the language of ALBA

In the present section, we will provide the algebraic semantic environment for the correspondence algorithm ALBA in the setting of possibility semantics. We will show the semantic properties which will be used for the interpretation of the expanded modal language of the algorithm ALBA in Section 3.4.1. The first notable feature of this language is that it includes special variables (besides the propositional variables), the so-called *nominals*, which in the original setting are interpreted as the *atoms* of the complex algebras of Kripke frames. This interpretation of nominals pivots on the fact that the complex algebras of Kripke frames are atomic, that is, are completely join-generated by their atoms. Likewise, in order to define a suitable interpretation for the nominals in the possibility setting, we need to find a class of elements which join-generate the complex algebra of any full possibility frame. Towards this goal, our strategy will consist in defining a BAO $\mathbb{B}_\mathsf{K}$ in which the BAO $\mathbb{B}_{\mathsf{RO}}$ can be order-embedded. This algebra will be used as an auxiliary tool to show that every element in $\mathbb{B}_{\mathsf{RO}}$ can be represented as the join of regular open closures of atoms in $\mathbb{B}_\mathsf{K}$. Hence, this will show that the regular open closures of atoms in $\mathbb{B}_\mathsf{K}$ are a suitable class of interpretants for nominal variables.

The second notable feature of the expanded language of ALBA is that it includes additional modal operators interpreted as the adjoints of the modal operators of the original language. In what follows, we will show that also these connectives have a natural interpretation in $\mathbb{B}_{\mathsf{RO}}$.

3.3.1 The auxiliary BAO $\mathbb{B}_{\mathsf{K}}$

Clearly, any full possibility frame $\mathbb{F}_{\mathsf{RO}} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$ can be associated with the *bimodal frame* $\mathbb{F}_{\mathsf{K}} = (W, \sqsubseteq, R)$, the complex algebra of which is the bimodal BAO $\mathbb{B}_{\mathsf{K}}$ (cf. [187]).

Diagrammatically, the $\mathcal{CAV}$-BAO $\mathbb{B}_{\mathsf{K}}$ dually corresponds to the Kripke frame $\mathbb{F}_{\mathsf{K}}$, and the $\mathcal{CV}$-BAO $\mathbb{B}_{\mathsf{RO}}$ to the full possibility frame $\mathbb{F}_{\mathsf{RO}}$, e is the order-embedding which sends a regular open subset in $\mathbb{B}_{\mathsf{RO}}$ to itself in $\mathbb{B}_{\mathsf{K}}$, and U sends a full possibility frame to its underlying bimodal Kripke frame "forgetting" the algebra $\mathsf{RO}(W, \tau_{\sqsubseteq})$ (and hence the restriction on the admissible valuations).

$$\begin{array}{ccc}
\mathbb{B}_{\mathsf{K}} & \xleftrightarrow{\;\cong^{\partial}\;} & \mathbb{F}_{\mathsf{K}} \\
{\scriptstyle e}\uparrow & & \uparrow{\scriptstyle U} \\
\mathbb{B}_{\mathsf{RO}} & \xleftrightarrow{\;\cong^{\partial}\;} & \mathbb{F}_{\mathsf{RO}}
\end{array}$$

The formal definition of the BAO $\mathbb{B}_{\mathsf{K}}$ is reported below:

3.3.1. Definition. [Full dual Boolean algebra with operators] For any full possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$, the *full dual Boolean algebra with operators (full dual BAO)* $\mathbb{B}_{\mathsf{K}}$[5] is defined as $\mathbb{B}_{\mathsf{K}} = (P(W), \varnothing, W, \cap, \cup, -, \Box_{\mathsf{K}}, \Box_{\sqsubseteq})$, where $\cap, \cup, -$ are set-theoretic intersection, union and complementation respectively, $\Box_{\mathsf{K}}(a) = \{w \in W \mid R[w] \subseteq a\}$, and $\Box_{\sqsubseteq}(a) = \{w \in W \mid (\forall v \sqsubseteq w)(v \in a)\}$.

It is easy to see that $\mathbb{B}_{\mathsf{K}}$ is a complete *atomic* Boolean algebra with complete operators. It is also clear that the carrier set of the regular open dual BAO is a subset of the full dual BAO, hence the natural embedding $e : \mathbb{B}_{\mathsf{RO}} \hookrightarrow \mathbb{B}_{\mathsf{K}}$ is well defined and is an order-embedding. Notice that by Proposition 3.2.1, arbitrary intersections of regular open sets in a downset topology are again regular open, therefore e is completely meet-preserving. Notice also that $\Box_{\mathsf{RO}}$ is the restriction of $\Box_{\mathsf{K}}$ to $\mathbb{B}_{\mathsf{RO}}$. All these observations can be summarized as follows:

[5] Notice that here the "full" means that the carrier set is the powerset of W, rather than the "full" in "full possibility frame".

3.3.2. LEMMA. *$e : \mathbb{B}_{\mathsf{RO}} \hookrightarrow \mathbb{B}_{\mathsf{K}}$ is a completely meet-preserving order-embedding such that $e \circ \Box_{\mathsf{RO}} = \Box_{\mathsf{K}} \circ e$.*

However, it is important to stress that, since $\mathbb{B}_{\mathsf{RO}}$ and $\mathbb{B}_{\mathsf{K}}$ have different definitions of join and complementation, $\mathbb{B}_{\mathsf{RO}}$ is *not* a subalgebra of $\mathbb{B}_{\mathsf{K}}$.

The next corollary follows immediately from the previous lemma (see e.g. [64, Proposition 7.34]):

3.3.3. COROLLARY. *$e : \mathbb{B}_{\mathsf{RO}} \hookrightarrow \mathbb{B}_{\mathsf{K}}$ has a left adjoint $c : \mathbb{B}_{\mathsf{K}} \to \mathbb{B}_{\mathsf{RO}}$ defined, for every $a \in \mathbb{B}_{\mathsf{K}}$,*

$$c(a) = \bigwedge\nolimits_{\mathsf{RO}}\{b \in \mathbb{B}_{\mathsf{RO}} \mid a \leq e(b)\}.$$

Clearly, $c(X) = \mathsf{ro}(X)$ for any $X \subseteq W$, Indeed, by definition $c(X) = \bigwedge_{\mathsf{RO}}\{Y \in \mathsf{RO}(W, \tau_{\sqsubseteq}) \mid X \leq e(Y)\} = \bigcap\{Y \in \mathsf{RO}(W, \tau_{\sqsubseteq}) \mid X \subseteq Y\}$, which is the least regular open set containing X. The closure operator c will be referred to as the *regular open closure map* and $c(a)$ as the *regular open closure* of a. We let $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) := \{c(x) \mid x \in \mathsf{At}(\mathbb{B}_{\mathsf{K}})\}$ (here PsAt stands for pseudo-atom, and $\mathsf{At}(\mathbb{B})$ denotes the set of atoms in the BAO $\mathbb{B}$) be the set of regular open closures of atoms in $\mathbb{B}_{\mathsf{K}}$, which will be shown to be the join-generators of $\mathbb{B}_{\mathsf{RO}}$.

A class of interpretants for nominals

As mentioned ealy on, the key requirement for a suitable class of interpretants for nominals is that it is join-dense in $\mathbb{B}_{\mathsf{RO}}$. In what follows, we give a proof of this property for $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$. This result has already been proved in [196, Lemma 3.7]; we give an alternative proof in the dual algebraic setting. Let us preliminarily recall that, by general facts of the theory of closure operators on posets, $c \circ e = Id_{\mathbb{B}_{\mathsf{RO}}}$ (cf. [64, Exercise 7.13]).

3.3.4. PROPOSITION. *For any $a \in \mathbb{B}_{\mathsf{RO}}$,*

$$a = \bigvee_{\mathsf{RO}}\{c(x) \mid x \in \mathsf{At}(\mathbb{B}_{\mathsf{K}}) \text{ and } x \leq e(a)\} = \bigvee_{\mathsf{RO}}\{y \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \mid y \leq a\}.$$

Proof:
The first equality follows from the fact that $c \circ e = Id_{\mathbb{B}_{\mathsf{RO}}}$ and that left adjoint preserve arbitrary existing joins; the second equality, from the definition of $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$ and adjunction between c and e. □

3.3.2 Interpreting the additional connectives of the expanded language of ALBA

As mentioned early on, $\mathbb{B}_{\mathsf{RO}}$ and $\mathbb{B}_{\mathsf{K}}$ are both complete, and $\Box_{\mathsf{RO}} : \mathbb{B}_{\mathsf{RO}} \to \mathbb{B}_{\mathsf{RO}}$ and $\Box_{\mathsf{K}} : \mathbb{B}_{\mathsf{K}} \to \mathbb{B}_{\mathsf{K}}$ are both completely meet-preserving. Thus both of them have left adjoints, which are denoted by $\blacklozenge_{\mathsf{RO}}$ and $\blacklozenge_{\mathsf{K}}$, respectively. They will be used as the semantic interpretation of the additional connective in the expanded modal language in the next section. In what follows we will explicitly compute the definitions of the adjoints. The computation of $\blacklozenge_{\mathsf{RO}}$ in Lemma 3.3.6 and Corollary 3.3.7 is essentially the algebraic counterpart of [196, Lemma 3.8].

3.3.5. Lemma. *(Folklore.)* $\blacklozenge_{\mathsf{K}}(X) = R[X]$ *for any* $X \subseteq W$.

Proof:
By definition of adjunction, for any $Y \in P(W)$, $w \in W$,

$$\begin{array}{rll} \blacklozenge_{\mathsf{K}}(\{w\}) \subseteq Y & \text{iff} & \{w\} \subseteq \Box_{\mathsf{K}}(Y) \\ & \text{iff} & \{w\} \subseteq \{v \in W \mid R[v] \subseteq Y\} \\ & \text{iff} & R[w] \subseteq Y, \end{array}$$

Therefore $\blacklozenge_{\mathsf{K}}(\{w\}) = R[w]$. Since left adjoints preserve existing joins,

$$\begin{array}{rcl} \blacklozenge_{\mathsf{K}}(X) & = & \blacklozenge_{\mathsf{K}}(\bigcup\{\{w\} \mid w \in X\}) \\ & = & \bigcup\{\blacklozenge_{\mathsf{K}}(\{w\}) \mid w \in X\} \\ & = & \bigcup\{R[w] \mid w \in X\} \\ & = & R[X]. \end{array}$$

□

3.3.6. Lemma. $\blacklozenge_{\mathsf{RO}}(a) = (c \circ \blacklozenge_{\mathsf{K}} \circ e)(a)$.

Proof:
We have the following chain of equalities:

$$\begin{array}{rcll} \blacklozenge_{\mathsf{RO}}(a) & = & \bigwedge_{\mathsf{RO}}\{b \in \mathbb{B}_{\mathsf{RO}} \mid a \leq \Box_{\mathsf{RO}}(b)\} & \text{(adjunction property)} \\ & = & \bigwedge_{\mathsf{RO}}\{b \in \mathbb{B}_{\mathsf{RO}} \mid e(a) \leq (\Box_{\mathsf{K}} \circ e)(b)\} & \text{(Lemma 3.3.2)} \\ & = & \bigwedge_{\mathsf{RO}}\{b \in \mathbb{B}_{\mathsf{RO}} \mid (\blacklozenge_{\mathsf{K}} \circ e)(a) \leq e(b)\} & \text{(definition of adjunction)} \\ & = & \bigwedge_{\mathsf{RO}}\{b \in \mathbb{B}_{\mathsf{RO}} \mid (c \circ \blacklozenge_{\mathsf{K}} \circ e)(a) \leq b\} & \text{(definition of adjunction)} \\ & = & (c \circ \blacklozenge_{\mathsf{K}} \circ e)(a). & \end{array}$$

□

3.3.7. Corollary. $\blacklozenge_{\mathsf{RO}}(X) = \mathsf{ro}(R[X])$ *for any* $X \in \mathsf{RO}(W, \tau_{\sqsubseteq})$.

Proof:
We have the following chain of equalities:

$$\begin{aligned}
\blacklozenge_{\mathsf{RO}}(X) &= (c \circ \blacklozenge_{\mathsf{K}} \circ e)(X) && \text{(Lemma 3.3.6)}\\
&= (c \circ \blacklozenge_{\mathsf{K}})(X) &&\\
&= c(R[X]) && \text{(Lemma 3.3.5)}\\
&= \mathsf{ro}(R[X]). && \text{(Corollary 3.3.3)}
\end{aligned}$$

□

3.4 Preliminaries on algorithmic correspondence

In the present section, we will collect preliminaries on algorithmic correspondence for possibility semantics. The theory of unified correspondence is based on duality and order-theoretic insights [48, 58], and distills the order-theoretic properties from concrete semantic settings. We will specialize it to the possibility semantics setting and explain how it works for correspondence over full possibility frames, and this methodology works in a similar way in the setting of filter-descriptive possibility frames.

$$\begin{array}{ccc}
\mathbb{B}_{\mathsf{RO}} \models \varphi(\vec{p}) & \Leftrightarrow & \mathbb{F}_{\mathsf{RO}} \Vdash \varphi(\vec{p}) \\
\Updownarrow & & \\
\mathbb{B}_{\mathsf{RO}} \models \mathrm{Pure}(\varphi(\vec{p})) & \Leftrightarrow & \mathbb{F}_{\mathsf{RO}} \models \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p})))
\end{array}$$

The argument works in three steps: the first step is to move from the relational semantics side to the dual algebraic side, i.e. understand the validity of $\varphi(\vec{p})$ from the validity in the full possibility frame $\mathbb{F}_{\mathsf{RO}}$ to the validity in the regular open dual BAO $\mathbb{B}_{\mathsf{RO}}$. The second step is to use an algorithm to transform the formula $\varphi(\vec{p})$ into an equivalent set of pure quasi-inequalities $\mathrm{Pure}(\varphi(\vec{p}))$ which does not contain propositional variables, but only nominals[6], which will be shown to be sound with respect to the algebraic semantics. The last step translates the pure quasi-inequalities into a first-order formula $\mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p})))$, which is the first-order correspondent of $\varphi(\vec{p})$ over full possibility frames.

Therefore, the ingredients for the algorithmic correspondence proof to go through can be listed as follows:

- An expanded modal language as the syntax of the algorithm, as well as their interpretations in $\mathbb{B}_{\mathsf{RO}}$;

[6] In lattice-based logic settings, there is another kind of variables called conominals (see e.g. [55]), which are interpreted as co-atoms or complete meet-irreducibles. Since in the Boolean setting, conominals can be interpreted as the negation of nominals, they are not really necessary here. In the remainder of the chapter, we will not use conominals.

- An algorithm which transforms a given modal formula $\varphi(\vec{p})$ into equivalent pure quasi-inequalities Pure($\varphi(\vec{p})$);
- A soundness proof of the algorithm with respect to $\mathbb{B}_{\mathsf{RO}}$;
- A syntactically identified class of formulas on which the algorithm is successful;
- A first-order correspondence language and first-order translation which transforms pure quasi-inequalities into their equivalent first-order correspondents.

In the remainder of the chapter, we will define an expanded modal language which the algorithm will manipulate (Section 3.4.1), define the first-order correspondence language of possibility frames (Section 3.4.1) and the counterpart of the standard translation into this language, which we refer to as *regular open translation* (Section 3.4.1). We report on the definition of Sahlqvist and inductive formulas (Section 3.4.2), and define a modified version of the algorithm ALBA suitable for the possibility semantic environment (Section 3.5), show its soundness over regular open dual BAOs and state its success on Sahlqvist and inductive formulas (Section 3.6). In Section 3.7 we show the soundness of the algorithm over the dual BAOs of filter-descriptive possibility frames.

3.4.1 The expanded modal language and the regular open translation

In the present section, we will define the expanded modal language for the algorithm, the first-order and second-order correspondence language, as well as the regular open translation of the expanded modal language into the correspondence language. We will also show that the translation preserves truth conditions. Our treatment is similar to [55].

The expanded modal language $\mathcal{L}^+$

The expanded modal language $\mathcal{L}^+$ is a proper expansion of the modal language. Apart from the propositional variables and connectives in the modal language, there are also a set Nom of *nominals*, a special kind of variables to be interpreted as elements in $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, and the *black connectives* $\blacklozenge, \blacksquare$, i.e. the unary connectives to be interpreted as the adjoints of $\Box$ and $\Diamond$ respectively. The formal definition of the formulas in the expanded modal language $\mathcal{L}^+$ is given as follows:

$$\varphi ::= p \mid \mathbf{i} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathsf{Prop}$ and $\mathbf{i} \in \mathsf{Nom}$. We also define $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$, and the other abbreviations are defined similar to the basic modal language. It will be convenient to use the abbreviations as primitive symbols in the definition of the rules in the algorithm.

In order to interpret the expanded modal language on possibility frames and the dual BAOs, we need to extend the valuation V and assignment θ also to nominals. As we have already seen in Section 3.3, every element in $\mathbb{B}_{\mathsf{RO}}$ can be represented as the join of elements in $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \subseteq \mathbb{B}_{\mathsf{RO}} = \mathsf{RO}(W, \tau_{\sqsubseteq})$. Therefore, we are going to interpret the nominals as elements in $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, i.e. $V(\mathbf{i}), \theta(\mathbf{i}) \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \subseteq \mathsf{RO}(W, \tau_{\sqsubseteq})$.

The satisfaction relation for the additional symbols is given as follows:

3.4.1. Definition. In any possibility model $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$,

$$\begin{array}{lll} \mathbb{M}, w \Vdash \mathbf{i} & \text{iff} & w \in V(\mathbf{i}); \\ \mathbb{M}, w \Vdash \blacklozenge\varphi & \text{iff} & (\forall v \sqsubseteq w)(\exists u \sqsubseteq v)(\exists t \sqsupseteq u)\exists s(Rst \text{ and } \mathbb{M}, s \Vdash \varphi). \end{array}$$

Notice that $V(\mathbf{i})$ is not necessarily in P, but it is always in $\mathsf{RO}(W, \tau_{\sqsubseteq})$. Similarly, P is not necessarily closed under $\blacklozenge_{\mathsf{RO}}$, but $\mathsf{RO}(W, \tau_{\sqsubseteq})$ is. Therefore, when interpreting formulas in the expanded modal language, we only restrict the interpretations of propositional variables to P (therefore also all formulas in the basic modal language), and allow formulas in the expanded modal language to be interpreted in $\mathsf{RO}(W, \tau_{\sqsubseteq})$. Truth set and validity are defined similarly to the basic modal language.

It is easy to check that the following facts hold for the expanded modal language:

3.4.2. Proposition. *In any possibility model $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$,*

- $\mathbb{M}, w \Vdash \blacklozenge\varphi$ *iff* $w \in \mathsf{ro}(R[[\![\varphi]\!]^{\mathbb{M}}])$;
- $\mathbb{M}, w \Vdash \blacksquare\varphi$ *iff* $(\forall v \sqsubseteq w)(\forall u \sqsupseteq v)\forall t(Rtu \Rightarrow (\exists s \sqsubseteq t)(\mathbb{M}, s \Vdash \varphi))$;
- $[\![\blacklozenge\varphi]\!]^{\mathbb{M}} = \blacklozenge_{\mathsf{RO}}[\![\varphi]\!]^{\mathbb{M}}$;
- $[\![\blacksquare\varphi]\!]^{\mathbb{M}} = (-_{\mathsf{RO}} \circ \blacklozenge_{\mathsf{RO}} \circ -_{\mathsf{RO}})([\![\varphi]\!]^{\mathbb{M}})$.

The next proposition shows that $\blacksquare$ is interpreted as the right adjoint of $\Diamond$:

3.4.3. Proposition. *For any $X, Y \in \mathsf{RO}(W, \tau_{\sqsubseteq})$,*

$$(-_{\mathsf{RO}} \circ \Box_{\mathsf{RO}} \circ -_{\mathsf{RO}})(X) \subseteq Y \textit{ iff } X \subseteq (-_{\mathsf{RO}} \circ \blacklozenge_{\mathsf{RO}} \circ -_{\mathsf{RO}})(Y).$$

For the algebraic semantics of the expanded modal language, we use a kind of hybrid algebraic structures obtained from arbitrary possibility frames: consider a possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$ (whose underlying full possibility frame is well-defined) and its underlying full possibility frame $\mathbb{F}^{\sharp} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$, we dualize the latter to obtain the regular open dual BAO $\mathbb{B}_{\mathsf{RO}} = (\mathsf{RO}(W, \tau_{\sqsubseteq}), \varnothing, W, \wedge_{\mathsf{RO}}, \vee_{\mathsf{RO}}, -_{\mathsf{RO}}, \Box_{\mathsf{RO}})$, and put an admissible set P on top of it and get a *hybrid dual BAO* $(\mathbb{B}_{\mathsf{RO}}, \mathsf{P})$, which restricts the assignment of propositional variables to P, but still allows formulas in the expanded modal language to range over $\mathsf{RO}(W, \tau_{\sqsubseteq})$. For the interpretation of the expanded modal language, we require that $\theta(\mathbf{i}) \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$ and $\blacklozenge$ is interpreted as $\blacklozenge_{\mathsf{RO}} : \mathbb{B}_{\mathsf{RO}} \to \mathbb{B}_{\mathsf{RO}}$.

For the definition of validity, we use the notation $\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{P}} \varphi$ to indicate that the assignments of propositional variables range over P (while the assignments of nominals range over $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$).

It is easy to check that Proposition 3.2.10 generalizes to the expanded modal language in the case of full possibility frames. For the case of arbitrary possibility frames, we use the hybrid dual BAO $(\mathbb{B}_{\mathsf{RO}}, \mathsf{P})$ and the adapted version of validity $\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{P}} \varphi$. The discussion above can be summarized as follows:

3.4.4. Proposition. *For any formula φ in the expanded modal language,*

- *For any full possibility frame $\mathbb{F}_{\mathsf{RO}}$ and its dual BAO $\mathbb{B}_{\mathsf{RO}}$,*
  - $\mathbb{F}_{\mathsf{RO}} \Vdash \varphi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models \varphi$;
  - $\mathbb{F}_{\mathsf{RO}} \Vdash \varphi \leq \psi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models \varphi \leq \psi$;
  - $\mathbb{F}_{\mathsf{RO}} \Vdash \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$.
- *For any possibility frame $\mathbb{F}_{\mathsf{P}}$ and its hybrid dual BAO $(\mathbb{B}_{\mathsf{RO}}, \mathsf{P})$,*
  - $\mathbb{F}_{\mathsf{P}} \Vdash \varphi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{P}} \varphi$;
  - $\mathbb{F}_{\mathsf{P}} \Vdash \varphi \leq \psi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{P}} \varphi \leq \psi$;
  - $\mathbb{F}_{\mathsf{P}} \Vdash \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$ *iff* $\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{P}} \&_{i=1}^{n}(\varphi_i \leq \psi_i) \Rightarrow \varphi \leq \psi$.

The correspondence languages

In order to express the first-order correspondents of modal formulas, we need to define the first-order and second-order correspondence language $\mathcal{L}^1$ and $\mathcal{L}^2$ (see e.g. [196]). The first-order correspondence language $\mathcal{L}^1$ consists of a set of unary predicate symbols P_n, each of which corresponds to a propositional variable p_n, two binary relation symbols $\sqsubseteq$ and R corresponding to the refinement relation and the accessibility relation respectively, a set of individual symbols i_n, each of which corresponds to a nominal $\mathbf{i}_n$, and the quantifiers $\forall x, \exists x$ are first-order, i.e. ranging over individual variables. The second-order correspondence language $\mathcal{L}^2$ contains all the symbols from $\mathcal{L}^1$ as well as second-order quantifiers $\forall^{\mathsf{P}} P, \exists^{\mathsf{P}} P$ over unary predicate variables. In addition, unary predicate symbols are interpreted as admissible subsets, and the second-order quantifiers range over admissible subsets.

The semantic structures to interpret the first-order and second-order formulas are the possibility models $\mathbb{M} = (W, \sqsubseteq, R, \mathsf{P}, V)$, where an individual symbol i_n is interpreted as a state $\underline{i_n} \in W$ such that $\mathsf{ro}(\{\underline{i_n}\}) = V(\mathbf{i}_n)$, a unary predicate symbols P_n is interpreted as $\underline{V(p_n)} \in \mathsf{P}$, and the binary relation symbols $\sqsubseteq$ and R are interpreted as the refinement relation and the accessibility relation denoted by the same symbol, respectively. At the level of possibility frames, we will abuse notation to take the unary predicate symbols P_n and the individual symbols i_n as variables and use quantifiers over them, and second-order quantifiers range over P. We use $[\![\alpha(\vec{x})]\!]^{\mathbb{M}}$ to denote the n-tuples $\vec{w} \in W^n$ that make $\alpha(\vec{x})$ true in the model $\mathbb{M}$, i.e. $[\![\alpha(\vec{x})]\!]^{\mathbb{M}} := \{\vec{w} \in W^n \mid \mathbb{M} \models \alpha[\vec{w}]\}$, which is called the *truth set* of $\alpha(\vec{x})$ in $\mathbb{M}$.

In the definition of correspondence between a modal formula and a first-order formula, we will require the first-order formula to contain only binary relation symbols, and not contain any unary predicate symbol. Now the definition can be given as follows in the setting of full possibility frames:

3.4.5. Definition. (cf. [196, Definition 2.7]) We say that a modal formula φ in the basic modal language $\mathcal{L}$ *locally corresponds* to a first-order formula $\alpha(x)$ in the first-order correspondence language with no occurence of unary predicate symbols if for any full possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$, any $w \in W$, $\mathbb{F}, w \Vdash \varphi$ iff $\mathbb{F} \models \alpha[w]$. We say that φ *globally corresponds* to a first-order sentence α with no occurence of unary predicate symbols if for any full possibility frame $\mathbb{F}$, $\mathbb{F} \Vdash \varphi$ iff $\mathbb{F} \models \alpha$. For inequalities and quasi-inequalities, the definition is similar.

The definition above can be easily adapted to the setting of filter-descriptive possibility frames and so on.

The regular open translation

In the present section, we will give the first-order translation of the expanded modal language into the first-order correspondence language, in the spirit of the standard translation in [21, Section 2.4]. Since the translation is based on the semantic interpretation of modal formulas on the relational structures, and in possibility semantics, the conditions about regular open sets play an important role, we will call our translation *regular open translation*.

For the sake of convenience, we give the following definition:

3.4.6. Definition. [Syntactic regular open closure](cf. [196, page 8]) Given a first-order formula $\alpha(x)$ with at most x free, the *syntactic regular open closure* $\mathsf{RO}_x(\alpha(x))$ is defined as $(\forall y \sqsubseteq x)(\exists z \sqsubseteq y)(\exists z' \sqsupseteq z)\alpha(z')$.

It is easy to see that the syntactic regular open closure of a formula is interpreted as the semantic regular open closure of its corresponding truth set:

3.4.7. Proposition. *(cf. [196, Lemma 3.8])* $[\![\mathsf{RO}_x(\alpha(x))]\!]^{\mathbb{M}} = \mathsf{ro}([\![\alpha(x)]\!]^{\mathbb{M}})$.

Since nominals are interpreted as elements in $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, i.e. regular open closures of singletons, we will translate nominals to the syntactic regular open closure of the identity $i = x$, where i is the individual symbol interpreted as a state $\underline{i_n} \in W$ such that $\mathsf{ro}(\{\underline{i_n}\}) = V(\mathbf{i}_n)$. The connectives are interpreted according to the definition of their satisfaction relations.

Now we are ready to give the regular open translation as follows:

3.4.8. Definition. [Regular open translation](cf. [196, Definition 2.6]) The regular open translation of a formula in the expanded modal language $\mathcal{L}^+$ into the first-order correspondence language $\mathcal{L}^1$ is given as follows:

$$\begin{array}{rcl}
ST_x(\mathbf{i}) & := & \mathsf{RO}_x(i = x);\\
ST_x(p_i) & := & P_i x;\\
ST_x(\neg\varphi) & := & \forall y(y \sqsubseteq x \to \neg ST_y(\varphi));\\
ST_x(\varphi_1 \wedge \varphi_2) & := & ST_x(\varphi_1) \wedge ST_x(\varphi_2);\\
ST_x(\Box\varphi) & := & \forall y(Rxy \to ST_y(\varphi));\\
ST_x(\blacklozenge\varphi) & := & \mathsf{RO}_x(\exists y(Ryx \wedge ST_y(\varphi))).
\end{array}$$

By Proposition 3.2.3 and 3.4.2, we can also take $\top, \bot, \vee, \to, \Diamond, \blacksquare$ as primitive connectives and define the following translation:

3.4.9. Definition. [Regular open translation continued]

$$\begin{array}{rcl}
ST_x(\top) & := & \top;\\
ST_x(\bot) & := & \bot;\\
ST_x(\varphi_1 \vee \varphi_2) & := & (\forall y \sqsubseteq x)(\exists z \sqsubseteq y)(ST_z(\varphi_1) \vee ST_z(\varphi_2));\\
ST_x(\varphi_1 \to \varphi_2) & := & (\forall y \sqsubseteq x)(ST_y(\varphi_1) \to ST_y(\varphi_2));\\
ST_x(\Diamond\varphi) & := & (\forall y \sqsubseteq x)\exists z(Ryz \wedge (\exists w \sqsubseteq z)(ST_w(\varphi));\\
ST_x(\blacksquare\varphi) & := & (\forall y \sqsubseteq x)(\forall z \sqsupseteq y)\forall w(Rwz \Rightarrow (\exists v \sqsubseteq w)(ST_v(\varphi)).
\end{array}$$

The following proposition justifies the translation defined above:

3.4.10. Proposition. *(cf. [196, Lemma 2.8]) For any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, any valuation V on $\mathbb{F}$, any $w \in W$ and any formula $\varphi(\vec{p})$ in $\mathcal{L}^+$,*

- $\mathbb{F}, V, w \Vdash \varphi(\vec{p})$ *iff* $\mathbb{F}, V \models ST_x(\varphi(\vec{p}))[w]$;
- $\mathbb{F}, V \Vdash \varphi(\vec{p})$ *iff* $\mathbb{F}, V \models \forall x ST_x(\varphi(\vec{p}))$;
- $\mathbb{F}, w \Vdash \varphi(\vec{p})$ *iff* $\mathbb{F} \models \forall^{\mathsf{P}}\vec{P}\forall\vec{i} ST_x(\varphi(\vec{p}))[w]$;
- $\mathbb{F} \Vdash \varphi(\vec{p})$ *iff* $\mathbb{F} \models \forall^{\mathsf{P}}\vec{P}\forall\vec{i}\forall x ST_x(\varphi(\vec{p}))$;
- $\mathbb{F}, V \Vdash \varphi(\vec{p}) \leq \psi(\vec{p})$ *iff* $\mathbb{F}, V \models \forall x(ST_x(\varphi(\vec{p})) \to ST_x(\psi(\vec{p})))$;
- $\mathbb{F} \Vdash \varphi(\vec{p}) \leq \psi(\vec{p})$ *iff* $\mathbb{F} \models \forall^{\mathsf{P}}\vec{P}\forall\vec{i}\forall x(ST_x(\varphi(\vec{p})) \to ST_x(\psi(\vec{p})))$;
- $\mathbb{F}, V \Vdash \&_{j=1}^{n}(\varphi_j(\vec{p}) \leq \psi_j(\vec{p})) \Rightarrow \varphi(\vec{p}) \leq \psi(\vec{p})$ *iff* $\mathbb{F}, V \models \bigwedge_{j=1}^{n} \forall x(ST_x(\varphi_j(\vec{p})) \to ST_x(\psi_j(\vec{p}))) \to \forall x(ST_x(\varphi(\vec{p})) \to ST_x(\psi(\vec{p})))$;
- $\mathbb{F} \Vdash \&_{j=1}^{n}(\varphi_j(\vec{p}) \leq \psi_j(\vec{p})) \Rightarrow \varphi(\vec{p}) \leq \psi(\vec{p})$ *iff* $\mathbb{F}, V \models \forall^{\mathsf{P}}\vec{P}\forall\vec{i}(\bigwedge_{j=1}^{n} \forall x(ST_x(\varphi_j(\vec{p})) \to ST_x(\psi_j(\vec{p}))) \to \forall x(ST_x(\varphi(\vec{p})) \to ST_x(\psi(\vec{p}))))$.

where $\vec{P}$ are the unary predicate symbols corresponding to $\vec{p}$.

Proof:
By induction on the structure of φ and the satisfaction relation for each connective and variable, as well as the semantics of the first-order correspondence language. □

3.4.11. Remark. In fact, since unary predicate symbols are interpreted only as admissible subsets (therefore regular open subsets), and the truth set of every formula φ is a regular open subset, there are some first-order formulas valid on all full possibility frames which are not first-order theorems (e.g. $ST_x(\varphi) \leftrightarrow (\forall y \sqsubseteq x)(\exists z \sqsubseteq y)(\exists w \sqsupseteq z)ST_w(\varphi)$). As a result, we have different ways to obtain translations. For example, if we obtain the translation of $\varphi \to \psi$ directly from the syntactic definition of $\to$, then we would have something different:

$$ST_x(\varphi_1 \to \varphi_2) := (\forall y \sqsubseteq x)(\exists z \sqsubseteq y)(((\forall w \sqsubseteq z)\neg ST_w(\varphi_1)) \vee ST_z(\varphi_2))$$

In fact, the translation given above is equivalent to the one in Definition 3.4.9:

$$[\![(\forall y \sqsubseteq x)(\exists z \sqsubseteq y)(((\forall w \sqsubseteq z)\neg ST_w(\varphi_1)) \vee ST_z(\varphi_2))]\!]^{\mathbb{M}} = [\![\neg\varphi_1 \vee \varphi_2]\!]^{\mathbb{M}},$$

and

$$[\![(\forall y \sqsubseteq x)(ST_y(\varphi_1) \to ST_y(\varphi_2))]\!]^{\mathbb{M}} = [\![\varphi_1 \to \varphi_2]\!]^{\mathbb{M}},$$

which are the same.

3.4.2 Sahlqvist and inductive formulas

In the present section, we will collect the preliminaries on Sahlqvist and inductive inequalities for classical modal formulas, which instantiate the general definitions given in Definition 2.5.3.

3.4.12. Definition. [Signed generation tree](cf. [60, Definition 4] and Definition 2.5.1) The *positive* (resp. *negative*) *generation tree* of any given formula φ is defined as follows: First of all, the root node of the generation tree of φ is labelled with sign + (resp. −). After this, the children nodes are labelled as follows:

- For $\Box, \Diamond, \vee, \wedge$, label the same sign to the child(ren) node(s);
- For $\neg$, label the opposite sign to the child node;
- For $\to$, label the opposite sign to the first child node and the same sign to the second child node.

3.4.13. Definition. (cf. [60, Definition 5] and Definition 2.5.2) In any signed generation tree, nodes will be respectively called *syntactically right adjoint (SRA)*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)* and Δ-*adjoints*[7], according to Table 3.1.

[7]For explanations of the terminologies here, we refer to [153, Remark 3.24].

<table>
<tr><th>Skeleton</th><th>PIA</th></tr>
<tr><td>Δ-adjoints
+ ∨ ∧
− ∧ ∨</td><td>SRA
+ ∧ □ ¬
− ∨ ◇ ¬</td></tr>
<tr><td>SLR
+ ∧ ◇ ¬
− ∨ □ ¬ →</td><td>SRR
+ ∨ →
− ∧</td></tr>
</table>

Table 3.1: Skeleton nodes and PIA nodes.

3.4.14. Definition. [Sahlqvist and inductive inequalities](cf. [60, Definition 6] and Definition 2.5.3) For any dependency order $<_\Omega$ on variables $p_1, \ldots p_n$ and any order-type $\varepsilon \in \{1, \partial\}^n$ (cf. Definition 2.5.3), the signed generation tree $*\varphi(p_1, \ldots p_n)$ (where $* \in \{-, +\}$) is (Ω, ε)-*inductive* if

(1) each ε-critical branch (cf. page 15) with leaf p_i is good (cf. Definition 2.5.2) for all $1 \leq i \leq n$;

(2) every SRR-node in the ε-critical branch with leaf p_i is either $\bigstar(\gamma, \beta)$ or $\bigstar(\beta, \gamma)$, where the ε-critical branch is in β, and

  (a) $\varepsilon^\partial(\gamma) \prec *\varphi$, and

  (b) $p_k <_\Omega p_i$ for every p_k that occurs in γ.

Given any order-type ε, $*\varphi(p_1, \ldots p_n)$ is ε-*Sahlqvist* if every ε-critical branch is excellent (cf. Definition 2.5.2).

An inequality $\varphi \leq \psi$ is (Ω, ε)-*inductive* (resp. ε-*Sahlqvist*) if the signed generation trees $+\varphi$ and $-\psi$ are (Ω, ε)-inductive (resp. ε-*Sahlqvist*). An inequality $\varphi \leq \psi$ is *inductive* (resp. Sahlqvist) if it is (Ω, ε)-inductive (ε-Sahlqvist) for some (Ω, ε).

3.5 The algorithm ALBA for possibility semantics

In the present section, we will give the algorithm ALBA for possibility semantics. This version of algorithm is a bit different from the version in Section 2.6, but similar to the version in [56, 57], especially when it comes to the approximation rules. This is because in the possibility semantics setting, nominals are interpreted as regular open closures of atoms in $\mathbb{B}_{\mathsf{K}}$, which are not necessarily complete join-primes in $\mathbb{B}_{\mathsf{RO}}$.

The input of the algorithm ALBA is an inequality $\varphi \leq \psi$. After receiving the input, ALBA executes in three stages:

The first stage preprocesses the input inequality. It eliminates all propositional variables with uniform occurence polarity, and apply the distribution and splitting rules exhaustively. After applying these rules, the input inequality is transformed into a finite set of inequalities $\{\varphi'_i \leq \psi'_i, 1 \leq i \leq n\}$. Then each inequality is rewritten as an *initial* *quasi-inequality* $\& S_i \Rightarrow \mathsf{Ineq_i}$, abbreviated as $(S_i, \mathsf{Ineq_i})$ called *systems*, where $S_i = \varnothing$ and $\mathsf{Ineq_i}$ is $\varphi'_i \leq \psi'_i$.

The second stage aims at transforming each (S_i, Ineq_i) into a *pure* system with no occurence of propositional variables, but only nominals. The rules which eliminate the propositional variables are called the Ackermann rules, and the other rules are auxiliary rules to make the Ackermann rules applicable.

The third stage outputs the result of the algorithm. If some of the systems cannot be transformed into pure systems, then the algorithm halts and reports failure. Otherwise, the algorithm outputs the conjunction of the pure quasi-inequalities $\& S_i \Rightarrow \mathsf{Ineq}_i$ as well as its regular open translation, which we denote by $\mathsf{ALBA}(\varphi \leq \psi)$ and $\mathsf{FO}(\varphi \leq \psi)$, respectively.

The details of the algorithm are given as follows:

Stage 1: Preprocessing In the first stage, the algorithm applies the following rules exhaustively to the input inequality $\varphi \leq \psi$ with signed generation trees $+\varphi$ and $-\psi$:

(1) Distribution rules:

(a) Push down $+\Diamond, +\wedge, -\neg$ and $-\rightarrow$ by distributing them over $+\vee$ which are Skeleton nodes, and

(b) Push down $-\Box, -\vee, +\neg$ and $-\rightarrow$ by distributing them over $-\wedge$ which are Skeleton nodes.

(2) Monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

where $\beta(p)$ (resp. $\alpha(p)$) is positive (resp. negative) in p.

(3) Splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

After exhaustively applying the rules above, the input inequality $\varphi \leq \psi$ is transformed into a set of inequalities $\{\varphi'_i \leq \psi'_i \mid 1 \leq i \leq n\}$. Then each inequality is rewritten as an *initial quasi-inequality* $\& S_i \Rightarrow \mathsf{Ineq_i}$, abbreviated as $(S_i, \mathsf{Ineq_i})$ called *systems*, where $S_i = \varnothing$ and $\mathsf{Ineq_i}$ is $\varphi'_i \leq \psi'_i$. Each system is processed in Stage 2.

Stage 2: Reduction and elimination This stage aims at eliminating all propositional variables from a system (S, Ineq) by the following *reduction rules*: *approximation rules*, *residuation rules*, *splitting rules*, and *Ackermann-rules*. The formulas and inequalities in this stage are from the expanded modal language with nominals and black connectives.

Approximation rules. There are four approximation rules. Each of these rules simplifies Ineq and adds an inequality to S. The notation $\varphi(!x)$ indicates that x occurs only once in φ, and the branch of $\varphi(!x)$ starting at x means the path from x to the root.

Left-positive approximation rule.

$$\frac{(S,\ \ \varphi'(\gamma/!x) \leq \psi')}{(S \cup \{\mathbf{i} \leq \gamma\},\ \ \varphi'(\mathbf{i}/!x) \leq \psi')}\ (L^{+}A)$$

where $+x \prec +\varphi'(!x)$, the branch of $+\varphi'(!x)$ starting at $+x$ consists of SLR nodes, γ belongs to the basic modal language and $\mathbf{i}$ is a nominal variable not occurring in $\varphi'(\gamma/!x) \leq \psi'$ or S.

Left-negative approximation rule.

$$\frac{(S,\ \ \varphi'(\gamma/!x) \leq \psi')}{(S \cup \{\gamma \leq \neg\mathbf{i}\},\ \ \varphi'(\neg\mathbf{i}/!x) \leq \psi')}\ (L^{-}A)$$

where $-x \prec +\varphi'(!x)$, the branch of $+\varphi'(!x)$ starting at $-x$ consists of SLR nodes, γ belongs to the basic modal language and $\mathbf{i}$ is a nominal variable not occurring in $\varphi'(\gamma/!x) \leq \psi'$ or S.

Right-positive approximation rule.

$$\frac{(S,\ \ \varphi' \leq \psi'(\gamma/!x))}{(S \cup \{\mathbf{i} \leq \gamma\},\ \ \varphi' \leq \psi'(\mathbf{i}/!x))}\ (R^{+}A)$$

where $+x \prec -\psi'(!x)$, the branch of $-\psi'(!x)$ starting at $+x$ consists of SLR nodes, γ belongs to the basic modal language and $\mathbf{i}$ is a nominal variable not occurring in $\varphi' \leq \psi'(\gamma/!x)$ or S.

Right-negative approximation rule.

$$\frac{(S,\ \ \varphi' \leq \psi'(\gamma/!x))}{(S \cup \{\gamma \leq \neg\mathbf{i}\},\ \ \varphi' \leq \psi'(\neg\mathbf{i}/!x))}\ (R^{-}A)$$

where $-x \prec -\psi'(!x)$, the branch of $-\psi'(!x)$ starting at $-x$ consists of SLR nodes, γ belongs to the basic modal language and $\mathbf{i}$ is a nominal variable not occurring in $\varphi' \leq \psi'(\gamma/!x))$ or S.

For the approximation rules, we will typically apply them *pivotally* to nodes $!x$ such that the branch starting at x is a *maximal* SLR branch, i.e. branch that cannot be extended further. If an execution of ALBA is such that the approximation rules are applied only pivotally, then it is called *pivotal*.

Residuation rules. Each residuation rule rewrite one inequality in S into another inequality:

$$\frac{\Diamond\gamma \leq \delta}{\gamma \leq \blacksquare\delta}\ (\Diamond\text{-Res}) \qquad \frac{\neg\gamma \leq \delta}{\neg\delta \leq \gamma}\ (\neg\text{-Res-Left})$$

$$\frac{\gamma \leq \Box\delta}{\blacklozenge\gamma \leq \delta}\ (\Box\text{-Res}) \qquad \frac{\gamma \leq \neg\delta}{\delta \leq \neg\gamma}\ (\neg\text{-Res-Right})$$

$$\frac{\gamma \wedge \delta \leq \beta}{\gamma \leq \delta \to \beta}\ (\wedge\text{-Res-1}) \qquad \frac{\gamma \leq \delta \vee \beta}{\gamma \wedge \neg\delta \leq \beta}\ (\vee\text{-Res-1})$$

$$\frac{\gamma \wedge \delta \leq \beta}{\delta \leq \gamma \to \beta}\ (\wedge\text{-Res-2}) \qquad \frac{\gamma \leq \delta \vee \beta}{\gamma \wedge \neg\beta \leq \delta}\ (\vee\text{-Res-2})$$

$$\frac{\gamma \leq \delta \to \beta}{\gamma \wedge \delta \leq \beta}\ (\to\text{-Res-1}) \qquad \frac{\gamma \leq \delta \to \beta}{\delta \leq \gamma \to \beta}\ (\to\text{-Res-2})$$

Right Ackermann-rule.

$$\frac{(\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq n\} \cup \{\alpha_i \leq p \mid 1 \leq i \leq m\},\ \ \mathsf{Ineq})}{(\{\beta_j(\bigvee_{i=1}^m \alpha_i) \leq \gamma_j(\bigvee_{i=1}^m \alpha_i) \mid 1 \leq j \leq n\},\ \ \mathsf{Ineq})}\ (RAR)$$

where:

- $\beta_j(p)$ (resp. $\gamma_j(p)$) are positive (resp. negative) in p for $1 \leq j \leq n$,
- p does not occur in Ineq or α_i for $1 \leq i \leq m$.

Left Ackermann-rule.

$$\frac{(\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq n\} \cup \{p \leq \alpha_i \mid 1 \leq i \leq m\},\ \ \mathsf{Ineq})}{(\{\beta_j(\bigwedge_{i=1}^m \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^m \alpha_i) \mid 1 \leq j \leq n\},\ \ \mathsf{Ineq})}\ (LAR)$$

where:

- $\beta_j(p)$ (resp. $\gamma_j(p)$) are negative (resp. positive) in p for $1 \leq j \leq n$,
- p does not occur in Ineq or α_i for $1 \leq i \leq m$.

Stage 3: Success, failure and output If in every system, all propositional variables are eliminated, then for each i, the system becomes (S_i, Ineq_i), where $S_i = \{\varphi_{i_k}(\vec{\mathbf{j}}_i) \leq \psi_{i_k}(\vec{\mathbf{j}}_i)\}_k$, Ineq_i is $\varphi_i(\vec{\mathbf{j}}_i) \leq \psi_i(\vec{\mathbf{j}}_i)$. The algorithm outputs the conjunction of their corresponding pure quasi-inequalities and its regular open translation

$$\bigwedge_i \forall \vec{j}_i(\bigwedge_k (\forall x(ST_x(\varphi_{i_k}(\vec{\mathbf{j}}_i)) \to ST_x(\psi_{i_k}(\vec{\mathbf{j}}_i)))) \to \forall x(ST_x(\varphi_i(\vec{\mathbf{j}}_i)) \to ST_x(\psi_i(\vec{\mathbf{j}}_i)))),$$

denoted by $\mathsf{ALBA}(\varphi \leq \psi)$ and $\mathsf{FO}(\varphi \leq \psi)$, respectively. Otherwise, the algorithm halts and reports failure.

3.5.1. Example. Let us consider the following inequality $\Box p \leq p$.

In the first stage, no rules are applied, so the initial system is

$$(\varnothing, \Box p \leq p).$$

In the second stage, by applying the left-positive approximation rule, we get

$$(\{\mathbf{i} \leq \Box p\}, \mathbf{i} \leq p);$$

then by the right-negative approximation rule, we get

$$(\{\mathbf{i} \leq \Box p, p \leq \neg\mathbf{j}\}, \mathbf{i} \leq \neg\mathbf{j});$$

then by the residuation rule, we get

$$(\{\blacklozenge\mathbf{i} \leq p, p \leq \neg\mathbf{j}\}, \mathbf{i} \leq \neg\mathbf{j});$$

then by the right Ackermann rule, we get

$$(\{\blacklozenge\mathbf{i} \leq \neg\mathbf{j}\}, \mathbf{i} \leq \neg\mathbf{j});$$

this system is equivalent to the following pure-inequality:

$$\mathbf{i} \leq \blacklozenge\mathbf{i}.$$

In the third stage, the first-order correspondent is given as follows:

$$\forall i \forall x(ST_x(\mathbf{i}) \to ST_x(\blacklozenge\mathbf{i}))$$
$$\forall i \forall x(\mathsf{RO}_x(x = i) \to ST_x(\blacklozenge\mathbf{i}))$$
$$\forall i \forall x(\mathsf{RO}_x(x = i) \to \mathsf{RO}_x(\exists y(Ryx \land \mathsf{RO}_y(y = i)))).$$

As we can see from this example, the first-order correspondent of modal formulas will become much more complicated even for very simple input.

3.6 Soundness and Success

In the present section, we will prove the soundness of the algorithm with respect to regular open dual BAOs of full possibility frames, and state the success of ALBA on inductive inequalities. The basic proof structure is similar to Section 2.7, but for the soundness proof of the rules, it is similar to other existing settings [56, 57], and for most of the rules, the proofs are the same, and hence omitted. We will focus on the approximation rules, which are the only rules dealing with the variant interpretations of nominals.

3.6.1. THEOREM (SOUNDNESS). *If* ALBA *runs successfully on* $\varphi \leq \psi$ *and outputs* $\mathsf{FO}(\varphi \leq \psi)$*, then for any full possibility frame* $\mathbb{F}_{\mathsf{RO}}$*,*

$$\mathbb{F}_{\mathsf{RO}} \Vdash \varphi \leq \psi \textit{ iff } \mathbb{F}_{\mathsf{RO}} \models \mathsf{FO}(\varphi \leq \psi).$$

Proof:
The proof goes similarly to [56, Theorem 5.1]. Let $\varphi_i \leq \psi_i$, $1 \leq i \leq n$ denote the inequalities produced by preprocessing $\varphi \leq \psi$ after Stage 1, and (S_i, Ineq_i), $1 \leq i \leq n$ denote the corresponding quasi-inequalities produced by ALBA after Stage 2. It suffices to show the equivalence from (3.1) to (3.7) given below:

(3.1)	$\mathbb{F}_{\mathsf{RO}} \Vdash \varphi \leq \psi$
(3.2)	$\mathbb{B}_{\mathsf{RO}} \models \varphi \leq \psi$
(3.3)	$\mathbb{B}_{\mathsf{RO}} \models \varphi_i \leq \psi_i$, for all $1 \leq i \leq n$
(3.4)	$\mathbb{B}_{\mathsf{RO}} \models \mathop{\&} \varnothing \Rightarrow \varphi_i \leq \psi_i$, for all $1 \leq i \leq n$
(3.5)	$\mathbb{B}_{\mathsf{RO}} \models \mathop{\&} S_i \Rightarrow \mathsf{Ineq}_i$, for all $1 \leq i \leq n$
(3.6)	$\mathbb{B}_{\mathsf{RO}} \models \mathsf{ALBA}(\varphi \leq \psi)$
(3.7)	$\mathbb{F}_{\mathsf{RO}} \models \mathsf{FO}(\varphi \leq \psi)$

- The equivalence between (3.1) and (3.2) follows from Proposition 3.4.4;
- to show the equivalence of (3.2) and (3.3), it suffices to show the soundness of the rules in Stage 1, which can be proved in the same way as in [56, Theorem 5.1];
- the equivalence between (3.3) and (3.4) is immediate;
- the equivalence between (3.4) and (3.5) follows from Propositions 3.6.3, 3.6.4, 3.6.5 and 3.6.6 below;
- the equivalence between (3.5) and (3.6) is again immediate;
- the equivalence between (3.6) and (3.7) follows from Propositions 3.4.4 and 3.4.10.

□

Therefore, it remains to show the soundness of Stage 2, for which it suffices to show the soundness of the approximation rules, the residuation rules, the Ackermann rules and the splitting rules. For the residuation rules, the only property needed is the adjunction property, and the proofs are similar to existing settings (see e.g. [55, Lemma 8.4]), and hence are omitted. For the splitting rules, their soundness are immediate. For the Ackermann rules, since the nominals and propositional variables are interpreted as elements in $\mathbb{B}_{\mathsf{RO}}$, and the basic and black connectives are all interpreted as operations from (products of) $\mathbb{B}_{\mathsf{RO}}$ to $\mathbb{B}_{\mathsf{RO}}$, the "minimal valuation" formula is always interpreted as an element in $\mathbb{B}_{\mathsf{RO}}$, therefore the soundness proof of the Ackermann lemmas are the same as [57, Lemma 6.3, 6.4]. The only rules which need special attention are the approximation rules. Since they are order-dual to each other, it suffices to focus on one of these rules.

Let us consider the left-positive approximation rule. Consider a system $(S, \varphi(\alpha, \gamma_1, \ldots, \gamma_n) \leq \psi)$ where α is the formula to be approximated, and the regular open dual BAO $\mathbb{B}_{\mathsf{RO}}$ and assignment θ where the system is interpreted. By Proposition 3.3.4, $\alpha^{(\mathbb{B}_{\mathsf{RO}},\theta)} = \bigvee_{\mathsf{RO}}\{x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \mid x \leq \alpha^{(\mathbb{B}_{\mathsf{RO}},\theta)}\}$.

The soundness of the left-positive approximation rule is justified by the following proposition, which is similar to [57, Lemma 6.2]:

3.6.2. Proposition. *Given a regular open dual BAO $\mathbb{B}_{\mathsf{RO}}$ and an assignment θ on $\mathbb{B}_{\mathsf{RO}}$,*

$$(\mathbb{B}_{\mathsf{RO}}, \theta) \models \,\&\, S \Rightarrow \varphi(\alpha, \gamma_1, \ldots, \gamma_n) \leq \psi$$

iff for any $x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$,

$$(\mathbb{B}_{\mathsf{RO}}, \theta'_x) \models \mathbf{j} \leq \alpha \,\&\, \&\, S \Rightarrow \varphi(\mathbf{j}, \gamma_1, \ldots, \gamma_n) \leq \psi,$$

where θ'_x is the $\mathbf{j}$-variant of θ such that $\theta'_x(\mathbf{j}) = x$ and θ'_x agrees with θ on other variables.

Proof:

First of all, by the requirement of the approximation rule, we have that $\varphi(\_, \gamma_1, \ldots, \gamma_n)$ is completely join-preserving in the empty coordinate. By Proposition 3.3.4, $\alpha^{(\mathbb{B}_{\mathsf{RO}},\theta)} = \bigvee_{\mathsf{RO}}\{x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \mid x \leq \alpha^{(\mathbb{B}_{\mathsf{RO}},\theta)}\}$. By complete distributivity, we have[8]:

[8] For simplicity of notation, we omit the superscript $(\mathbb{B}_{\mathsf{RO}}, \theta)$ of formulas and inequalities except for the last but one line; indeed, in the last two lines, the interpretation is in $(\mathbb{B}_{\mathsf{RO}}, \theta'_x)$, and in other lines, the interpretation is in $(\mathbb{B}_{\mathsf{RO}}, \theta)$.

$$
\begin{array}{ll}
& (\mathbb{B}_{\mathsf{RO}},\theta) \models (\&S) \Rightarrow \varphi(\alpha,\gamma_1,\ldots,\gamma_n) \leq \psi \\
\text{iff} & (\&S) \Rightarrow \varphi(\bigvee_{\mathsf{RO}}\{x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \mid x \leq \alpha\},\gamma_1,\ldots,\gamma_n) \leq \psi \\
\text{iff} & (\&S) \Rightarrow \bigvee_{\mathsf{RO}}\{\varphi(x,\gamma_1,\ldots,\gamma_n) \mid x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \text{ and } x \leq \alpha\} \leq \psi \\
\text{iff} & (\&S) \Rightarrow (\forall x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \text{ s.t. } x \leq \alpha)(\varphi(x,\gamma_1,\ldots,\gamma_n) \leq \psi) \\
\text{iff} & (\forall x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \text{ s.t. } x \leq \alpha)(\&S \Rightarrow \varphi(x,\gamma_1,\ldots,\gamma_n) \leq \psi) \\
\text{iff} & (\forall x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}))(x \leq \alpha \& \&S \Rightarrow \varphi(x,\gamma_1,\ldots,\gamma_n) \leq \psi) \\
\text{iff} & (\forall x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}))(x \leq \alpha^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)} \& (\&S)^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)} \Rightarrow \varphi^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)}(x,\gamma_1^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)},\ldots,\gamma_n^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)}) \leq \psi^{(\mathbb{B}_{\mathsf{RO}},\theta'_x)}) \\
\text{iff} & (\forall x \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}))((\mathbb{B}_{\mathsf{RO}},\theta'_x) \models (\mathbf{j} \leq \alpha \& \&S \Rightarrow \varphi(\mathbf{j},\gamma_1,\ldots,\gamma_n) \leq \psi)).
\end{array}
$$

Notice that $\mathbf{j}$ does not occur in $S,\alpha,\varphi,\psi,\gamma_1,\ldots,\gamma_n$. □

Notice again that here we interpret the nominals as regular opens closures of atoms in the full dual BAO rather than atoms or complete join-irreducibles since they might not be available in $\mathbb{B}_{\mathsf{RO}}$.

By the proposition above, we have the soundness of the left-positive approximation rule as an easy corollary:

3.6.3. Proposition. *The left-positive approximation rule is sound.*

By order-dual arguments, the other three approximation rules are sound:

3.6.4. Proposition. *The left-negative approximation rule, the right-positive approximation rule and the right-negative approximation rule are sound.*

Following from standard soundness proof of ALBA [55, Lemma 8.3, 8.4], we have the following:

3.6.5. Proposition. *The distribution rules, the splitting rules, the monotone and antitone rules and the residuation rules are sound.*

For the Ackermann rules, the proof is similar to [55, Lemma 4.2-4.3]. It is worth mentioning that all the formulas in the expanded modal language are interpreted in $\mathbb{B}_{\mathsf{RO}}$. Moreover, we are working at the discrete duality level, therefore the topological Ackermann lemmas (cf. [55, Lemma 9.3, 9.4]) are not needed in the soundness proof here.

3.6.6. Proposition. *The Ackermann rules are sound.*

The proof that ALBA succeeds on inductive inequalities goes similarly to [57], therefore in what follows we will only state the main result without giving details, and refer the reader to [57] for an exhaustive treatment.

3.6.7. Theorem. ALBA *succeeds on all inductive inequalities.*

3.6.8. Corollary. *Every inductive inequality $\varphi \leq \psi$ has a first-order correspondent* $\mathsf{FO}(\varphi \leq \psi)$.

3.7 Canonicity

In the present section, we prove that inductive inequalities are filter-canonical. Our proof is alternative to the one given by Holliday [118, Theorem 7.20]. Indeed, Holliday's proof follows from the constructive canonicity of inductive inequalities proved in [56], observing that the filter completions of BAOs coincide with their constructive canonical extensions (cf. [118, Theorem 5.46]). In this section, we make use of the possibility semantics counterpart of the canonicity-via-correspondence argument, which is a variation of the standard U-shaped argument (cf. [55] and Section 2.7) represented in the diagram below. For this U-shaped argument to be supported, we need to prove that ALBA is sound also with respect to the filter-descriptive possibility frames (see Definition 3.7.2), i.e. the dual relational structure of Boolean algebras with operators in the framework of possibility semantics. Hence, in what follows, we will mainly focus on this aspect.

$$
\begin{array}{lll}
\mathbb{F}_{\mathsf{FD}} \Vdash \varphi(\vec{p}) \leq \psi(\vec{p}) & & \mathbb{F}_{\mathsf{RO}} \Vdash \varphi(\vec{p}) \leq \psi(\vec{p}) \\
\Updownarrow & & \Updownarrow \\
\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{FD}} \varphi(\vec{p}) \leq \psi(\vec{p}) & & \mathbb{B}_{\mathsf{RO}} \models \varphi(\vec{p}) \leq \psi(\vec{p}) \\
\Updownarrow & & \Updownarrow \\
\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{FD}} (\forall \vec{p} \in \mathbb{B}_{\mathsf{FD}})(\varphi(\vec{p}) \leq \psi(\vec{p})) & & \mathbb{B}_{\mathsf{RO}} \models (\forall \vec{p} \in \mathbb{B}_{\mathsf{RO}})(\varphi(\vec{p}) \leq \psi(\vec{p})) \\
\Updownarrow & & \Updownarrow \\
\mathbb{B}_{\mathsf{RO}} \models_{\mathsf{FD}} (\forall \vec{\mathbf{i}} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}))\mathrm{Pure}(\varphi(\vec{p}) \leq \psi(\vec{p})) & & \mathbb{B}_{\mathsf{RO}} \models (\forall \vec{\mathbf{i}} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}))\mathrm{Pure}(\varphi(\vec{p}) \leq \psi(\vec{p})) \\
\Updownarrow & & \Updownarrow \\
\mathbb{F}_{\mathsf{FD}} \models \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p}) \leq \psi(\vec{p}))) & \Leftrightarrow & \mathbb{F}_{\mathsf{RO}} \models \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p}) \leq \psi(\vec{p})))
\end{array}
$$

Here $\mathbb{B}_{\mathsf{FD}}$ denote the dual BAO of the filter-descriptive possibility frame $\mathbb{F}_{\mathsf{FD}}$, $\mathbb{F}_{\mathsf{RO}}$ is the underlying full possibility frame of $\mathbb{F}_{\mathsf{FD}}$, and $\mathbb{B}_{\mathsf{RO}}$ is the dual BAO of $\mathbb{F}_{\mathsf{RO}}$.

We will first provide the semantic environment of the present section, i.e. filter-descriptive possibility frames and canonical extensions in Section 3.7.1. Section 3.7.2 gives the soundness proof of the algorithm with respect to (the dual BAOs of) filter-descriptive possibility frames, where the topological Ackermann lemmas are given. The correspondence and canonicity results are collected in Section 3.7.3.

3.7.1 Filter-descriptive possibility frames and canonical extensions

In the present section, we will collect basic definitions of filter-descriptive possibility frames as well as canonical extensions. For more details, the reader is referred to [118] and [22].

Filter-descriptive possibility frames

In [118], Holliday introduced filter-descriptive possibility frames, i.e. the possibility semantics counterpart of descriptive general (Kripke) frames [21, Section 5.5], in which restrictions on the admissible set are imposed. The following definition is one of these restrictions.

3.7.1. Definition. [Tightness](cf. [118, Definition 4.31]) A possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$ is said to be

- *R-tight* if $(\forall w, v \in W)(\forall X \in \mathsf{P}(w \in \Box_P(X) \Rightarrow v \in X) \Rightarrow wRv)$;
- $\sqsubseteq$*-tight* if $(\forall w, v \in W)(\forall X \in \mathsf{P}(w \in X \Rightarrow v \in X) \Rightarrow v \sqsubseteq w)$;
- *tight* if it is both R-tight and $\sqsubseteq$-tight.

The filter-descriptive possibility frames are introduced as below:

3.7.2. Definition. [Filter-descriptive possibility frames] (cf. [118, Definition 5.39]) A possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$ is said to be *filter-descriptive* if the following conditions hold:

- it is tight;
- for every proper filter[9] F in $\mathbb{B}_{\mathsf{P}}$, there exists an element $w \in W$ such that $F = P(w) = \{X \in \mathsf{P} \mid w \in X\}$.

3.7.3. Remark. As we mentioned on page 29, for any possibility frame $\mathbb{F} = (W, \sqsubseteq, R, \mathsf{P})$, $\mathsf{RO}(W, \tau_{\sqsubseteq})$ might not be closed under the conditions in Definition 3.2.2. However, if $\mathbb{F}$ is a filter-descriptive possibility frame, $\mathsf{RO}(W, \tau_{\sqsubseteq})$ is always closed under the conditions, i.e. given any filter-descriptive possibility frame $\mathbb{F}_{\mathsf{FD}} = (W, \sqsubseteq, R, \mathsf{P})$, its underlying full possibility frame $\mathbb{F}_{\mathsf{RO}} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$ is well-defined (see [118, Theorem 5.32.1, Proposition 5.40]).

In this section, we will mainly work on the dual algebraic side, i.e. the dual BAOs of filter-descriptive possibility frames and the dual BAOs of their underlying full possibility frames. Indeed, the latter are the *constructive canonical extensions* of the former, which we will discuss below.

[9] A *proper filter* in a BAO $\mathbb{B}$ is a non-empty subset $F \subseteq B$ such that

- for any $a, b \in F$, $a \wedge b \in F$;
- for any $a, b \in B$, if $a \leq b$ and $a \in B$, then $b \in B$;
- $\bot \notin F$.

Canonical extensions

As is known in the setting of Kripke semantics and its algebraic counterpart, given a descriptive general frame $\mathbb{G}$ and its underlying Kripke frame $\mathbb{F}$, the dual BAO of $\mathbb{F}$ is the canonical extension of the dual BAO of $\mathbb{G}$. It is natural to ask what is the relation between the dual BAO of a filter-descriptive possibility frame $\mathbb{F}_{\mathsf{FD}}$ and the dual BAO of its underlying full possibility frame $\mathbb{F}_{\mathsf{RO}}$. Indeed, by [118, Theorem 5.46], given any filter-descriptive possibility frame $\mathbb{F}_{\mathsf{FD}} = (W, \sqsubseteq, R, \mathsf{P})$ and its underlying full possibility frame $\mathbb{F}_{\mathsf{RO}} = (W, \sqsubseteq, R, \mathsf{RO}(W, \tau_{\sqsubseteq}))$, the dual BAO of $\mathbb{F}_{\mathsf{RO}}$ is the *constructive canonical extension* of $\mathbb{F}_{\mathsf{FD}}$. In what follows, we will collect the basic definitions about canonical extensions. We will refer the reader to [22, Chapter 6] and Section 2.7 for more details.

3.7.4. Definition. [Canonical extensions of Boolean algebras] (cf. [22, Chapter 6, Definition 104], page 11) The *canonical extension* of a Boolean algebra $\mathbb{B}$ is a complete Boolean algebra $\mathbb{B}^{\delta}$ containing $\mathbb{B}$ as a sub-Boolean algerba, and such that the following two conditions hold:

(*denseness*) each element of $\mathbb{B}^{\delta}$ can be represented both as a join of meets and as a meet of joins of elements from $\mathbb{B}$;

(*compactness*) for all $X, Y \subseteq \mathbb{B}$ with $\bigwedge X \leq \bigvee Y$ in $\mathbb{B}^{\delta}$, there are finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\bigwedge X_0 \leq \bigvee Y_0$.[10]

An element $x \in \mathbb{B}^{\delta}$ is *open* (resp. *closed*)[11] if it is the join (resp. meet) of some $X \subseteq \mathbb{B}$. We use $\mathsf{O}(\mathbb{B}^{\delta})$ (resp. $\mathsf{K}(\mathbb{B}^{\delta})$) to denote the set of open (resp. closed) elements of $\mathbb{B}^{\delta}$. It is easy to see that elements in $\mathbb{B}$ are exactly the ones which are both closed and open (i.e. *clopen*).

It is well-known that for any given $\mathbb{B}$, its canonical extension is unique up to isomorphism, and that assuming the axiom of choice, the canonical extension of a Boolean algebra is a perfect Boolean algebra, i.e. a complete and atomic Boolean algebra (cf. e.g. [118, page 90-91]). Let $\mathbb{A}, \mathbb{B}$ be Boolean algebras. There are two canonical ways to extend an order-preserving map $f : \mathbb{A} \to \mathbb{B}$ to a map $\mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$:

3.7.5. Definition. [σ- and π-extension]([22, page 375] and page 12) For any order-preserving map $f : \mathbb{A} \to \mathbb{B}$ and any $u \in \mathbb{A}^{\delta}$, we define

$$f^{\sigma}(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in \mathbb{A}\} : u \geq x \in \mathsf{K}(\mathbb{A}^{\delta})\}$$

$$f^{\pi}(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in \mathbb{A}\} : u \leq y \in \mathsf{O}(\mathbb{A}^{\delta})\}.$$

[10] In fact, this is an equivalent formulation of the definition in [22].

[11] Notice that here the definition of closedness and openness is different from the ones in the order topology introduced by the refinement order. In the remainder of the present section, closedness and openness refer to this definition.

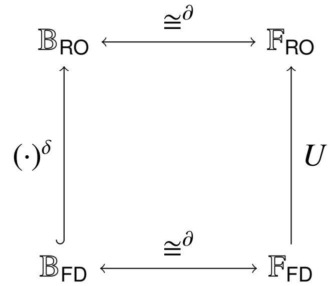
Since in a BAO, $\Box$ is meet-preserving, it is *smooth*, i.e. $\Box^\sigma = \Box^\pi$ (cf. [22, Proposition 111(3)]). Therefore, the canonical extension of a BAO can be defined as follows:

3.7.6. Definition. [Canonical extensions of BAOs](cf. e.g. [123, page 474]) The canonical extension of any BAO $(\mathbb{B}, \Box)$ is $(\mathbb{B}^\delta, \Box^\sigma) = (\mathbb{B}^\delta, \Box^\pi)$.

3.7.7. Definition. [Perfect Boolean algebra](cf. [22, Chapter 6, Definition 40]) A BAO $\mathbb{B} = (B, \bot, \top, \wedge, \vee, -, \Box)$ is said to be *perfect* if $\mathbb{B}$ is complete, atomic and completely multiplicative.

As is argued in [118, page 90-91], with the axiom of choice, it can be shown that if $\mathbb{B}$ is a BAO, then $\mathbb{B}^\delta$ is a perfect BAO. When the axiom of choice is not available, the canonical extension of a BAO cannot be shown to be perfect in general. What can be shown is that the canonical extension of a BAO is the constructive canonical extension defined in e.g. [87], which is complete and completely multiplicative.

As is shown in [118, Theorem 5.46], $\mathbb{B}_{\mathsf{RO}}$ is in fact the constructive canonical extension of $\mathbb{B}_{\mathsf{FD}}$. Therefore, the following diagram describes the relation between filter-descriptive possibility frames and their underlying full possibility frames, as well as their duals:



Here $\cong^\partial$ means dual equivalence, U is the forgetful functor dropping the admissible condition and replacing the admissible set to the set of regular opens in the downset topology, $(\cdot)^\delta$ is taking the constructive canonical extension.

Using the definitions and constructions given above, it is possible to define the notion of filter-canonicity:

3.7.8. Definition. [Filter-canonicity](see [118, Definition 7.15]) We say that an inequality $\varphi \leq \psi$ is *filter-canonical* if whenever it is valid on a filter-descriptive possibility frame, it is also valid on its underlying full possibility frame.[12]

By the duality theory of possibility semantics (see [118, Section 5]), filter-canonicity above is equivalent to the preservation under taking constructive canonical extensions.

Now we can come back to the U-shaped argument given on page 51. This argument starts from the top-left corner with the validity of the input inequality $\varphi \leq \psi$

[12]Notice that here our definition is different from [118, Definition 7.15], which is based on the notion of canonical possibility models and frames.

on $\mathbb{F}_{\mathsf{FD}}$, then reformulate it as the validity of the inequality in $\mathbb{B}_{\mathsf{RO}}$ with propositional variables interpreted as elements in $\mathbb{B}_{\mathsf{FD}}$, and use the algorithm ALBA to transform the inequality into an equivalent (set of) quasi-inequality(-ies) Pure($\varphi \leq \psi$) as well as its first-order translation, and then go back to the dual filter-descriptive possibility frame. Since the validity of the first-order formulas does not depend on the admissible set, the bottom equivalence is obvious. The right half of the argument goes on the side of full possibility frames and their duals, the soundness of which is already shown in Section 3.6.

Indeed, the U-shaped argument on page 51 gives the following results:

- Correspondence results with respect to filter-descriptive frames, which only uses the left arm of the U-shaped argument;
- Canonicity results, which uses the whole U-shaped argument[13];

Since the equivalences of the right arm of the U-shaped argument is already shown in Section 3.6, and the bottom equivalence is obvious, we will focus on the equivalences of the left-arm, i.e. the soundness of the algorithm with respect to filter-descriptive possibility frames and their duals.

3.7.2 Soundness over filter-descriptive possibility frames

In the present section we will prove the soundness of the algorithm ALBA with respect to the dual BAOs of filter-descriptive possibility frames. Indeed, similar to other semantic settings (see e.g. [55] and Section 2.7.2), the soundness proof of the filter-descriptive possibility frame side goes similar to that of the full possibility frame side (i.e. Theorem 3.6.1), and for every rule except for the Ackermann rules, the proof goes without modification, thus we will only focus on the Ackermann rules here, which is justified by the topological Ackermann lemmas given below. The proof is similar to e.g. [55], therefore we will only expand on the parts which are different.

Topological Ackermann lemmas

In the present section we prove the topological Ackermann lemmas, which is the technical core of the soundness proof of the Ackermann rules with respect to filter-descriptive possibility frames. The proof is analogous to the topological Ackermann lemmas in the existing literature (e.g. [152]), and we only expand on the parts of the proof which are different.

For the Ackermann rules, the soundness proof with respect to full possibility frames is justified by the following Ackermann lemmas (cf. e.g. [55, Lemma 4.2-4.3]):

[13] In fact, as is mentioned in [118, Section 7], using techniques from [56], the canonicity results can also be obtained via another U-shaped argument where nominals are interpreted as closed elements. Therefore, our proof here can be regarded as an alternative proof which has its relational counterpart.

3.7.9. Lemma (Right-handed Ackermann lemma). *Let α be a formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be positive (resp. negative) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$) be all the propositional variables (resp. nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{B}_{\mathsf{RO}}$, $\vec{x} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x})) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}))$ *for* $1 \leq i \leq n$;

(2) *There exists* $a_0 \in \mathbb{B}_{\mathsf{RO}}$ *such that* $\alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}) \leq a_0$ *and* $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, a_0) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, a_0)$ *for* $1 \leq i \leq n$.

3.7.10. Lemma (Left-handed Ackermann lemma). *Let α be a formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be negative (resp. positive) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$) be all the propositional variables (resp. nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{a} \in \mathbb{B}_{\mathsf{RO}}$, $\vec{x} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x})) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}))$ *for* $1 \leq i \leq n$;

(2) *There exists* $a_0 \in \mathbb{B}_{\mathsf{RO}}$ *such that* $a_0 \leq \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x})$ *and* $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, a_0) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x}, a_0)$ *for* $1 \leq i \leq n$.

As is similar to what is discussed in the existing literature (e.g. [55, Section 9]) and in Section 2.7.2, the soundness proof of the Ackermann rules, namely the Ackermann lemmas, cannot be straightforwardly adapted to the setting of admissible assignments, since formulas in the expanded modal language $\mathcal{L}^+$ might be interpreted as elements in $\mathbb{B}_{\mathsf{RO}} \setminus \mathbb{B}_{\mathsf{FD}}$ even if all the propositional variables are interpreted in $\mathbb{B}_{\mathsf{FD}}$, thus we cannot just take $a_0 = \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{a}, \vec{x})$ to be the element in $\mathbb{B}_{\mathsf{RO}}$ in the setting of admissible assignments. Therefore, some adaptations are necessary based on the syntactic shape of the formulas, the definitions of which are analogous to [152, Definition B.3]:

3.7.11. Definition. [Syntactically closed and open formulas]

(1) A formula in $\mathcal{L}^+$ is *syntactically closed* if all occurrences of nominals and $\blacklozenge$ are positive, and all occurrences of $\blacksquare$ are negative;

(2) A formula in $\mathcal{L}^+$ is *syntactically open* if all occurrences of nominals and $\blacklozenge$ are negative, and all occurrences of $\blacksquare$ are positive.

As is discussed in [55, Section 9], the underlying idea of the definitions above is that given an admissible assignment, the value of a syntactically closed (resp. open) formula is always an closed (resp. open) element in $\mathbb{B}_{\mathsf{RO}}$, i.e., in $\mathsf{K}(\mathbb{B}_{\mathsf{RO}})$ (resp. $\mathsf{O}(\mathbb{B}_{\mathsf{RO}})$), therefore by compactness, we can get an admissible a_0 as required by the topological Ackermann lemmas stated below, which are analogous to [152, Lemma B.4, B.5]:

3.7.12. Lemma (Right-handed topological Ackermann lemma). *Let α be a syntactically closed formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be syntactically closed (resp. open) and positive (resp. negative) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$) be all the propositional variables (resp. nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{d} \in \mathbb{B}_{\mathsf{FD}}, \vec{x} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x})) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}))$ *for* $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{B}_{\mathsf{FD}}$ *such that* $\alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}) \leq a_0$ *and* $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, a_0) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, a_0)$ *for* $1 \leq i \leq n$.

3.7.13. Lemma (Left-handed topological Ackermann lemma). *Let α be a syntactically open formula which does not contain p, let $\beta_i(p)$ (resp. $\gamma_i(p)$) be syntactically closed (resp. open) and negative (resp. positive) in p for $1 \leq i \leq n$, and let $\vec{q}$ (resp. $\vec{\mathbf{j}}$) be all the propositional variables (resp. nominals) occurring in $\beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p), \alpha$ other than p. Then for all $\vec{d} \in \mathbb{B}_{\mathsf{FD}}, \vec{x} \in \mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$, the following are equivalent:*

(1) $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x})) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}))$ *for* $1 \leq i \leq n$;

(2) There exists $a_0 \in \mathbb{B}_{\mathsf{FD}}$ *such that* $a_0 \leq \alpha^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x})$ *and* $\beta_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, a_0) \leq \gamma_i^{\mathbb{B}_{\mathsf{RO}}}(\vec{d}, \vec{x}, a_0)$ *for* $1 \leq i \leq n$.

Indeed, the proof of the topological Ackermann lemmas is similar to [152, Section B], and most of the lemmas and steps are similar, and hence are omitted. We only state and prove one lemma which is different. The reason that the lemma needs a different treatment is that elements in $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}})$ are not necessarily complete join-irreducible elements.

3.7.14. Lemma. $\mathsf{PsAt}(\mathbb{B}_{\mathsf{RO}}) \subseteq \mathsf{K}(\mathbb{B}_{\mathsf{RO}})$.

Proof:
It suffices to show that for any element $w \in W$, $\mathsf{ro}(\{w\}) = \bigwedge\{a \in \mathbb{B}_{\mathsf{FD}} \mid w \in a\}$. It is easy to see that $\mathsf{ro}(\{w\}) \leq \bigwedge\{a \in \mathbb{B}_{\mathsf{FD}} \mid w \in a\}$. Suppose that the inequality is strict, then there exists a $v \in W$ such that $v \notin \mathsf{ro}(\{w\})$ and $v \in \bigwedge\{a \in \mathbb{B}_{\mathsf{FD}} \mid w \in a\}$. By Definition 3.7.2, every filter-descriptive possibility frame is $\sqsubseteq$-tight, therefore $v \sqsubseteq w$. Since $\mathsf{ro}(\{w\})$ is downward closed, we have $v \in \mathsf{ro}(\{w\})$, a contradiction. □

The next lemma shows that the syntactic requirement by the topological Ackermann lemmas on the formulas is always satisfied when ALBA is executed pivotally, and together with the two topological Ackermann lemmas above, the soundness of the Ackermann rules in pivotal executions of ALBA with respect to filter-descriptive possibility frames are obtained (the argument is essentially the same as in the proof of [57, Proposition 7.6]). The proof of this lemma is similar to [57, Lemma 7.5], and is hence omitted.

3.7.15. Lemma. *If* ALBA *is executed pivotally (see page 45) on an input inequality* $\varphi \leq \psi$*, then for any system* (S, Ineq) *obtained during the execution, for any non-pure inequality in* S *and for* Ineq*, the left-hand side (resp. right-hand side) is syntactically closed (resp. open).*

3.7.3 Results

Using the soundness of the algorithm ALBA with respect to filter-descriptive possibility frames, we have the following results:

3.7.16. Theorem. *For any inequality* $\varphi \leq \psi$ *in the basic modal language* $\mathcal{L}$ *on which* ALBA *succeeds and outputs* $\mathsf{FO}(\varphi \leq \psi)$*,* $\varphi \leq \psi$ *corresponds to* $\mathsf{FO}(\varphi \leq \psi)$ *in the sense that for any filter-descriptive possibility frame* $\mathbb{F}$*,* $\mathbb{F} \Vdash \varphi \leq \psi$ *if and only if* $\mathbb{F} \models \mathsf{FO}(\varphi \leq \psi)$*.*

Proof:
By the left-arm of the U-shaped argument given on page 51. □

3.7.17. Theorem. *Any inequality* $\varphi \leq \psi$ *in the basic modal language* $\mathcal{L}$ *on which* ALBA *succeeds is preserved under taking constructive canonical extensions, and hence filter-canonical.*

Proof:
By the whole U-shaped argument given on page 51. □

3.8 Conclusions and future directions

In the present section, we will discuss some aspects of correspondence and canonicity in possibility semantics, and give some future directions.

3.8.1 The variation of interpretations: nominals and approximation rules in different ALBAs

The SQEMA[14]-ALBA line of algorithmic correspondence research starts from Boolean algebra based modal logics [51], and later on the underlying semantic environment generalizes to distributive lattices [55], general lattices [57], constructive extensions of lattices [56] and possibility semantics. Along the line of generalizations, properties specific to certain settings are separated from the more general properties. Here we are going to discuss the rules of nominals, i.e. the approximation rules in detail.

[14] SQEMA stands for **S**econd-order **Q**uantifier **E**limination for **M**odal formulae using **A**ckermann's lemma (see [51]).

Boolean algebras with operators. In the setting of Boolean algebras with operators, every $\mathcal{CAV}$-BAO has atoms, which are complete join-irreducibles, complete join-primes and join-generators. Therefore, the following kind of approximation rule is available (see [51]):

$$\frac{\mathbf{i} \leq \Diamond\gamma}{\mathbf{i} \leq \Diamond\mathbf{j} \quad \mathbf{j} \leq \gamma} \text{ (Left-}\Diamond\text{-Appr)}$$

where the nominals are interpreted as atoms. To guarantee the soundness of the rule above, the following properties are used (in the remainder of this subsection, we will abuse notation and identify formulas with their interpretations on algebras):

- First of all, $\Diamond\gamma$ is represented as $\Diamond \bigvee\{\mathbf{j} \in \mathsf{At}(\mathbb{B}) \mid \mathbf{j} \leq \gamma\}$, which uses the fact that the atoms in the $\mathcal{CAV}$-BAOs are join-generators;
- secondly, since $\Diamond$ is completely join-preserving, it can be equivalently represented as $\bigvee\{\Diamond\mathbf{j} \mid \mathbf{j} \in \mathsf{At}(\mathbb{B}) \text{ and } \mathbf{j} \leq \gamma\}$;
- finally, $\mathbf{i} \leq \bigvee\{\Diamond\mathbf{j} \mid \mathbf{j} \in \mathsf{At}(\mathbb{B}) \text{ and } \mathbf{j} \leq \gamma\}$ iff there exists a $\mathbf{j} \in \mathsf{At}(\mathbb{B})$ such that $\mathbf{j} \leq \gamma$ and $\mathbf{i} \leq \Diamond\mathbf{j}$, which uses the fact that the atoms are completely join-prime.

Therefore, the semantic properties used in the soundness proof of (Left-$\Diamond$-Appr) are the following:

- Atoms are join-generators;
- $\Diamond$ is completely join-preserving;
- Atoms are completely join-prime.

Therefore, the atomicity of $\mathcal{CAV}$-BAOs are not essentially used in the soundness proof of the rule above.

Distributive lattice expansions. Going from Boolean algebras with operators to distributive lattice expansions, the atomicity is not available anymore. However, in perfect distributive lattices, there are complete join-irreducibles which are also complete join-primes and join-generators, although not necessarily atomic. Therefore, as we analyzed in the Boolean setting, if we interpret the nominals as complete join-irreducibles (i.e. complete join-primes), these properties are enough to guarantee the rule (Left-$\Diamond$-Appr) to be sound on perfect distributive lattice expansions.

General lattice expansions. In the further general setting of perfect general lattice expansions without assuming distributivity, complete join-irreducibles are not the same as complete join-primes anymore. The remaining property in general lattice expansions is that the complete join-irreducibles (here we denote the set of complete join-irreducibles of the lattice $\mathbb{L}$ as $\mathsf{J}(\mathbb{L})$) are still join-generators. Consider the following rule:

$$\frac{(S,\ \ \varphi(\gamma/!x) \leq \psi)}{(S \cup \{\mathbf{i} \leq \gamma\},\ \ \varphi(\mathbf{i}/!x) \leq \psi)}\ (L^{+}A)$$

with $+x \prec +\varphi(!x)$, the branch of $+\varphi(!x)$ starting at $+x$ being SLR, γ belonging to the original modal language and $\mathbf{i}$ being the first nominal variable not occurring in S or $\varphi(\gamma/!x) \leq \psi$.

To guarantee the soundness of the rule above, the following properties are used:

- First of all, $\varphi(\gamma)$ is represented as $\varphi(\bigvee\{\mathbf{j} \in \mathsf{J}(\mathbb{L}) \mid \mathbf{j} \leq \gamma\})$, which uses the fact that the complete join-irreducibles in prefect general lattices are join-generators;

- secondly, since $\varphi(!x)$ is completely join-preserving, it can be equivalently represented as $\bigvee\{\varphi(\mathbf{j}) \mid \mathbf{j} \in \mathsf{J}(\mathbb{L})$ and $\mathbf{j} \leq \gamma\}$;

- finally, $\bigvee\{\varphi(\mathbf{j}) \mid \mathbf{j} \in \mathsf{J}(\mathbb{L})$ and $\mathbf{j} \leq \gamma\} \leq \psi$ iff for all $\mathbf{j} \in \mathsf{J}(\mathbb{L})$ s.t. $\mathbf{j} \leq \gamma$, it holds that $\varphi(\mathbf{j}) \leq \psi$, which does not use special properties of perfect general lattice expansions.

Therefore, in an approximation rule like $(L^{+}A)$, the semantic property essentially used is that the complete join-irreducibles are join-generators. As a result, any complete lattice-like structures which have join-generators can have an approximation rule in the style of $(L^{+}A)$ to be sound, once the nominals are interpreted as the join-generators.

For example, in the constructive canonical extensions of general lattice expansions, they are not necessarily perfect, so there are not "enough" complete join-irreducibles. However, in canonical extensions, every element can be represented as the join of closed elements, therefore if we interpret the nominals as closed elements, the approximation rule above is sound.

In possibility semantics, the regular open dual BAOs of full possibility frames are $\mathcal{CV}$-BAOs, thus do not have "enough" atoms. However, in this setting, the regular open closures of singletons can serve as the join-generators, which gives the semantic environment for the rule $(L^{+}A)$ to be sound.

To sum up, the following table shows the semantic properties available in each setting, which justifies the use of different approximation rules in each setting:

Propositional base	Nominals/join-generators
perfect Boolean algebras	atoms
perfect distributive lattices	complete join-primes
perfect general lattices	complete join-irreducibles
constructive canonical extensions	closed elements
possibility semantics	regular open closures of singletons

3.8.2 The essence of minimal valuation

As we can see from the previous subsection, nominals are not necessarily interpreted as atoms (singletons) or complete join-irreducibles, anything that can be taken as the join-generators can serve as the interpretation. Therefore, the execution of the algorithm ALBA provides quasi-inequalities equivalent to the input formula (or inequality), which are in the language with nominals and the connectives in the expanded modal language. Therefore, a successful ALBA reduction of a modal formula to the pure language means that the input modal formula can be expressed by the join-generators. In the setting of constructive canonical extensions of general lattices, there is not (yet) an obvious way of expressing closed elements in the first-order correspondence language. However, in other settings, we can express the atoms, complete join-irreducibles or regular open closures of singletons in the first-order correspondence language. Since we also have the first-order translation of the connectives both in the basic modal language and in the expanded modal language, we can thus translate the pure quasi-inequalities into first-order formulas.

3.8.3 Translation method and its limitations

Since possibility frames have two binary relations, it is natural to view them as bimodal Kripke frames with additional restrictions on the valuations of propositional variables (see [187] for a detailed discussion of this bimodal perspective). Therefore, a natural question is whether we can use this view to reduce correspondence problems in possibility semantics to correspondence problems in the bimodal language, like using the Gödel translation from intuitionistic logic to S4 modal logic (see [60] for a detailed discussion of the power and limits of translation method in correspondence theory). As we can see in e.g. [187], when we try to reduce correspondence problems in possibility semantics to correspondence problems in the bimodal language, a problem arises due to the complication of the formula structure of the bimodal formulas after the translation. However, since there are additional properties satisfied in the bimodal language (e.g. the interaction axioms between the accessibility relation and the refinement relation), there are additional order-theoretic properties satisfied, which can make more connective combinations having the nice order-theoretic properties. One example is $\Box_{\sqsubseteq}\Diamond_{\sqsubseteq}$, which is meet-preserving. If we make the analysis from the prespective of the order-theoretic properties of the individual connectives only, then $\Box_{\sqsubseteq}\Diamond_{\sqsubseteq}$ is of the shape of the antecedent of the McKinsey formula, which has a very bad combination pattern.

Therefore, a natural question is: can we use the order-theoretic properties of the combinations of connectives in correspondence theory for superintuitionistic logics, in addition to the properties of the individual connectives, to obtain results like in [158]?

3.8.4 Constructive canonical extensions

As we can see in [118, Theorem 5.46] and the canonicity proof in the present chapter, the persistence notion corresponding to the validity preservation from filter-descriptive possibility frames to their underlying full possibility frames is exactly the same as the validity preservation under taking constructive canonical extensions. Therefore, possibility semantics provides a relational semantic environment to give canonicity proofs without appealing to the axiom of choice and its equivalent forms. In addition, possibility frames could be recognized as the dual relational semantic environment to study constructive canonical extensions [87, 91]. In this sense, we can say that possibility semantics is the constructive counterpart of Kripke semantics.

A future question related to the constructive feature is about the frame-theoretic counterpart of constructive canonical extensions in lattice-based settings, i.e. non-classical versions of possibility semantics as the duals of distributive lattices and their canonical extensions, which is expected to "relationalize" the constructive canonical extensions in a "pointed" way, where "points" refer to filters rather than prime filters or ultrafilters. For the intuitionistic generalization of possibility semantics, it is already studied in [18] as Dragalin semantics.

Chapter 4

Algorithmic Sahlqvist preservation for modal compact Hausdorff spaces

In the present chapter, which is a revised version of the paper [199], we apply unified correspondence theory to modal compact Hausdorff spaces, providing alternative canonicity-type preservation results in [14] and [19].

4.1 Introduction

Canonicity, i.e. the preservation of validity of formulas from descriptive general frames to their underlying Kripke frames, is an important notion in modal logic, since it provides a uniform strategy for proving the strong completeness of axiomatic extensions of a basic (normal modal) logic. Thanks to its importance, the notion of canonicity has been explored also for other non-classical logics. In [123], Jónsson gave a purely algebraic reformulation of the frame-theoretic notion of canonicity, which he defined as the preservation of validity under taking canonical extensions, and proved the canonicity of Sahlqvist identities in a purely algebraic way. The construction of canonical extension was introduced by Jónsson and Tarski [124] as a purely algebraic encoding of the Stone spaces dual to Boolean algebras. In particular, the denseness requirement directly relates to the zero-dimensionality of Stone spaces. A natural question is then for which classes of formulas do canonicity-type preservation results hold in topological settings in which compactness is maintained and zero-dimensionality is generalized to the Hausdorff separation condition. This question has been addressed in [14, 19]. Specifically, in [14], Bezhanishvili, Bezhanishvili and Harding gave a canonicity-type preservation result for Sahlqvist formulas from modal compact Hausdorff spaces to their underlying Kripke frames, and in [19], Bezhanishvili and Sourabh generalized this result to modal fixed point formulas.

In the present chapter, some preliminary results are collected which reformulate in an algebraic and algorithmic way the canonicity-type preservation results in [14, 19]. We define the algorithm ALBA for modal compact Hausdorff spaces, show the soundness of the algorithm with respect to the interpretation over modal compact Hausdorff spaces. In particular, an adapted version of the topological Ackermann lemma is proved using the Esakia-type lemma for the modal language over modal compact Hausdorff spaces. The results of the present chapter pave the way to extending the tools of unified correspondence to canonicity-type preservation results based on different dualities than Stone duality.

This chapter is organized as follows: Section 4.2 collects preliminaries on modal compact Hausdorff spaces and the semantic interpretation for the modal language. Section 4.3 discusses the main ideas for the preservation results. Section 4.4 provides the expanded modal language of the algorithm as well as its interpretations, together with the syntactic definition of Sahlqvist sequents. The Ackermann Lemma Based Algorithm (ALBA) for modal compact Hausdorff space is given in Section 4.5. Section 4.6 and 4.7 respectively shows the soundness of the algorithm with respect to modal compact Hausdorff spaces and the success of the algorithm on Sahlqvist sequents.

4.2 Preliminaries

4.2.1 Modal compact Hausdorff spaces

In the present subsection, we collect the preliminaries for modal compact Hausdorff spaces. For more details, the readers are referred to [14, 19, 122].

We will use the following notations: given a binary relation R on W, we denote $R[X] = \{w \in W \mid (\exists x \in X)Rxw\}$ and $R^{-1}[X] = \{w \in W \mid (\exists x \in X)Rwx\}$, $R[w] := R[\{w\}]$ and $R^{-1}[w] := R^{-1}[\{w\}]$, respectively.

A topological space $\mathcal{T} = (W, \tau)$ is

(1) *compact* if for any collection $\{X_i\}_{i\in I}$ of open sets, if $W = \bigcup_{i\in I} X_i$, then there is a finite subset $I_0 \subseteq I$ such that $W = \bigcup_{i\in I_0} X_i$;

(2) *Hausdorff* if for any two distinct points $x, y \in W$, there exist $X, Y \in \tau$ such that $x \in X$, $y \in Y$ and $X \cap Y = \varnothing$.

It is well-known that singletons are closed in Hausdorff spaces.

4.2.1. Definition. (cf. e.g. [14, Definition 2.14]) A *modal compact Hausdorff space* is a triple $\mathcal{T} = (W, R, \tau)$ such that (W, τ) is a compact Hausdorff space and R is *continuous*, i.e.

(1) $R[w]$ is closed for any $w \in W$;

(2) $R^{-1}[X]$ is closed for any closed set X;

(3) $R^{-1}[X]$ is open for any open set X.

We let $\mathbb{F}_{\mathcal{T}} = (W, R)$ denote the underlying Kripke frame[1] of $\mathcal{T}$.

As is well known, open sets of topological spaces are captured algebraically by the notion of *frame*. A frame $\mathbb{L}$ is a complete lattice validating the following identity: $a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\}$ for any $X \subseteq \mathbb{L}$. A frame is *compact* if for any $X \subseteq \mathbb{L}$ such that $\bigvee X = \top$, there is a finite subset $Y \subseteq X$ such that $\bigvee Y = \top$. For any frame $\mathbb{L}$ and any $a \in \mathbb{L}$, the *pseudocomplement* of a is $\neg a := \bigvee\{b \mid b \wedge a = \bot\}$. For $a, b \in \mathbb{L}$, a is *well inside* b (notation: $a \prec b$) if $\neg a \vee b = \top$. A frame $\mathbb{L}$ is *regular* if $a = \bigvee\{b \mid b \prec a\}$ for all $a \in \mathbb{L}$. For any topological space $\mathcal{T} = (W, \tau)$, its associated frame is defined as $\mathbb{L}_{\mathcal{T}} := (\tau, \cap, \bigcup)$. If $\mathcal{T}$ is compact Hausdorff, then $\mathbb{L}_{\mathcal{T}}$ is compact regular.

4.2.2. Definition. (cf. e.g. [14, Definition 3.5]) A *modal compact regular frame* is a triple $\mathbb{L} = (L, \Box, \Diamond)$ where L is a compact regular frame, and $\Box, \Diamond$ are unary operations on L such that:[2]

(1) $\Box\top = \top$ and $\Box(a \wedge b) = \Box a \wedge \Box b$;

(2) $\Diamond\bot = \bot$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$;

(3) $\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$;

(4) $\Diamond \bigvee X = \bigvee\{\Diamond x \mid x \in X\}$ and $\Box \bigvee X = \bigvee\{\Box x \mid x \in X\}$ for any upward directed $X \subseteq L$.

One can readily show (cf. [14, Proposition 3.10]) that if $\mathcal{T} = (W, R, \tau)$ is a modal compact Hausdorff space, then $\mathbb{L}_{\mathcal{T}} := (\tau, \Box_{\mathcal{T}}, \Diamond_{\mathcal{T}})$ is a modal compact regular frame where $\Box_{\mathcal{T}} X = (R^{-1}[X^c])^c$ and $\Diamond_{\mathcal{T}} X = R^{-1}[X]$.

4.2.2 Language and interpretation

Given a set Prop of propositional variables, the positive modal language $\mathcal{L}$ is recursively defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. We let $\mathsf{Prop}(\alpha)$ denote the set of propositional variables occuring in α.

In [69], the positive fragment of basic normal modal logic is completely axiomatized as follows.

[1]Notice that the name "frame" occurs in two different ways in the present chapter, one is in point-free topology, the other is in modal logic. Here we use the name "Kripke frame" to refer to the notion in modal logic and "frame" to refer to the notion in point-free topology.

[2]The condition 3 is well-known in [69].

$$\varphi \vdash \varphi \quad \varphi \vdash \varphi \vee \psi \quad \psi \vdash \varphi \vee \psi \quad \varphi \wedge \psi \vdash \varphi \quad \varphi \wedge \psi \vdash \psi$$

$$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\Diamond(\varphi \vee \psi) \dashv\vdash \Diamond\varphi \vee \Diamond\psi \quad \Box(\varphi \wedge \psi) \dashv\vdash \Box\varphi \wedge \Box\psi$$

$$\Diamond\varphi \wedge \Box\psi \vdash \Diamond(\varphi \wedge \psi) \quad \Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \quad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \quad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi} \quad \frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}$$

In the following sections we will typically work with *inequalities* $\varphi \leq \psi$, and *quasi-inequalities* $\varphi_1 \leq \psi_1$ & ... & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ (cf. [55]), where & is the meta-conjunction and $\Rightarrow$ is the meta-implication.

Interpretation on modal compact Hausdorff spaces.

Modal compact Hausdorff spaces play the role played by descriptive general frames in the Stone-based setting. Accordingly, the counterparts of admissible valuations are the open valuations defined below.

A *modal compact Hausdorff model* is a pair $\mathbb{M} = (\mathcal{T}, V)$ where $\mathcal{T} = (W, R, \tau)$ is a modal compact Hausdorff space, and $V : \mathsf{Prop} \to \tau$ is an *open valuation* on $\mathcal{T}$. The satisfaction relation on modal compact Hausdorff models is defined as standard in modal logic. We let $[\![\varphi]\!]^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \Vdash \varphi\}$ denote the *truth set* of φ in $\mathbb{M}$.

An inequality $\varphi \leq \psi$ is valid on a modal compact Hausdorff space $\mathcal{T}$ if $[\![\varphi]\!]^{\mathbb{M}} \subseteq [\![\psi]\!]^{\mathbb{M}}$ for every model $\mathbb{M}$ based on $\mathcal{T}$ (i.e. for every open valuation into τ). A quasi-inequality $\varphi_1 \leq \psi_1$ & ... & $\varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ is valid on $\mathcal{T}$ if, for every model $\mathbb{M}$ based on $\mathcal{T}$, if $[\![\varphi_i]\!]^{\mathbb{M}} \subseteq [\![\psi_i]\!]^{\mathbb{M}}$ for all i then $[\![\varphi]\!]^{\mathbb{M}} \subseteq [\![\psi]\!]^{\mathbb{M}}$.

Interpretation on algebras.

In what follows, we let $\mathbb{B}$ denote a Boolean algebra with operator (BAO). We let $\theta : \mathsf{Prop} \to \mathbb{B}$ denote an *assignment* on $\mathbb{B}$, and let $\varphi^{(\mathbb{B},\theta)}$ or $\theta(\varphi)$ denote the value of φ in $\mathbb{B}$ under θ. We write $(\mathbb{B}, \theta) \models \varphi \leq \psi$ to indicate that $\varphi \leq \psi$ is *true* on $\mathbb{B}$ under θ, and $\mathbb{B} \models \varphi \leq \psi$ to indicate that $\varphi \leq \psi$ is *valid* on $\mathbb{B}$. Notations for truth and validity for quasi-inequalities are similar.

Another useful way to look at a formula $\varphi(p_1, \ldots, p_n)$ is to interpret it as an n-ary function $\varphi^{\mathbb{B}} : \mathbb{B}^n \to \mathbb{B}$ such that $\varphi^{\mathbb{B}}(a_1, \ldots, a_n) = \theta(\varphi)$ where $\theta : \mathsf{Prop} \to \mathbb{B}$ satisfies $\theta(p_i) = a_i$, $i = 1, \ldots, n$.

For any Kripke frame $\mathbb{F} = (W, R)$, we let $\mathbb{B}_{\mathbb{F}} = (P(W), \varnothing, W, \cap, \cup, (\cdot)^c, \Box_{\mathbb{B}_{\mathbb{F}}})$ denote the *dual BAO* of $\mathbb{F}$ (i.e. the complex algebra of $\mathbb{F}$), where $\Box_{\mathbb{B}_{\mathbb{F}}} X = (R^{-1}[X^c])^c$ for any $X \in P(W)$. A Kripke frame $\mathbb{F}$ and its dual BAO validate the same (quasi-)inequalities. In what follows, we let $\mathsf{At}(\mathbb{B}_{\mathbb{F}}) = \{\{w\} \mid w \in W\}$ and $\mathsf{CoAt}(\mathbb{B}_{\mathbb{F}}) = \{W - \{w\} \mid w \in W\}$ denote the set of atoms and coatoms of $\mathbb{B}_{\mathbb{F}}$ respectively.

Analogous notions and notations also apply to modal compact regular frames (cf. Definition 4.2.2). In particular, the dual algebra of the modal compact Hausdorff space

$\mathcal{T} = (W, R, \tau)$ is the modal compact regular frame $\mathbb{L}_{\mathcal{T}} = (\tau, \Box_{\mathcal{T}}, \Diamond_{\mathcal{T}})$, which provides a natural interpretation for the positive modal language. In addition, $\mathbb{L}_{\mathcal{T}}$ can be naturally embedded as a modal subframe[3] into the complex algebra $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ of the underlying Kripke frame $\mathbb{F}_{\mathcal{T}}$ of $\mathcal{T}$. Hence, all connectives in the positive modal language are interpreted in the same way when restricting the valuation of propositional variables to open subsets. Therefore, validity in $\mathbb{L}_{\mathcal{T}}$ (denoted $\mathbb{L}_{\mathcal{T}} \models \varphi \leq \psi$) coincides with validity in $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ relative to *open assignments* (i.e. assignments into open subsets), denoted $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models_{\mathbb{L}_{\mathcal{T}}} \varphi \leq \psi$.

4.3 Main ideas

4.3.1 From Stone to modal compact Hausdorff

As is well-known [95, 124], every Boolean algebra $\mathbb{B}$ is dually equivalent to a descriptive general frame $\mathbb{G}$, and the underlying Kripke frame $\mathbb{F}_{\mathbb{G}}$ of $\mathbb{G}$ is dually equivalent to the canonical extension $\mathbb{B}^{\delta}$ of $\mathbb{B}$, as illustrated in the left diagram below. The canonicity of an inequality (i.e., the preservation of its validity from any $\mathbb{G}$ to its $\mathbb{F}_{\mathbb{G}}$) can be equivalently rephrased as the preservation of its validity from any $\mathbb{B}$ to $\mathbb{B}^{\delta}$. This picture can be analogously generalized to the setting of modal compact Hausdorff spaces. In the right diagram, in the bottom line, every modal compact Hausdorff space $\mathcal{T}$ is dually equivalent to its dual modal compact regular frame $\mathbb{L}_{\mathcal{T}}$ (cf. [14, Theorem 3.14]), and the forgetful functor U maps any $\mathcal{T}$ to its underlying Kripke frame $\mathbb{F}_{\mathcal{T}}$. On the dual algebraic side, $\mathbb{L}_{\mathcal{T}}$ is embedded into the dual BAO $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ of $\mathbb{F}_{\mathcal{T}}$. The canonical embedding $\mathbb{B} \hookrightarrow \mathbb{B}^{\delta}$ encodes the Stone-type duality between BAOs and descriptive general frames in a purely algebraic way. Likewise, the Isbell duality can be encoded in the purely algebraic properties of the embedding $e : \mathbb{L} \hookrightarrow \mathbb{B}$ of a compact regular frame $\mathbb{L}$ into a complete atomic Boolean algebra $\mathbb{B}$, namely, that e be a frame homomorphism such that the following conditions hold (where we suppress the embedding):

> (compactness) For any $S \subseteq \mathbb{L}$, if $\bigvee S = \top$, then $\bigvee S' = \top$ for some finite $S' \subseteq S$;
>
> (Hausdorff) For any $x, y \in \mathsf{At}(\mathbb{B})$, if $x \neq y$, then $x \leq a, y \leq b$ for some $a, b \in \mathbb{L}$ such that $a \wedge b = \bot$.

In particular, $\mathbb{L}_{\mathcal{T}}$ can be identified with the collection $\mathsf{O}(\mathbb{B}_{\mathbb{F}_{\mathcal{T}}})$ of open subsets in $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$. The collection $\mathsf{K}(\mathbb{B}_{\mathbb{F}_{\mathcal{T}}})$ of closed subsets in $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ can be then identified as the relative complements of elements in $\mathbb{L}_{\mathcal{T}}$. We omit the subscripts when they are clear from the context.

[3]That is, not only finite meets and complete joins are preserved, but also the modal operators, i.e. $\Box_{\mathbb{B}_{\mathbb{F}}} X = \Box_{\mathcal{T}} X$ and $(\Box_{\mathbb{B}_{\mathbb{F}}} X^c)^c = \Diamond_{\mathcal{T}} X$ for all $X \in \tau$.

$$
\begin{array}{ccc}
\mathbb{B}^{\delta} & \xleftrightarrow{\cong^{\partial}} & \mathbb{F}_{\mathbb{G}} \\
e \uparrow & & \uparrow U \\
\mathbb{B} & \xleftrightarrow{\cong^{\partial}} & \mathbb{G}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} & \xleftrightarrow{\cong^{\partial}} & \mathbb{F}_{\mathcal{T}} \\
e \uparrow & & \uparrow U \\
\mathbb{L}_{\mathcal{T}} & \xleftrightarrow{\cong^{\partial}} & \mathcal{T}
\end{array}
$$

4.3.2 Basic proof strategy for preservation

In the present section, we explain the basic proof structure we will implement in Section 4.5. We will treat the preservation results as a generalized canonicity result which, using the algorithmic canonicity strategy, are typically proved by a "U-shaped" argument (cf. page 21) described in the figure below (see [48] for a more detailed discussion): In the present setting, the U-shaped argument can be sketched as follows:

$$
\begin{array}{lcl}
\mathbb{L}_{\mathcal{T}} \Vdash \varphi \leq \psi & & \mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models_{\mathbb{L}_{\mathcal{T}}} \varphi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models_{\mathbb{L}_{\mathcal{T}}} \mathrm{Pure}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models \mathrm{Pure}(\varphi \leq \psi)
\end{array}
$$

Assume that the inequality $\varphi \leq \psi$ is valid on the modal compact regular frame $\mathcal{L}_{\mathcal{T}}$. This is equivalent to the validity on the BAO $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ over all open assignments. Then the algorithm ALBA can equivalently transform the input inequality into a set of pure quasi-inequalities $\mathrm{Pure}(\varphi \leq \psi)$ which contain no propositional variables, therefore their validity is invariant under replacing open assignments by arbitrary assignments. Then by the soundness of ALBA on perfect BAOs, the validity of $\mathrm{Pure}(\varphi \leq \psi)$ is equivalent to the validity of $\varphi \leq \psi$.

4.4 Language and interpretation for ALBA

4.4.1 The expanded language for the algorithm

In the present subsection, we will define the expanded modal language for the algorithm. Our treatment is similar to [55] and Section 2.2.

The expanded positive modal language $\mathcal{L}^+$ contains, in addition to the symbols in the positive modal language, two sets of special variables Nom of *nominals* and CoNom of *conominals*, and *connectives* $\blacklozenge$ and $\blacksquare$. The nominals and conominals are interpreted as atoms and coatoms in $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}}$ respectively, and $\blacklozenge$ (resp. $\blacksquare$) is interpreted as the left (resp. right) adjoint of the operations interpreting $\Box$ (resp. $\Diamond$).

The formulas in the expanded modal language $\mathcal{L}^+$ is given as follows:

$$\varphi ::= p \mid \mathbf{i} \mid \mathbf{m} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \blacklozenge\varphi \mid \blacksquare\varphi,$$

where $p \in \mathsf{Prop}$, $\mathbf{i} \in \mathsf{Nom}$ and $\mathbf{m} \in \mathsf{CoNom}$.

For the expanded positive modal language $\mathcal{L}^+$, the valuation V and assignment θ extend to the nominals and conominals, such that $V(\mathbf{i}), \theta(\mathbf{i}) \in \mathsf{At}(\mathbb{B})$ and $V(\mathbf{m}), \theta(\mathbf{m}) \in \mathsf{CoAt}(\mathbb{B})$. The satisfaction relation for the additional symbols is given as follows:

4.4.1. Definition. In any Kripke model $\mathbb{M} = (W, R, V)$ or any modal compact Hausdorff models $\mathbb{M} = (W, R, \tau, V)$,

$$\begin{array}{llllll}
\mathbb{M}, w \Vdash \mathbf{i} & \text{iff} & w \in V(\mathbf{i}) & \text{iff} & V(\mathbf{i}) = \{w\}; \\
\mathbb{M}, w \Vdash \mathbf{m} & \text{iff} & w \in V(\mathbf{m}) & \text{iff} & V(\mathbf{m}) \neq W - \{w\}; \\
\mathbb{M}, w \Vdash \blacklozenge\varphi & \text{iff} & \exists v(Rvw & \text{and} & \mathbb{M}, v \Vdash \varphi); \\
\mathbb{M}, w \Vdash \blacksquare\varphi & \text{iff} & \forall v(Rvw & \Rightarrow & \mathbb{M}, v \Vdash \varphi).
\end{array}$$

Algebraically, $\blacklozenge^{\mathbb{B}} X = R[X]$ and $\blacksquare^{\mathbb{B}} X = (R[X^c])^c$.

4.4.2 1-Sahlqvist inequalities

In the present section, we define the class of inequalities for which we prove the p-reservation result in Section 4.6.

4.4.2. Definition. [1-Sahlqvist inequalities] The $\mathcal{L}$-inequality $\varphi \leq \psi$ is 1-*Sahlqvist* if $\varphi = \varphi'(\chi_1/z_1, \ldots \chi_n/z_n)$ such that

(1) $\varphi'(z_1, \ldots, z_n)$ is built out of $\wedge, \vee, \Diamond$;

(2) every χ is of the form $\Box^n p$, $\Box^n \top$, $\Box^n \bot$ for $n \geq 0$.

4.4.3. Remark. As its name suggests, the definition above is the restriction of the general definition of ε-Sahlqvist inequalities of Section 2.5.3 to the order type ε which assigns every variable to 1. In the general notation of unified correspondence, the formula φ' corresponds to the Skeleton of φ, and the χ-formulas correspond to its PIA parts. This definition is slightly more general than [14, Definition 7.12] since $\vee$ is allowed to occur in φ'. The inequalities captured by [14, Definition 7.12] correspond to those referred to as *definite* 1-Sahlqvist inequalities in [55].

4.5 The algorithm ALBA

In the present section, we will give the algorithm ALBA for modal compact Hausdorff spaces, which is similar to the version in [55] and Section 2.6.

ALBA receives an inequality $\varphi \leq \psi$ as input. Then the algorithm proceeds in three stages:

The first stage is the preprocessing stage, which eliminates all uniformly occurring propositional variables, and exhaustively apply the distribution and splitting rules. This stage produces a finite number of inequalities, $\varphi_i \leq \psi_i$, $1 \leq i \leq n$. Then for each inequality, the first approximation rule is applied, which produces a set of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$.

The second stage is the reduction and elimination stage, which aims at rewrite the set $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ into a set of inequalities which has no occurence of propositional variables. In particular, the step which eliminates all propositional variables is called the Ackermann rule. After this stage, the algorithm produces a set S_i of inequalities.

The third stage is the output stage. If for some set $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$, the propositional variables cannot be eliminated, then the algorithm stops and output failure. Otherwise, the algorithm outputs the conjunction of the pure quasi-inequalities $\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}(\& S_i \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0)$.

(1) **Preprocessing and first approximation**:

In the generation tree of φ,

(a) Apply the distribution rules: Push down $\Diamond$ and $\wedge$, by distributing them over nodes labelled with $\vee$;

(b) Apply the splitting rule 1:

$$\frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

(c) Apply the variable-elimination rules:

$$\frac{\alpha \leq \beta(p)}{\alpha \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha}{\beta(\top) \leq \alpha}$$

for $\beta(p)$ containing p and α not containing p.

We denote by Preprocess($\varphi \leq \psi$) the finite set $\{\varphi_i \leq \psi_i\}_{i \in I}$ of inequalities obtained after the exhaustive application of the previous rules. Then we apply the first approximation rule to each inequality in Preprocess($\varphi \leq \psi$) :

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \quad \psi_i \leq \mathbf{m}_0}$$

Here, $\mathbf{i}_0$ and $\mathbf{m}_0$ are special nominals and co-nominals. Now we get a set of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}_{i \in I}$.

(2) **Reduction and elimination**:

In this stage, for each $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$, we apply the following rules in the previous stage to eliminate all the proposition variables in $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$:

Residuation rule	Approximation rule	Splitting rule 2
$\dfrac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta}$	$\dfrac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j}}$	$\dfrac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma}$

The nominals introduced by the approximation rule must not occur in the system before applying the rule.

The right-handed Ackermann rule. This is the core rule of ALBA, which eliminates propositional variables. This rule operates on all inequalities in the system, instead of on a single inequality.

$$\frac{S_1 \cup \ldots \cup S_k \cup P \cup \{\psi_i(p_1, \ldots, p_k) \leq \mathbf{m}_0\}}{P \cup \{\psi_i(\alpha_{(1,1)} \vee \ldots \vee \alpha_{(1,n_1)}, \ldots, \alpha_{(k,1)} \vee \ldots \vee \alpha_{(k,n_k)}) \leq \mathbf{m}_0\}}$$

where $S_l = \{\alpha_{(l,j)} \leq p_l \mid 1 \leq j \leq n_l\}$, $P = \{\beta_l \leq \gamma_l \mid 1 \leq l \leq m\}$, and $\alpha_{(1,1)}, \ldots, \alpha_{(k,n_k)}, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_m$ do not contain propositional variables.

(3) **Output**: If in the previous stage, some proposition variables cannot be eliminated by the application of the reduction rules, then the algorithm halts and outputs "failure". Otherwise, each initial tuple $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ of inequalities after the first approximation has been reduced to a set Reduce($\varphi_i \leq \psi_i$) of pure inequalities, and then the output is a set of quasi-inequalities $\{\&\mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0 : \varphi_i \leq \psi_i \in \mathrm{Preprocess}(\varphi \leq \psi)\}$, which we denote as Pure($\varphi \leq \psi$).

4.6 Main result

In the present section, we prove the preservation of the validity of 1-Sahlqvist inequalities we are after. This result follows from the soundness and success of ALBA. Specifically, we prove the soundness of ALBA with respect to the dual BAOs of the Kripke frames, both for open assignments and for arbitrary assignments. For the soundness with respect to arbitrary assignments and most of the rules with respect to open assignments, the argument is similar to existing settings (cf. [55] and Section 2.7.1), and hence omitted. We will focus on the right-handed Ackermann rule with respect to open assignments.

4.6.1. Theorem (Soundness with respect to arbitrary assignments). *If* ALBA *succeeds on an input inequality $\varphi \leq \psi$ and outputs Pure($\varphi \leq \psi$), then for any modal compact Hausdorff space $\mathcal{T}$,*

$$\mathbb{B}_{\mathbb{F}_\mathcal{T}} \models \varphi \leq \psi \textit{ iff } \mathbb{B}_{\mathbb{F}_\mathcal{T}} \models Pure(\varphi \leq \psi).$$

Proof:
The proof goes similarly to [55, Theorem 8.1]. Let $\varphi_i \leq \psi_i$, $1 \leq i \leq n$ denote the inequalities produced by preprocessing $\varphi \leq \psi$ after Stage 1, and (S_i, Ineq_i), $1 \leq i \leq n$

denote the corresponding quasi-inequalities produced by ALBA after Stage 2. It suffices to show the equivalence from (4.1) to (4.4) given below:

(4.1) $\mathbb{B}_{\mathbb{F}_\mathcal{T}} \models \varphi \leq \psi$

(4.2) $\mathbb{B}_{\mathbb{F}_\mathcal{T}} \models \varphi_i \leq \psi_i$, for all $1 \leq i \leq n$

(4.3) $\mathbb{B}_{\mathbb{F}_\mathcal{T}} \models \mathbf{i}_0 \leq \varphi_i \ \& \ \psi_i \leq \mathbf{m}_0 \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0$, for all $1 \leq i \leq n$

(4.4) $\mathbb{B}_{\mathbb{F}_\mathcal{T}} \models \& \mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0$, for all $1 \leq i \leq n$

- for the equivalence of (4.1) and (4.2), it suffices to show the soundness of the rules in Stage 1, which can be proved in the same way as in [55, Lemma 8.3];
- the equivalence between (4.2) and (4.3) follows from the soundness of the first-approximation rule, which is similar to [55, Theorem 8.1];
- the equivalence between (4.3) and (4.4) follows from the soundness of rules in Stage 2, i.e. the soundness of the approximation rule, the residuation rule, the right-handed Ackermann rule and the splitting rule, which is similar to [55, Lemma 8.4].

□

Similar to Section 2.7.2, for the soundness with respect to open assignments, most of the arguments are the same as the case for arbitrary assignments except for the right-handed Ackermann rule. The soundness of the right-handed Ackermann rule with respect to arbitrary assignments is justified by the following lemma:

4.6.2. Lemma (Right-handed Ackermann lemma). *Let $\varphi_1, \ldots, \varphi_n$ be pure formulas, $\psi(p_1, \ldots, p_n)$ be an $\mathcal{L}$-formula, $a \in \mathbb{B}$. Then for any arbitrary assignment θ, the following are equivalent:*

(1) $\psi^{\mathbb{B}}(\alpha_1^{\mathbb{B},h}, \ldots, \alpha_n^{\mathbb{B},h}) \leq a$;

(2) There exist $b_1, \ldots, b_n \in \mathbb{B}$ s.t. $\alpha_i^{\mathbb{B},h} \leq b_i$ for $1 \leq i \leq n$ and $\psi^{\mathbb{B}}(b_1, \ldots, b_n) \leq a$.

As is discussed in e.g. [55, Section 9], the lemma above cannot be applied immediately to the setting of open assignments, since formulas in the expanded modal language $\mathcal{L}^+$ might be interpreted as non-open elements, thus the elements $b_1, \ldots, b_n$ might not be in $\mathsf{O}(\mathbb{B})$. We are going to apply similar adaptation strategies as in [55] in the current setting, namely adapt the Ackermann lemma based on syntactic restrictions of the formulas.

4.6.3. Definition. [Syntactically closed and open formulas](cf. Definition 2.7.7)

(1) A formula in $\mathcal{L}^+$ is *syntactically closed* if it does not contain occurences of conominals;

(2) A formula in $\mathcal{L}^+$ is *syntactically open* if it does not contain occurences of nominals or $\blacklozenge$.

4.6.4. Lemma. *(cf. e.g. [14, Lemma 7.10]) For any modal compact Hausdorff space $\mathcal{T} = (W, R, \tau)$, if X is closed, then $R[X]$ is also closed.*

4.6.5. Lemma. *If $\varphi(\vec{\mathbf{i}}, \vec{p})$ and $\psi(\vec{\mathbf{m}}, \vec{p})$ are syntactically closed and open respectively, then*

(1) $\varphi^{\mathbb{B}}(\vec{i}, \vec{c}) \in \mathsf{K}(\mathbb{B})$ for any $\vec{i} \in \mathsf{At}(\mathbb{B})$, $\vec{c} \in \mathsf{K}(\mathbb{B})$.

(2) $\psi^{\mathbb{B}}(\vec{m}, \vec{o}) \in \mathsf{O}(\mathbb{B})$ for any $\vec{m} \in \mathsf{CoAt}(\mathbb{B})$, $\vec{o} \in \mathsf{O}(\mathbb{B})$.

Proof:
By induction on the structure of the formulas. The basic case follows from the fact that singletons are closed in Hausdorff spaces, and their complements are open. The cases of $\wedge$ and $\vee$ are easy. The cases of $\Diamond$ and $\Box$ follow from Definition 4.2.1. The case of $\blacklozenge$ follows from Lemma 4.6.4. □

4.6.6. Lemma. *(cf. e.g. [14, Lemma 7.8]) For any modal compact Hausdorff space $\mathcal{T} = (W, R, \tau)$, any $\mathcal{L}$-formula $\varphi(p_1, \ldots, p_n)$, any $c_1, \ldots, c_n \in \mathsf{K}(\mathbb{B})$,*

(1) $\varphi(c_1, \ldots, c_n) = \bigwedge\{\varphi(o_1, \ldots, o_n) \mid c_i \leq o_i$ for $1 \leq i \leq n$ and $o_i \in \mathsf{O}(\mathbb{B})\}$;

(2) $\varphi(c_1, \ldots, c_n) = \bigwedge\{\varphi(\mathsf{cl}(o_1), \ldots, \mathsf{cl}(o_n)) \mid c_i \leq o_i$ for $1 \leq i \leq n$ and $o_i \in \mathsf{O}(\mathbb{B})\}$, where $\mathsf{cl}(a)$ denotes the least closed element $\geq a$.

The lemma below justifies the soundness of right-handed Ackermann rule with respect to open assignments:

4.6.7. Lemma (Right-handed topological Ackermann lemma). *Let $\varphi_1, \ldots, \varphi_n$ be pure and syntactically closed formulas, $\psi(p_1, \ldots, p_n)$ be an $\mathcal{L}$-formula, $o \in \mathsf{O}(\mathbb{B})$. Then for any open assignment θ, the following are equivalent:*

(1) $\psi^{\mathbb{B}}(\alpha_1^{\mathbb{B},\theta}, \ldots, \alpha_n^{\mathbb{B},\theta}) \leq o$;

(2) There exist $b_1, \ldots, b_n \in \mathsf{O}(\mathbb{B})$ such that $\alpha_i^{\mathbb{B},\theta} \leq b_i$ for $1 \leq i \leq n$ and $\psi^{\mathbb{B}}(b_1, \ldots, b_n) \leq o$.

Proof:
1⇐ 2 : By the monotonicity of $\psi^{\mathbb{B}}(p_1, \ldots, p_n)$ together with $\alpha_i^{\mathbb{B},\theta} \leq b_i$ for $1 \leq i \leq n$, we have that $\psi^{\mathbb{B}}(\alpha_1^{\mathbb{B},\theta}, \ldots, \alpha_n^{\mathbb{B},\theta}) \leq \psi^{\mathbb{B}}(b_1, \ldots, b_n) \leq \mathbf{m}^{\mathbb{B}}$.

2 ⇒ 1 : Suppose that $\psi^{\mathbb{B}}(\alpha_1^{\mathbb{B},\theta}, \ldots, \alpha_n^{\mathbb{B},\theta}) \leq o$. By Lemma 4.6.5, $\alpha_1^{\mathbb{B},\theta}, \ldots, \alpha_n^{\mathbb{B},\theta} \in \mathsf{K}(\mathbb{B})$. By Lemma 4.6.6, $o \geq \psi^{\mathbb{B}}(\alpha_1^{\mathbb{B},\theta}, \ldots, \alpha_n^{\mathbb{B},\theta}) = \bigwedge\{\psi^{\mathbb{B}}(\mathsf{cl}(o_1), \ldots, \mathsf{cl}(o_n)) \mid \alpha_i^{\mathbb{B},\theta} \leq o_i$ and $o_i \in \mathsf{O}(\mathbb{B})$ for $1 \leq i \leq n\}$. Since $\mathsf{cl}(o_1), \ldots, \mathsf{cl}(o_n) \in \mathsf{K}(\mathbb{B})$, by Lemma 4.6.5, $\psi^{\mathbb{B}}(\mathsf{cl}(o_1), \ldots, \mathsf{cl}(o_n)) \in \mathsf{K}(\mathbb{B})$. By compactness, there exist $o_{1,j} \ldots, o_{n,j}$, $1 \leq j \leq m$ such that $o \geq \bigwedge_j\{\psi^{\mathbb{B}}(\mathsf{cl}(o_{1,j}), \ldots, \mathsf{cl}(o_{n,j})) \mid \alpha_i^{\mathbb{B},\theta} \leq o_{i,j}$ and $o_{i,j} \in \mathsf{O}(\mathbb{B})$ for $1 \leq i \leq n\}$. Then

$$\begin{array}{rcll} o & \geq & \bigwedge_j \psi^{\mathbb{B}}(\mathsf{cl}(o_{1,j}), \ldots, \mathsf{cl}(o_{n,j})) & \\ & \geq & \psi^{\mathbb{B}}(\bigwedge_j \mathsf{cl}(o_{1,j}), \ldots, \bigwedge_j \mathsf{cl}(o_{n,j})) & \text{(Monotonicity of } \psi^{\mathbb{B}}) \\ & \geq & \psi^{\mathbb{B}}(\bigwedge_j o_{1,j}, \ldots, \bigwedge_j o_{n,j}), & \text{(Monotonicity of } \psi^{\mathbb{B}} \text{ and } \mathsf{cl}) \end{array}$$

Take $b_i := \bigwedge_j o_{i,j}$, then b_i is a finite meet of open elements, therefore $b_i \in \mathsf{O}(\mathbb{B})$. Since $\alpha_i^{\mathbb{B},\theta} \leq o_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, it follows that $\alpha_i^{\mathbb{B},\theta} \leq b_i$. □

The lemma above is formulated independently of the specific language. As to the language treated by this chapter, in any concrete application of the Ackermann rule (see descriptions in Lemma 4.7.3), the inequalities $\alpha \leq p$ have the shape $\blacklozenge^n \mathbf{j} \leq p$, with $\blacklozenge^n \mathbf{j}$ syntactically closed by definition.

Main result

As is shown in Section 4.7, we have:

4.6.8. Theorem. *(Success)* ALBA *succeeds on 1-Sahlqvist inequalities.*

As discussed in Section 4.3.2, the preservation result follows from Theorem 4.6.8 above and the soundness of ALBA with respect to both open assignments and arbitrary assignments:

4.6.9. Theorem. *For any 1-Sahlqvist inequality $\varphi \leq \psi$, if $\mathbb{L}_{\mathcal{T}} \models \varphi \leq \psi$, then $\mathbb{B}_{\mathbb{F}_{\mathcal{T}}} \models \varphi \leq \psi$.*

4.7 ALBA succeeds on 1-Sahlqvist inequalities

In the present section, we sketch the proof of Theorem 4.6.8. In the following lemmas, we will track the shape of term inequalities in each stage of execution of ALBA. The proofs of the lemmas are similar to those given in [55, Section 10], therefore we only report on the main line of argument and omit proofs.

4.7.1. Lemma. *Let $\varphi \leq \psi$ be a 1-Sahlqvist inequality. After stage 1, it becomes sets of inequalities $\{\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\} \mid i \in I\}$ where ψ_i is a formula in the positive language $\mathcal{L}$, and every φ_i is built from $\Box^n p$, $\Box^n \top$, $\Box^n \bot$ by applying $\wedge$ and $\Diamond$.*

Proof:
For an input 1-Sahlqvist inequality $\varphi \leq \psi$, by applying the distribution rules, $\vee$ is pushed towards the root of φ, and therefore φ is transformed into a disjunction of formulas built from $\Box^n p$, $\Box^n \top$, $\Box^n \bot$ by applying $\wedge$ and $\Diamond$. By applying the splitting rule, the formula φ on the left-hand side of the inequality become splitted, and the inequality $\varphi \leq \psi$ is transformed into a set of inequalities $\{\varphi_i \leq \psi_i\}_{i \in I}$ where each φ_i satisfies the conditions stated in the lemma. By applying the first approximation rule, we get the sets of inequalities required. □

4.7.2. Lemma. *Let* $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ *be as described in Lemma 4.7.1. By applying the approximation rule and the splitting rule 2 exhaustively, the system is transformed into one which contains the following types of inequalities:*

- $\psi_i \leq \mathbf{m}_0$,
- $\mathbf{j} \leq \Diamond\mathbf{k}$ *where* $\mathbf{j}, \mathbf{k}$ *are nominals,*
- $\mathbf{j} \leq \Box^n p$,
- $\mathbf{j} \leq \beta$ *where* β *is pure, i.e.* β *contains no propositional variables.*

Proof:
Given the set of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ described in the lemma, by applying the approximation rule and the splitting rule 2, the inequality $\psi_i \leq \mathbf{m}_0$ is not rewritten, and for each inequality $\mathbf{j} \leq \varphi$ in the set, if $\varphi = \gamma \wedge \theta$, then this inequality is transformed into $\mathbf{j} \leq \gamma$ and $\mathbf{j} \leq \theta$; if $\varphi = \Diamond\gamma$, then this inequality is transformed into $\mathbf{j} \leq \Diamond\mathbf{k}$ and $\mathbf{k} \leq \gamma$. By applying the approximation rule and the splitting rule 2 exhaustively, the $\wedge, \Diamond$s are eliminated from φ_i, and $\mathbf{i}_0 \leq \varphi_i$ becomes $\mathbf{j} \leq \Box^n p$, $\mathbf{j} \leq \Box^n \top$ or $\mathbf{j} \leq \Box^n \bot$, with side conditions of the form $\mathbf{j} \leq \Diamond\mathbf{k}$. Therefore, after applying the rules mentioned, the system is as described in the lemma. □

4.7.3. Lemma. *Given a system as described in Lemma 4.7.2, by applying the residuation rule exhaustively, the system is transformed into one which contains the following types of inequalities:*

- $\psi_i \leq \mathbf{m}_0$,
- $\mathbf{j} \leq \Diamond\mathbf{k}$ *where* $\mathbf{j}, \mathbf{k}$ *are nominals,*
- $\blacklozenge^n\mathbf{j} \leq p$,
- $\beta \leq \gamma$ *where* β, γ *are pure.*

Proof:
Given a system as described in Lemma 4.7.2, the only inequalities that the residuation rule operates on are of the form $\mathbf{j} \leq \Box^n p$ or $\mathbf{j} \leq \beta$ where β contains no propositional variables. By applying the residuation rule exhaustively, $\mathbf{j} \leq \Box^n p$ becomes $\blacklozenge^n\mathbf{j} \leq p$, and $\mathbf{j} \leq \beta$ becomes an inequality which is again pure. □

The system described in Lemma 4.7.3 is in a shape in which the right-handed Ackermann rule can be applied and all propositional variables can be eliminated. Therefore the algorithm succeeds and we have proven Theorem 4.6.8.

4.8 Conclusion

In this chapter, we give an algorithmic account of the preservation results proved in [14]. The preservation result in the present chapter concerns a slight generalization (cf. Remark 4.4.3) of the class of inequalities treated in [14] over the same language of positive modal logic. The algorithmic approach adopted here emphasizes the algebraic side of this preservation result and makes it more similar to the way in which Sahlqvist canonicity has been presented in an algebraic way in [123]. In particular, just in the same way in which the embedding map of algebras into their canonical extensions encodes Stone-type dualities, the canonical embedding of modal compact regular frames into the complex algebras of the underlying Kripke frames of their dual spaces encodes Isbell-type dualities. How to optimally characterize this embedding in a way which is aligned with the definition of Jónsson and Tarski [124] is ongoing work. Building on this algebraic perspective, the ALBA approach has unified many different strategies for canonicity, e.g. those of Jónsson [123], Sambin–Vaccaro [160], Ghilardi–Meloni [91], and Venema's pseudo-correspondence [192]. Having extended the algorithmic approach to the Isbell-type dualities paves the way to several different generalizations and extensions: to richer languages such as arbitrary distributive lattice expansions, fixed point expansions of positive modal logics [19], but also to a non-distributive setting, to a constructive meta-theory, to more general syntactic shapes than 1-Sahlqvist, and so on.

Chapter 5

Sahlqvist via translation

In the present chapter, which is based on an older version of the paper [60][1], we investigate to what extent Sahlqvist-type correspondence and canonicity results for non-classical logics can be obtained via translation into classical modal logics.

5.1 Introduction

Notwithstanding the new insights and connections with various areas of logic brought about by the developments of unified correspondence theory, when limiting attention just to the Salhqvist results for non-classical logics, a natural question to ask is whether these results could have been obtained by embedding into classical logic. Indeed, it is well known that intuitionistic logic can be interpreted into the classical modal logic S4 via the famous Gödel-McKinsey-Tarski translation [94, 147], henceforth simply the Gödel-Tarski or Gödel translation. There exist various extensions of the Gödel translation, like the one used by Wolter and Zakharyaschev [194, 195] to translate intuitionistic modal logic with one $\Box$ connective into suitable polymodal logics on a classical propositional base. Since validity is preserved and reflected under this translation, it is possible to use it to transfer many results from classical to intuitionistic modal logic. This translation has linear complexity, and would make available (an adaptation of) the SQEMA technology for Boolean polymodal logic [51] also for intuitionistic modal logic. The idea of developing Sahlqvist theory via translation has been around for a long time (besides [194, 195], see also [187, 88] and more recently, for correspondence only, [187]). In the present chapter we investigate to what extent this is realizable, given the current state-of-the-art.

A hurdle that immediately presents itself is the fact that, in general, translations of the Gödel-type can run into difficulties when trying to derive correspondence results for intuitionistic modal logics, particularly if both $\Box$ and $\Diamond$ occur as primitive connectives. For instance, even a minimal extension of the Gödel translation to such an

[1] My specific contribution to this research has been to refine the canonicity methodology and write a preliminary version of the paper.

intuitionistc modal setting would transform the Sahlqvist inequality $\Box\Diamond p \leq \Diamond p$ into $\Box\Diamond\Box_G p \leq \Diamond\Box_G p$, which is not Sahlqvist, and in fact does not even have a first-order correspondent [177]. Any translation which 'boxes' propositional variables would suffer from this problem (for further discussion see [48, Section 36.9]). There are more subtle translations which avoid this problem and may therefore be used as a way to fall back on the classical Sahlqvist theorem. This is done, for example, by Gehrke, Nagahashi and Venema in [88] to obtain the correspondence part of their Sahlqvist theorem for distributive modal logic.

In the present chapter, the question of determining the extent to which Sahlqvist correspondence and canonicity for non-classical logics can be obtained via translation is systematically investigated in various settings, the most general of which is given by logics algebraically captured by normal distributive lattice expansions (DLEs). The starting point of our analysis is an order-theoretic reformulation of the main semantic property of the Gödel-Tarski translation. Our main conclusions are twofold. Firstly, that the *correspondence-via-translation* methodology in [88] straightforwardly generalizes to arbitrary signatures of normal distributive lattice expansions. Secondly, that the proof of *canonicity-via-translation* can be obtained in a similarly straightforward manner, but *only* in the special setting of normal *bi-Heyting algebra* expansions. As discussed in Section 5.5, proving canonicity *via translation* in the general normal DLE setting, if possible at all, would require techniques that are not currently available. Therefore the existing unified correspondence techniques remain the most economical route to these results.

Overall, the translation method seems inadequate to provide autonomous foundations for a general Sahlqvist theory. Besides the technical difficulties in implementing canonicity-via-translation, proceeding via translation suffers from certain inherent methodological drawbacks. In particular, any general development of Sahlqvist theory requires a uniform way to recognize Sahlqvist-type classes across logical signatures. For a treatment via translation to be significant, the specification of these syntactic classes cannot be derived from the translation itself, but should be independent from it. The definitions of Sahlqvist and inductive inequalities provided by unified correspondence are based on the order-theoretic analysis of the connectives of the logic under consideration, and are able to provide the required independent background. Thus, the Sahlqvist via translation methodology as developed in the present chapter is, in fact, yet another application of unified correspondence.

The chapter is structured as follows. In Section 5.2 we provide an order-theoretic analysis of the semantic underpinnings of the Gödel-Tarski translation. Then, in Section 5.3, we extend the insights gained in the previous section to a class of Gödel type translations parameterized with order-types. This sets the stage for Sections 5.4 and 5.5 where we collect our results on correspondence- and canonicity-via-translation. In particular, in Section 5.5.2 we discuss the difficulties in extending canonicity-via-translation beyond the bi-intuitionistic setting. We conclude in Section 5.6.

For the language, axiomatization, algebraic semantics and relational semantics of basic DLE-logics, inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities and auxiliary notations, we refer the readers to Chapter 2. In the present chapter we also find it convenient to talk of (normal) Boolean algebra expansions (BAEs) (respectively, Heyting algebra expansions (HAEs), bi-Heyting algebra expansions (bHAEs)) which are structures defined as in Definition 2.1.2, but replacing the distributive lattice $\mathbb{L}$ with a Boolean algebra (respectively, Heyting algebra, bi-Heyting algebra).

5.2 The semantic environment of the Gödel-Tarski translation

In the present section, we give a semantic analysis of the Gödel-Tarski translation in an algebraic way. In what follows, for any partial order $(W, \leq)$, we let $w{\uparrow} := \{v \in W \mid w \leq v\}$, $w{\downarrow} := \{v \in W \mid w \geq v\}$ for every $w \in W$, and for every $X \subseteq W$, we let $X{\uparrow} := \bigcup_{x\in X} x{\uparrow}$ and $X{\downarrow} := \bigcup_{x\in X} x{\downarrow}$. *Up-sets* (resp. *down-sets*) of $(W, \leq)$ are subsets $X \subseteq W$ such that $X = X{\uparrow}$ (resp. $X = X{\downarrow}$). We denote by $\mathcal{P}(W)$ the Boolean algebra of subsets of W, and by $\mathcal{P}^{\uparrow}(W)$ (resp. $\mathcal{P}^{\downarrow}(W)$) the (bi-)Heyting algebra of up-sets (resp. down-sets) of $(W, \leq)$. Finally we let X^c denote the relative complement $W \setminus X$ of every subset $X \subseteq W$.

5.2.1 Semantic analysis of the Gödel-Tarski translation

Fix a denumerable set Atprop of propositional variables. The language of intuitionistic logic over Atprop is given by

$$\mathcal{L}_I \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

The language of the normal modal logic S4 over Atprop is given by

$$\mathcal{L}_{S4\Box} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \Box_{\leq}\alpha.$$

The Gödel-Tarski translation is the map $\tau\colon \mathcal{L}_I \to \mathcal{L}_{S4\Box}$ defined by the following recursion:

$$\begin{array}{rcl}
\tau(p) & = & \Box_{\leq} p \\
\tau(\bot) & = & \bot \\
\tau(\top) & = & \top \\
\tau(\varphi \wedge \psi) & = & \tau(\varphi) \wedge \tau(\psi) \\
\tau(\varphi \vee \psi) & = & \tau(\varphi) \vee \tau(\psi) \\
\tau(\varphi \to \psi) & = & \Box_{\leq}(\neg\tau(\varphi) \vee \tau(\psi)).
\end{array}$$

The present subsection is aimed at analyzing the semantic underpinning of the Gödel-Tarski translation. This analysis will provide the insights motivating the uniform extension of the Gödel-Tarski translation to arbitrary normal DLEs.

Both intuitionistic and S4-formulas can be interpreted on partial orders $\mathbb{F} = (W, \leq)$, as follows: an S4-model is a tuple $(\mathbb{F}, U)$ where $U : \mathsf{AtProp} \to \mathcal{P}(W)$ is a valuation. The interpretation $\Vdash^*$ of S4-formulas on S4-models is defined recursively as follows: for an $w \in W$,

$\mathbb{F}, w, U \Vdash^* p$	iff $p \in U(p)$
$\mathbb{F}, w, U \Vdash^* \bot$	never
$\mathbb{F}, w, U \Vdash^* \top$	always
$\mathbb{F}, w, U \Vdash^* \alpha \wedge \beta$	iff $\mathbb{F}, w, U \Vdash^* \alpha$ and $\mathbb{F}, w, U \Vdash^* \beta$
$\mathbb{F}, w, U \Vdash^* \alpha \vee \beta$	iff $\mathbb{F}, w, U \Vdash^* \alpha$ or $\mathbb{F}, w, U \Vdash^* \beta$
$\mathbb{F}, w, U \Vdash^* \neg\alpha$	iff $\mathbb{F}, w, U \nVdash^* \alpha$
$\mathbb{F}, w, U \Vdash^* \Box_{\leq}\alpha$	iff $\mathbb{F}, v, U \Vdash^* \alpha$ for any $v \in w{\uparrow}$.

For any S4-formula α we let $(\!|\alpha|\!)_U := \{w \mid \mathbb{F}, w, U \Vdash^* \alpha\}$. It is not difficult to verify that for every $\alpha \in \mathcal{L}_{S4}$ and any valuation U,

$$(\!|\Box_{\leq}\alpha|\!)_U = (\!|\alpha|\!)_U^c{\downarrow}^c. \tag{5.1}$$

An intuitionistic model is a tuple $(\mathbb{F}, V)$ where $V : \mathsf{AtProp} \to \mathcal{P}^{\uparrow}(W)$ is a *persistent* valuation. The interpretation $\Vdash^*$ of S4-formulas on S4-models is defined recursively as follows: for an $w \in W$,

$\mathbb{F}, w, V \Vdash p$	iff $p \in V(p)$
$\mathbb{F}, w, V \Vdash \bot$	never
$\mathbb{F}, w, V \Vdash \top$	always
$\mathbb{F}, w, V \Vdash \varphi \wedge \psi$	iff $\mathbb{F}, w, V \Vdash \varphi$ and $\mathbb{F}, w, V \Vdash \psi$
$\mathbb{F}, w, V \Vdash \varphi \vee \psi$	iff $\mathbb{F}, w, V \Vdash \varphi$ or $\mathbb{F}, w, V \Vdash \psi$
$\mathbb{F}, w, V \Vdash \varphi \to \psi$	iff either $\mathbb{F}, v, V \nVdash \varphi$ or $\mathbb{F}, v, V \Vdash \psi$ for any $v \in w{\uparrow}$.

For any intuitionistic formula φ we let $[\![\varphi]\!]_V := \{w \mid \mathbb{F}, w, V \Vdash \varphi\}$. It is not difficult to verify that for all $\varphi, \psi \in \mathcal{L}_I$ and any persistent valuation V,

$$[\![\varphi \to \psi]\!]_V = ([\![\varphi]\!]_V^c \cup [\![\psi]\!]_V)^c{\downarrow}^c. \tag{5.2}$$

Clearly, every persistent valuation V on $\mathbb{F}$ is also a valuation on $\mathbb{F}$. Moreover, for every valuation U on $\mathcal{F}$, the assignment mapping every $p \in \mathsf{AtProp}$ to $U(p)^c{\downarrow}^c$ defines a persistent valuation $U^{\uparrow}$ on $\mathbb{F}$. The main semantic property of the Gödel-Tarski translation is given by the following

5.2.1. Proposition. *For every intuitionistic formula φ and every partial order $\mathbb{F} = (W, \leq)$,*

$$\mathbb{F} \Vdash \varphi \quad \textit{iff} \quad \mathbb{F} \Vdash^* \tau(\varphi).$$

Proof:
If $\mathbb{F} \nVdash \varphi$, then $\mathbb{F}, w, V \nVdash \varphi$ for some persistent valuation V and $w \in W$. That is, $w \notin [\![\varphi]\!]_V = (\!|\tau(\varphi)|\!)_V$, the last identity holding by item 1 of Lemma 5.2.2. Hence, $\mathbb{F}, w, V \nVdash^* \tau(\varphi)$, i.e. $\mathbb{F} \nVdash^* \tau(p)$. Conversely, if $\mathbb{F} \nVdash^* \tau(\varphi)$, then $\mathbb{F}, w, U \nVdash \tau(\varphi)$ for some valuation U and $w \in W$. That is, $w \notin (\!|\tau(\varphi)|\!)_U = [\![\varphi]\!]_{U^{\uparrow}}$, the last identity holding by item 2 of Lemma 5.2.2. Hence, $\mathbb{F}, w, U^{\uparrow} \nVdash \varphi$, yielding $\mathbb{F} \nVdash \varphi$. ☐

5.2.2. LEMMA. *For every intuitionistic formula φ and every partial order $\mathbb{F} = (W, \leq)$,*

(1) $[\![\varphi]\!]_V = (\![\tau(\varphi)]\!)_V$ *for every persistent valuation V on $\mathbb{F}$;*

(2) $(\![\tau(\varphi)]\!)_U = [\![\varphi]\!]_{U^\uparrow}$ *for every valuation U on $\mathbb{F}$.*

Proof:
1. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{AtProp}$. Then, for any persistent valuation V,

$$\begin{array}{rcll} [\![p]\!]_V & = & V(p) & \text{(def. of } [\![\cdot]\!]_V) \\ & = & V(p)^c{\downarrow}^c & (V \text{ persistent}) \\ & = & (\![\Box_\leq p]\!)_V & \text{(equation (5.1))} \\ & = & (\![\tau(p)]\!)_V, & \text{(def. of } \tau) \end{array}$$

as required. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any persistent valuation V,

$$\begin{array}{rcll} [\![\psi \to \chi]\!]_V & = & ([\![\psi]\!]_V^c \cup [\![\chi]\!]_V)^c{\downarrow}^c & \text{(equation (5.1))} \\ & = & ((\![\tau(\psi)]\!)_V^c \cup (\![\tau(\chi)]\!)_V)^c{\downarrow}^c & \text{(induction hypothesis)} \\ & = & (\![\Box_\leq(\neg\tau(\psi) \vee \tau(\chi))]\!)_V & \text{(equation (5.1), def. of } (\![\cdot]\!)_V) \\ & = & (\![\tau(\psi \to \chi)]\!)_V, & \text{(def. of } \tau) \end{array}$$

as required. The remaining cases are omitted.

2. By induction on φ. As for the base case, let $\varphi := p \in \mathsf{AtProp}$. Then, for any valuation U,

$$\begin{array}{rcll} (\![\tau(p)]\!)_U & = & (\![\Box_\leq p]\!)_U & \text{(def. of } \tau) \\ & = & (\![p]\!)_U^c{\downarrow}^c & \text{(equation (5.1))} \\ & = & U(p)^c{\downarrow}^c & \text{(def. of } (\![\cdot]\!)_U) \\ & = & [\![p]\!]_{U^\uparrow}, & \text{(def. of } U^\uparrow) \end{array}$$

as required. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any valuation U,

$$\begin{array}{rcll} (\![\tau(\psi \to \chi)]\!)_U & = & (\![\Box_\leq(\neg\tau(\psi) \vee \tau(\chi))]\!)_U & \text{(def. of } \tau) \\ & = & (\![\neg\tau(\psi) \vee \tau(\chi)]\!)_U^c{\downarrow}^c & \text{(equation (5.1))} \\ & = & ((\![\tau(\psi)]\!)_U^c \cup (\![\tau(\chi)]\!)_U)^c{\downarrow}^c & \text{(def. of } (\![\cdot]\!)_U) \\ & = & ([\![\psi]\!]_{U^\uparrow}^c \cup [\![\chi]\!]_{U^\uparrow})^c{\downarrow}^c & \text{(induction hypothesis)} \\ & = & [\![\psi \to \chi]\!]_{U^\uparrow}, & \text{(equation (5.2), } U^\uparrow \text{ persistent)} \end{array}$$

as required. The remaining cases are omitted. □

We saw that the key to the main semantic property of Gödel-Tarski translation, stated in Proposition 5.2.1, is the interplay between persistent and nonpersistent valuations, as captured in the above lemma. This interplay is in fact a byproduct of a more basic relationship, which we are going to analyze more in general and abstractly in the framework of interior operators.

5.2.2 An algebraic template for preservation and reflection of validity under translation

In the present subsection, we are going to generalize the key mechanism captured in the previous subsection, guaranteeing the preservation and reflection of validity under the Gödel-Tarski translation. Being able to identify this pattern in generality will make it possible to extend this mechanism to other Gödel-Tarski type translations.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages over a given set X, and let $\mathbb{A}$ and $\mathbb{B}$ be ordered $\mathcal{L}_1$- and $\mathcal{L}_2$-algebras respectively, such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists. For each $V \in \mathbb{A}^X$ and $U \in \mathbb{B}^X$, let $[\![\cdot]\!]_V$ and $(\![\cdot]\!)_U$ denote their unique homomorphic extensions to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Clearly, $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ lifts to a map $\overline{e} : \mathbb{A}^X \to \mathbb{B}^X$ by the assignment $V \mapsto e \circ V$.

5.2.3. Proposition. *Let $\tau\colon \mathcal{L}_1 \to \mathcal{L}_2$ and $r\colon \mathbb{B}^X \to \mathbb{A}^X$ be such that the following conditions hold for every $\varphi \in \mathcal{L}_1$:*

(a) $e([\![\varphi]\!]_V) = (\![\tau(\varphi)]\!)_{\overline{e}(V)}$ for every $V \in \mathbb{A}^X$;

(b) $(\![\tau(\varphi)]\!)_U = e([\![\varphi]\!]_{r(U)})$ for every $U \in \mathbb{B}^X$.

Then, for all $\varphi, \psi \in \mathcal{L}_1$,

$$\mathbb{A} \models \varphi \leq \psi \quad \text{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi).$$

Proof:
From left to right, suppose contrapositively that $(\mathbb{B}, U) \not\models \tau(\varphi) \leq \tau(\psi)$ for some $U \in \mathbb{B}^X$, that is, $(\![\tau(\varphi)]\!)_U \not\leq (\![\tau(\psi)]\!)_U$. By item (b) above, this non-inequality is equivalent to $e([\![\varphi]\!]_{r(U)}) \not\leq e([\![\psi]\!]_{r(U)})$, which, by the monotonicity of e, implies that $[\![\varphi]\!]_{r(U)} \not\leq [\![\psi]\!]_{r(U)}$, that is, $(\mathbb{A}, r(U)) \not\models \varphi \leq \psi$, as required. Conversely, if $(\mathbb{A}, V) \not\models \varphi \leq \psi$ for some $V \in \mathbb{A}^X$, then $[\![\varphi]\!]_V \not\leq [\![\psi]\!]_V$, and hence, since e is an order-embedding and by item (a) above, $(\![\tau(\varphi)]\!)_{\overline{e}(V)} = e([\![\varphi]\!]_V) \not\leq e([\![\psi]\!]_V) = (\![\tau(\psi)]\!)_{\overline{e}(V)}$, that is $(\mathbb{B}, \overline{e}(V)) \not\models \tau(\varphi) \leq \tau(\psi)$, as required. □

Notice that in the proof above we have only made use of the assumption that e is an order-embedding, but have not needed to assume any property of r. Notice also that the proposition above is independent of the logical/algebraic signature of choice, and holds for *arbitrary* algebras. This latter point will be key to the treatment of Sahlqvist canonicity via translation.

5.2.3 Interior operator analysis of the Gödel-Tarski translation

As observed above, Proposition 5.2.3 generalizes Proposition 5.2.1 in more than one way. In the present subsection, we show that the Gödel-Tarski translation fits the strengthening given by Proposition 5.2.3. Towards this goal, we let $X :=$ AtProp, $\mathcal{L}_1 := \mathcal{L}_I$, and $\mathcal{L}_2 := \mathcal{L}_{S4}$. Moreover, we let $\mathbb{A}$ be a Heyting algebra, and $\mathbb{B}$ a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a right adjoint[2] $\iota\colon \mathbb{B} \to \mathbb{A}$ such that for all $a, b \in \mathbb{A}$,

$$a \to^{\mathbb{A}} b = \iota(\neg^{\mathbb{B}} e(a) \vee^{\mathbb{B}} e(b)). \tag{5.3}$$

Then $\mathbb{B}$ can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining $\Box^{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$. The following is a well known fact in algebraic modal logic:

5.2.4. Proposition. *The BAE* $(\mathbb{B}, \Box^{\mathbb{B}})$*, with* $\Box^{\mathbb{B}}$ *defined above, is normal and is also an S4-modal algebra.*

Proof:
The fact that $\Box^{\mathbb{B}}$ preserves finite (hence empty) meets readily follows from the fact that ι is a right adjoint, and hence preserves existing (thus all finite) meets of $\mathbb{B}$, and e is a lattice homomorphism. For every $b \in \mathbb{B}$, $\iota(b) \leq \iota(b)$ implies that $\Box^{\mathbb{B}} b = e(\iota(b)) \leq b$, which proves (T). For every $b \in \mathbb{B}$, $e(\iota(b)) \leq e(\iota(b))$ implies that $\iota(b) \leq \iota(e(\iota(b)))$ and hence $\Box^{\mathbb{B}} b = e(\iota(b)) \leq e(\iota(e(\iota(b)))) = (e \circ \iota)((e \circ \iota)(b)) = \Box^{\mathbb{B}}\Box^{\mathbb{B}} b$, which proves K4. □
Finally, we let $r : \mathbb{B}^X \to \mathbb{A}^X$ be defined by the assignment $U \mapsto (\iota \circ U)$.

5.2.5. Proposition. *Let* $\mathbb{A}$*,* $\mathbb{B}$*,* $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ *and* $r\colon \mathbb{B}^X \to \mathbb{A}^X$ *be as above.*[3] *Then the Gödel-Tarski translation* τ *satisfies conditions (a) and (b) of Proposition 5.2.3 for any formula* $\varphi \in \mathcal{L}_I$.

Proof:
By induction on φ. As for the base case, let $\varphi := p \in \mathsf{AtProp}$. Then, for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

[2] That is, $e(a) \leq b$ iff $a \leq \iota(b)$ for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$. By well known order-theoretic facts (cf. [64]), $e \circ \iota$ is an *interior operator*, that is, for every $b, b' \in \mathbb{B}$,

i1. $(e \circ \iota)(b) \leq b$;

i2. if $b \leq b'$ then $(e \circ \iota)(b) \leq (e \circ \iota)(b')$;

i3. $(e \circ \iota)(b) \leq (e \circ \iota)((e \circ \iota)(b))$.

Moreover, $e \circ \iota \circ e = e$ and $\iota = \iota \circ e \circ \iota$ (cf. [64, Lemma 7.26]).

[3] The assumption that e is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ is needed for the inductive steps relative to $\bot, \top, \wedge, \vee$ in the proof this proposition, while condition (5.3) is needed for the step relative to $\to$.

$$\begin{aligned}
e([\![p]\!]_{r(U)}) &= e((\iota \circ U)(p)) \\
&= (e \circ \iota)((\!\![p]\!\!)_U) && \text{assoc. of } \circ \\
&= \Box^{\mathbb{B}}(\!\![p]\!\!)_U \\
&= (\!\![\Box_{\leq} p]\!\!)_U \\
&= (\!\![\tau(p)]\!\!)_U,
\end{aligned}$$

$$\begin{aligned}
(\!\![\tau(p)]\!\!)_{\overline{e}(V)} &= (\!\![\Box_{\leq} p]\!\!)_{\overline{e}(V)} \\
&= \Box^{\mathbb{B}}(\!\![p]\!\!)_{\overline{e}(V)} \\
&= \Box^{\mathbb{B}}((e \circ V)(p)) \\
&= (e \circ \iota)((e \circ V)(p)) \\
&= e((\iota \circ e)(V(p))) && \text{assoc. of } \circ \\
&= e(V(p)) && e \circ (\iota \circ e) = e \\
&= e([\![p]\!]_V),
\end{aligned}$$

which proves the base cases of (b) and (a) respectively. As for the inductive step, let $\varphi := \psi \to \chi$. Then, for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{aligned}
e([\![\psi \to \chi]\!]_{r(U)}) &= e([\![\psi]\!]_{r(U)} \to^{\mathbb{A}} [\![\chi]\!]_{r(U)}) \\
&= e(\iota(\neg^{\mathbb{B}} e([\![\psi]\!]_{r(U)}) \vee^{\mathbb{B}} e([\![\chi]\!]_{r(U)}))) && \text{assumption (5.3)} \\
&= e(\iota(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_U \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_U)) && \text{(induction hypothesis)} \\
&= (e \circ \iota)(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_U \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_U) \\
&= \Box^{\mathbb{B}}(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_U \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_U) \\
&= (\!\![\Box_{\leq}(\neg\tau(\psi) \vee \tau(\chi))]\!\!)_U \\
&= (\!\![\tau(\psi \to \chi)]\!\!)_U.
\end{aligned}$$

$$\begin{aligned}
e([\![\psi \to \chi]\!]_V) &= e([\![\psi]\!]_V \to^{\mathbb{A}} [\![\chi]\!]_V) \\
&= e(\iota(\neg^{\mathbb{B}} e([\![\psi]\!]_V) \vee^{\mathbb{B}} e([\![\chi]\!]_V))) && \text{assumption (5.3)} \\
&= e(\iota(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_{\overline{e}(V)})) && \text{(induction hypothesis)} \\
&= (e \circ \iota)(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_{\overline{e}(V)}) \\
&= \Box^{\mathbb{B}}(\neg^{\mathbb{B}}(\!\![\tau(\psi)]\!\!)_{\overline{e}(V)} \vee^{\mathbb{B}} (\!\![\tau(\chi)]\!\!)_{\overline{e}(V)}) \\
&= (\!\![\Box_{\leq}(\tau(\psi) \vee \tau(\chi))]\!\!)_{\overline{e}(V)} \\
&= (\!\![\tau(\psi \to \chi)]\!\!)_{\overline{e}(V)}.
\end{aligned}$$

The remaining cases are straightforward, and are left to the reader. □

The following strengthening of Proposition 5.2.1 immediately follows from Propositions 5.2.3 and 5.2.5:

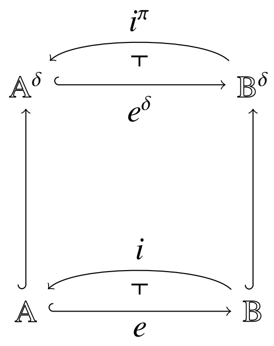
5.2.6. COROLLARY. *Let $\mathbb{A}$ be a Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ exist as above. Then for all intuitionistic formulas φ and ψ,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau(\varphi) \leq \tau(\psi),$$

where τ is the Gödel-Tarski translation.

We finish this subsection by showing that every Heyting algebra $\mathbb{A}$ embeds into a Boolean algebra $\mathbb{B}$ in the way described at the beginning of the present subsection:

5.2.7. Proposition. *For every Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ verifying condition* (5.3)*. Finally, these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



Proof:
Via Esakia duality [73], the Heyting algebra $\mathbb{A}$ can be identified with the algebra of clopen up-sets of its associated Esakia space $\mathbb{X}_{\mathbb{A}}$, which is a Priestley space, hence a Stone space. Let $\mathbb{B}$ be the Boolean algebra of the clopen sub*sets* of $\mathbb{X}_{\mathbb{A}}$. Since any clopen up-set is in particular a clopen subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim.

As to the second part, notice that Esakia spaces are Priestley spaces in which the downward-closure of a clopen set is a clopen set.

Therefore, we can define the map $\iota\colon \mathbb{B} \to \mathbb{A}$ by the assignment $b \mapsto \neg((\neg b)\downarrow)$ where b is identified with its corresponding clopen set in $\mathbb{X}_{\mathbb{A}}$, $\neg b$ is identified with the relative complement of the clopen set b, and $(\neg b)\downarrow$ is defined using the order in $\mathbb{X}_{\mathbb{A}}$. It can be readily verified that ι is the right adjoint of e and that moreover condition (5.3) holds.

Finally, $e : \mathbb{A} \to \mathbb{B}$ being also a homomorphism between the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ implies that e is smooth and its canonical extension $e^\delta\colon \mathbb{A}^\delta \to \mathbb{B}^\delta$, besides being an order-embedding, is a complete homomorphism between the lattice reducts of $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$ (cf. [87, Corollary 4.8]), and hence is endowed with both a left and a right adjoint. Furthermore, the right adjoint of e^δ coincides with ι^π (cf. [89, Proposition 4.2]). Hence, $\mathbb{B}^\delta$ can be endowed with a natural structure of S4 bi-modal algebra by defining $\Box_{\leq}^{\mathbb{B}^\delta} : \mathbb{B}^\delta \to \mathbb{B}^\delta$ by the assignment $u \mapsto (e^\delta \circ \iota^\pi)(u)$, and $\Diamond_{\geq}^{\mathbb{B}^\delta} : \mathbb{B}^\delta \to \mathbb{B}^\delta$ by the assignment $u \mapsto (e^\delta \circ c)(u)$. □

5.3 Gödel-Tarski type translations

As discussed in the introduction, only the fragment of the ε-Sahlqvist and inductive inequalities of intuitionistic logic for ε constantly equal to 1 are translated into Sahlqvist and inductive S4-formulas via Gödel-Tarski translation. Thus, the Gödel-Tarski translation alone is not enough to account for the full Sahlqvist and inductive correspondence theory. In the present section, we look into a family of *Gödel-Tarski type* translations, defined for different languages, to which we apply the template of Section 5.2.2. The first of them naturally arises by dualizing the setting of Section 5.2.1

5.3.1 The co-Gödel-Tarski translation

Fix a denumerable set Atprop of propositional variables. The language of co-intuitionistic logic over Atprop is given by

$$\mathcal{L}_C \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \succ\!\!- \varphi.$$

The target language for translating co-intuitionistic logic is that of the normal modal logic S4$\Diamond$ over Atprop, given by

$$\mathcal{L}_{S4\Diamond} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \Diamond_{\geq}\alpha.$$

Just like intuitionistic logic, formulas of co-intuitionistic logic can be interpreted on partial orders $\mathbb{F} = (W, \leq)$ with persistent valuations. Here we only report on the interpretation of $\Diamond_{\geq}$-formulas in $\mathcal{L}_{S4\Diamond}$ and $\succ\!\!-$-formulas in $\mathcal{L}_C$:

$$\begin{array}{lll} \mathbb{F}, w, U \Vdash^* \Diamond_{\geq}\varphi & \text{iff} & \mathbb{F}, v, U \Vdash^* \varphi \text{ for some } v \in w{\downarrow}. \\ \mathbb{F}, w, V \Vdash \varphi \succ\!\!- \psi & \text{iff} & \mathbb{F}, v, V \nVdash \varphi \text{ and } \mathbb{F}, v, V \Vdash \psi \text{ for some } v \in w{\downarrow}. \end{array}$$

The language $\mathcal{L}_C$ is naturally interpreted in co-Heyting algebras. The connective $\succ\!\!-$ is interpreted as the left residual of $\vee$. The co-Gödel-Tarski translation is the map $\sigma\colon \mathcal{L}_C \to \mathcal{L}_{S4\Diamond}$ defined by the following recursion:

$$\begin{array}{rcl} \sigma(p) & = & \Diamond_{\geq} p \\ \sigma(\bot) & = & \bot \\ \sigma(\top) & = & \top \\ \sigma(\varphi \wedge \psi) & = & \sigma(\varphi) \wedge \sigma(\psi) \\ \sigma(\varphi \vee \psi) & = & \sigma(\varphi) \vee \sigma(\psi) \\ \sigma(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\sigma(\varphi) \wedge \sigma(\psi)) \end{array}$$

Next, we show that Proposition 5.2.3 applies to the co-Gödel-Tarski translation. We let $X :=$ AtProp, $\mathcal{L}_1 := \mathcal{L}_C$, and $\mathcal{L}_2 := \mathcal{L}_{S4\Diamond}$. Moreover, we let $\mathbb{A}$ be a co-Heyting algebra, and $\mathbb{B}$ a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and has a left adjoint[4]

[4]That is, $c(b) \leq a$ iff $b \leq e(a)$ for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$. By well known order-theoretic facts (cf. [64]), $e \circ c$ is an *interior operator*, that is, for every $b, b' \in \mathbb{B}$,

$c\colon \mathbb{B} \to \mathbb{A}$ such that for all $a, b \in \mathbb{A}$,

$$a \succ\!\!-^{\mathbb{A}} b = c(\neg^{\mathbb{B}} e(a) \wedge^{\mathbb{B}} e(b)). \tag{5.4}$$

Then $\mathbb{B}$ can be endowed with a natural structure of Boolean algebra expansion (BAE) by defining $\Diamond^{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$. The following is the dual of Proposition 5.2.4 and its proof is omitted.

5.3.1. Proposition. *The BAE* $(\mathbb{B}, \Diamond^{\mathbb{B}})$*, with* $\Diamond^{\mathbb{B}}$ *defined above, is normal and is also an* $S4\Diamond$*-modal algebra.*

Finally, we let $r : \mathbb{B}^X \to \mathbb{A}^X$ be defined by the assignment $U \mapsto (c \circ U)$. The proof of the following proposition is similar to that of Proposition 5.2.5, and its proof is omitted.

5.3.2. Proposition. *Let* $\mathbb{A}$, $\mathbb{B}$, $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ *and* $r\colon \mathbb{B}^X \to \mathbb{A}^X$ *be as above.*[5] *Then the co-Gödel-Tarski translation* σ *satisfies conditions (a) and (b) of Proposition 5.2.3 for any formula* $\varphi \in \mathcal{L}_C$.

The following corollary immediately follows from Propositions 5.2.3 and 5.3.2:

5.3.3. Corollary. *Let* $\mathbb{A}$ *be a co-Heyting algebra and* $\mathbb{B}$ *a Boolean algebra such that an order-embedding* $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ *exists, which is a homomorphism of the lattice reducts of* $\mathbb{A}$ *and* $\mathbb{B}$*, and has a left adjoint* $c\colon \mathbb{B} \to \mathbb{A}$ *such that condition* (5.4) *holds for all* $a, b \in \mathbb{A}$*. Then for all* $\varphi, \psi \in \mathcal{L}_C$,

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \sigma(\varphi) \leq \sigma(\psi),$$

where σ *is the co-Gödel-Tarski translation.*

We finish this subsection by showing that every co-Heyting algebra $\mathbb{A}$ embeds into a a Boolean algebra $\mathbb{B}$ in the way described in Corollary 5.3.3. The proof of the following proposition is similar to the proof of Proposition 5.2.7. We include it nonetheless for the reader's convenience.

5.3.4. Proposition. *For every co-Heyting algebra* $\mathbb{A}$*, there exists a Boolean algebra* $\mathbb{B}$ *such that* $\mathbb{A}$ *embeds into* $\mathbb{B}$ *via some order-embedding* $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ *which is a homomorphism of the lattice reducts of* $\mathbb{A}$ *and* $\mathbb{B}$*, and has a left adjoint* $c\colon \mathbb{B} \to \mathbb{A}$ *verifying condition* (5.4)*. Finally, these facts lift to the canonical extensions of* $\mathbb{A}$ *and* $\mathbb{B}$ *as in the following diagram:*

---

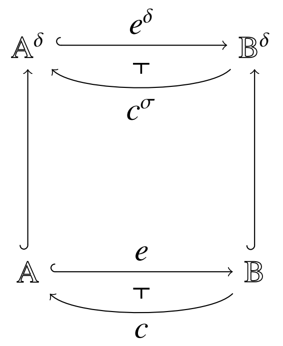
c1. $b \leq (e \circ c)(b)$;

c2. if $b \leq b'$ then $(e \circ c)(b) \leq (e \circ c)(b')$;

c3. $(e \circ c)((e \circ c)(b)) \leq (e \circ c)(b)$.

Moreover, $e \circ c \circ e = e$ and $c = c \circ e \circ c$ (cf. [64, Lemma 7.26]).

[5]The assumption that e is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ is needed for the inductive steps relative to $\bot, \top, \wedge, \vee$ in the proof this proposition, while condition (5.4) is needed for the step relative to $\succ\!\!-$.



Proof:
Similar to Proposition 5.2.7. □

5.3.2 Extending the Gödel and co-Gödel translations to bi-intuitionistic logic

The language of bi-intuitionistic logic is given by

$$\mathcal{L}_B \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \succ\!\!- \varphi$$

The language of the normal bi-modal logic S4 is given by

$$\mathcal{L}_{S4B} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \neg\alpha \mid \Box_{\leq}\alpha \mid \Diamond_{\geq}\alpha$$

The Gödel-Tarski and the co-Gödel-Tarski translations τ and σ can be extended to the bi-intuitionistic language as the maps $\tau', \sigma' \colon \mathcal{L}_B \to \mathcal{L}_{S4B}$ defined by the following recursions:

$$\begin{array}{rcl}
\tau'(p) & = & \Box_{\leq} p \\
\tau'(\bot) & = & \bot \\
\tau'(\top) & = & \top \\
\tau'(\varphi \wedge \psi) & = & \tau'(\varphi) \wedge \tau'(\psi) \\
\tau'(\varphi \vee \psi) & = & \tau'(\varphi) \vee \tau'(\psi) \\
\tau'(\varphi \rightarrow \psi) & = & \Box_{\leq}(\neg\tau'(\varphi) \vee \tau'(\psi)) \\
\tau'(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\tau'(\varphi) \wedge \tau'(\psi))
\end{array}
\qquad
\begin{array}{rcl}
\sigma'(p) & = & \Diamond_{\geq} p \\
\sigma'(\bot) & = & \bot \\
\sigma'(\top) & = & \top \\
\sigma(\varphi \wedge \psi) & = & \sigma'(\varphi) \wedge \sigma'(\psi) \\
\sigma'(\varphi \vee \psi) & = & \sigma'(\varphi) \vee \sigma'(\psi) \\
\sigma'(\varphi \rightarrow \psi) & = & \Box_{\leq}(\neg\sigma'(\varphi) \vee \sigma'(\psi)). \\
\sigma'(\varphi \succ\!\!- \psi) & = & \Diamond_{\geq}(\neg\sigma'(\varphi) \wedge \sigma'(\psi)).
\end{array}$$

Notice that τ' and σ' agree on each defining clause but those relative to the proposition variables. Let $\mathbb{A}$ be a bi-Heyting algebra and $\mathbb{B}$ a Boolean algebra such that $e \colon \mathbb{A} \hookrightarrow \mathbb{B}$ is an order-embedding and a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$. Suppose that e has both a left adjoint $c \colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota \colon \mathbb{B} \to \mathbb{A}$ such that identities (5.3) and (5.4) hold for every $a, b \in \mathbb{A}$. Then $\mathbb{B}$ can be endowed with a natural structure of bi-modal S4-algebra by defining $\Box^{\mathbb{B}} \colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$ and $\Diamond^{\mathbb{B}} \colon \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$.

5.3.5. Proposition. *The BAE $(\mathbb{B}, \Box^{\mathbb{B}}, \Diamond^{\mathbb{B}})$, with $\Box^{\mathbb{B}}, \Diamond^{\mathbb{B}}$ defined as above, is normal and an S4-bimodal algebra.*

The proof of the following proposition is similar to those of Propositions 5.2.5 and 5.3.2, and is omitted.

5.3.6. Proposition. *The translation τ' (resp. σ') defined above satisfies conditions (a) and (b) of Proposition 5.2.3 relative to $r : \mathbb{B}^X \to \mathbb{A}^X$ defined by $U \mapsto (\iota \circ U)$ (resp. defined by $U \mapsto (c \circ U)$).*

Thanks to the proposition above, Proposition 5.2.3 applies to both τ' and σ', which provides us with two equally well behaved ways of defining Gödel-Tarski-type translations for the bi-intuitionistic language in a way which retains the main property of the original Gödel-Tarski translation, namely the preservation and reflection of validity over S4-frames. In the light of this result, a natural question is whether τ' and σ' are the only two translations with this property. In the following subsection we will answer this question in the negative.

5.3.3 Parametric Gödel-Tarski-type translations for bi-intuitionistic logic

Let $X :=$ AtProp. For any order-type ε on X, define the translation $\tau_\varepsilon : \mathcal{L}_B \to \mathcal{L}_{S4B}$ by the following recursion:

$$\tau_\varepsilon(p) = \begin{cases} \Box_\leq p & \text{if } \varepsilon(p) = 1 \\ \Diamond_\geq p & \text{if } \varepsilon(p) = \partial \end{cases}.$$

A similar definition appears in [88]. The remaining defining clauses for τ_ε are analogous to those for τ' (see above).[6] Clearly, $\tau' = \tau_\varepsilon$ for ε constantly 1, and $\sigma' = \tau_\varepsilon$ for ε constantly ∂.

Let $\mathbb{A}$ be a bi-Heyting algebra and $\mathbb{B}$ be a Boolean algebra such that an order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice-reducts of $\mathbb{A}$ and $\mathbb{B}$, is endowed with both right and left adjoints, and satisfies (5.3) and (5.4) for every $a, b \in \mathbb{A}$ as described in the previous subsection. For every order-type ε on X, consider the map $r_\varepsilon : \mathbb{B}^X \to \mathbb{A}^X$ defined, for any $U \in \mathbb{B}^X$ and $p \in X$, by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

5.3.7. Proposition. *For every order-type ε on X, the translation τ_ε defined above satisfies conditions (a) and (b) of Proposition 5.2.3 relative to r_ε.*

[6] Dually, we could also define the parametric generalization σ_ε of σ. Since $\sigma_\varepsilon = \tau_{\varepsilon^\partial}$, this definition would be redundant.

Proof:
By induction on φ. As for the base case, let $\varphi := p \in \mathsf{AtProp}$. If $\varepsilon(p) = \partial$, then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll} e([\![p]\!]_{r_\varepsilon(U)}) & = & e((c \circ U)(p)) & \text{(def. of } r_\varepsilon) \\ & = & (e \circ c)(\!(p)\!)_U & \text{(assoc. of } \circ) \\ & = & \Diamond^{\mathbb{B}}(\!(p)\!)_U & \text{(def. of } \Diamond^{\mathbb{B}}) \\ & = & (\!(\Diamond_{\geq} p)\!)_U & \text{(def. of } (\!(\cdot)\!)_U) \\ & = & (\!(\tau_\varepsilon(p))\!)_U, & \text{(def. of } \tau_\varepsilon) \end{array}$$

$$\begin{array}{rcll} (\!(\tau_\varepsilon(p))\!)_{\overline{e}(V)} & = & (\!(\Diamond_{\geq} p)\!)_{\overline{e}(V)} & \text{(def. of } \tau_\varepsilon) \\ & = & \Diamond^{\mathbb{B}}(\!(p)\!)_{\overline{e}(V)} & \text{(def. of } (\!(\cdot)\!)_U) \\ & = & \Diamond^{\mathbb{B}}((e \circ V)(p)) & \text{(def. of } \overline{e}(V)) \\ & = & (e \circ c)((e \circ V)(p)) & \text{(def. of } \Diamond^{\mathbb{B}}) \\ & = & e((c \circ e)(V(p))) & \text{(assoc. of } \circ) \\ & = & e(V(p)) & (e \circ (c \circ e) = e) \\ & = & e([\![p]\!]_V). & \text{(def. of } [\![\cdot]\!]_V) \end{array}$$

If $\varepsilon(p) = 1$, then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

$$\begin{array}{rcll} e([\![p]\!]_{r_\varepsilon(U)}) & = & e((\iota \circ U)(p)) & \text{(def. of } r_\varepsilon) \\ & = & (e \circ \iota)(\!(p)\!)_U & \text{(assoc. of } \circ) \\ & = & \Box^{\mathbb{B}}(\!(p)\!)_U & \text{(def. of } \Box^{\mathbb{B}}) \\ & = & (\!(\Box_{\leq} p)\!)_U & \text{(def. of } (\!(\cdot)\!)_U) \\ & = & (\!(\tau_\varepsilon(p))\!)_U, & \text{(def. of } \tau_\varepsilon) \end{array}$$

$$\begin{array}{rcll} (\!(\tau_\varepsilon(p))\!)_{\overline{e}(V)} & = & (\!(\Box_{\leq} p)\!)_{\overline{e}(V)} & \text{(def. of } \tau_\varepsilon) \\ & = & \Box^{\mathbb{B}}(\!(p)\!)_{\overline{e}(V)} & \text{(def. of } (\!(\cdot)\!)_U) \\ & = & \Box^{\mathbb{B}}((e \circ V)(p)) & \text{(def. of } \overline{e}(V)) \\ & = & (e \circ \iota)((e \circ V)(p)) & \text{(def. of } \Box^{\mathbb{B}}) \\ & = & e((\iota \circ e)(V(p))) & \text{(assoc. of } \circ) \\ & = & e(V(p)) & (e \circ (\iota \circ e) = e) \\ & = & e([\![p]\!]_V). & \text{(def. of } [\![\cdot]\!]_V) \end{array}$$

The remainder of the proof is similar to that of Proposition 5.3.6 for τ', and is omitted.
□

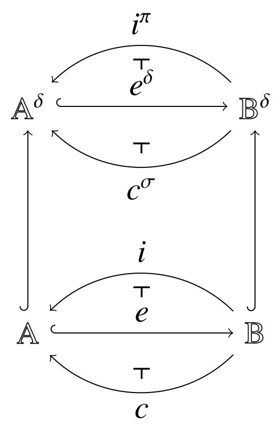
As a consequence of the proposition above, Proposition 5.2.3 applies to τ_ε for any order-type ε on X. Hence:

5.3.8. Corollary. *Let $\mathbb{A}$ be a bi-Heyting algebra. If an embedding $e : \mathbb{A} \to \mathbb{B}$ exists into a Boolean algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts and e has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ satisfying* (5.3) *and* (5.4) *for every $a, b \in \mathbb{A}$, then for any bi-intuitionistic inequality $\varphi \leq \psi$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

We finish this subsection by showing that every bi-Heyting algebra $\mathbb{A}$ embeds into a a Boolean algebra $\mathbb{B}$ in the way described in Corollary 5.3.8. The proof of the following proposition is similar to the proofs of Proposition 5.2.7 and 5.3.4. We include it nonetheless for the reader's convenience.

5.3.9. PROPOSITION. *For every bi-Heyting algebra $\mathbb{A}$, there exists a Boolean algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ verifying conditions* (5.3) *and* (5.4). *Finally, all these facts lift to the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ as in the following diagram:*



Proof:
Similar to Proposition 5.2.7. □

5.3.4 Parametric Gödel-Tarski-type translations for normal DLEs

Throughout the present section, let us fix a normal DLE-signature $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F},\mathcal{G})$. The present section is aimed at extending the definition of parametric Gödel-Tarski-type translations from the bi-intuitionistic setting to the general DLE-setting. Towards this aim, we need to define the target language for these translations. This is given in two steps: firstly, we define the normal BAE signature $\overline{\mathcal{L}}_{\mathrm{BAE}} = \mathcal{L}_{\mathrm{BAE}}(\overline{\mathcal{F}},\overline{\mathcal{G}})$, where $\overline{\mathcal{F}} := \{\overline{f} \mid f \in \mathcal{F}\}$, and $\overline{\mathcal{G}} := \{\overline{g} \mid g \in \mathcal{G}\}$, and for every $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), the connective $\overline{f}$ (resp. $\overline{g}$) is such that $n_{\overline{f}} = n_f$ (resp. $n_{\overline{g}} = n_g$) and $\varepsilon_{\overline{f}}(i) = 1$ for each $1 \leq i \leq n_f$ (resp. $\varepsilon_{\overline{g}}(i) = \partial$ for each $1 \leq i \leq n_g$).

Secondly, we assume that an order embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, is such that both the left and right adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ exist and moreover the following diagrams commute for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$:[7]

$$\begin{array}{ccccccc}
\mathbb{A}^{\varepsilon_f} & \xrightarrow{e^{\varepsilon_f}} & \mathbb{B}^{\varepsilon_{\overline{f}}} & \qquad & \mathbb{A}^{\varepsilon_g} & \xrightarrow{e^{\varepsilon_g}} & \mathbb{B}^{\varepsilon_{\overline{g}}} \\
\Big\downarrow f^{\mathbb{A}} & & \Big\downarrow \overline{f}^{\mathbb{B}} & & \Big\downarrow g^{\mathbb{A}} & & \Big\downarrow \overline{g}^{\mathbb{B}} \\
\mathbb{A} & \xleftarrow{c} & \mathbb{B} & & \mathbb{A} & \xleftarrow{\iota} & \mathbb{B}
\end{array} \tag{5.5}$$

Then, as discussed early on, the Boolean reduct of $\mathbb{B}$ can be endowed with a natural structure of bi-modal S4-algebra by defining $\Box^{\mathbb{B}}: \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ \iota)(b)$ and $\Diamond^{\mathbb{B}}: \mathbb{B} \to \mathbb{B}$ by the assignment $b \mapsto (e \circ c)(b)$.

The target language for the parametrized Gödel-Tarski type translations over Atprop is given by

$$\mathcal{L}^*_{BAE} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid \overline{f}(\overline{\alpha}) \mid \overline{g}(\overline{\alpha}) \mid \Diamond_{\geq}\alpha \mid \Box_{\leq}\alpha.$$

Let $X :=$ AtProp. For any order-type ε on X, define the translation $\tau_\varepsilon : \mathcal{L}_{\mathrm{DLE}} \to \mathcal{L}^*_{\mathrm{BAE}}$ by the following recursion:

$$\tau_\varepsilon(p) = \begin{cases} \Box_{\leq} p & \text{if } \varepsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \varepsilon(p) = \partial, \end{cases} \qquad \begin{array}{rcl}
\tau_\varepsilon(\bot) & = & \bot \\
\tau_\varepsilon(\top) & = & \top \\
\tau_\varepsilon(\varphi \wedge \psi) & = & \tau_\varepsilon(\varphi) \wedge \tau_\varepsilon(\psi) \\
\tau_\varepsilon(\varphi \vee \psi) & = & \tau_\varepsilon(\varphi) \vee \tau_\varepsilon(\psi) \\
\tau_\varepsilon(f(\overline{\varphi})) & = & \overline{f}(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_f}) \\
\tau_\varepsilon(g(\overline{\varphi})) & = & \overline{g}(\overline{\tau_\varepsilon(\varphi)}^{\varepsilon_g})
\end{array}$$

where for each order-type η on n and any n-tuple $\overline{\psi}$ of $\mathcal{L}_{\mathrm{BAE}}$-formulas, $\overline{\psi}^{\eta}$ denotes the n-tuple $(\psi'_i)_{i=1}^n$, where $\psi'_i = \psi_i$ if $\eta(i) = 1$ and $\psi'_i = \neg\psi_i$ if $\eta(i) = \partial$.

Let $\mathbb{A}$ be a $\mathcal{L}_{\mathrm{DLE}}$-algebra and $\mathbb{B}$ be a $\mathcal{L}^*_{\mathrm{BAE}}$-algebra such that an order-embedding $e: \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is a homomorphism of the lattice-reducts of $\mathbb{A}$ and $\mathbb{B}$, is endowed with both right and left adjoints, and satisfies the commutativity of the diagrams (5.5) for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$. For every order-type ε on X, consider the map $r_\varepsilon : \mathbb{B}^X \to \mathbb{A}^X$ defined, for any $U \in \mathbb{B}^X$ and $p \in X$, by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

5.3.10. Proposition. *For every order-type ε on X, the translation τ_ε defined above satisfies conditions (a) and (b) of Proposition 5.2.3 relative to r_ε.*

Proof:
By induction on φ. The base cases are analogous to those in the proof of Proposition 5.3.7. Let $\varphi := f(\overline{\varphi})$. Then for any $U \in \mathbb{B}^X$ and $V \in \mathbb{A}^X$,

[7] Notice that equations (5.3) and (5.4) encode the special cases of the commutativity of the diagrams (5.5) for $f(\varphi,\psi) := \varphi \succ\!\!- \psi$ (in which case, $\overline{f}(\alpha,\beta) := \neg\alpha \wedge \beta$) and $g(\varphi,\psi) := \varphi \to \psi$ (in which case, $\overline{g}(\alpha,\beta) := \neg\alpha \vee \beta$).

$$
\begin{array}{rcll}
e(\llbracket f(\overline{\varphi})\rrbracket_{r_\varepsilon(U)}) & = & e(f(\overline{\llbracket\varphi\rrbracket_{r_\varepsilon(U)}})) & (\text{def. of } \llbracket\cdot\rrbracket_{r_\varepsilon(U)}) \\
 & = & \overline{f}(\overline{e(\llbracket\varphi\rrbracket_{r_\varepsilon(U)})}) & (\text{assump. } (5.5)) \\
 & = & \overline{f}(\overline{\llparenthesis\tau_\varepsilon(\varphi)\rrparenthesis_U}) & (\text{IH}) \\
 & = & \llparenthesis\overline{f}(\overline{\tau_\varepsilon(\varphi)})\rrparenthesis_U & (\text{def. of } \llparenthesis\cdot\rrparenthesis_U) \\
 & = & \llparenthesis\tau_\varepsilon(f(\overline{\varphi}))\rrparenthesis_U, & (\text{def. of } \tau_\varepsilon)
\end{array}
$$

$$
\begin{array}{rcll}
\llparenthesis\tau_\varepsilon(f(\overline{\varphi}))\rrparenthesis_{\overline{e}(V)} & = & \llparenthesis\overline{f}(\overline{\tau_\varepsilon(\varphi)})\rrparenthesis_{\overline{e}(V)} & (\text{def. of } \tau_\varepsilon) \\
 & = & \overline{f}(\overline{\llparenthesis\tau_\varepsilon(\varphi)\rrparenthesis_{\overline{e}(V)}}) & (\text{def. of } \llparenthesis\cdot\rrparenthesis_{\overline{e}(V)}) \\
 & = & \overline{f}(\overline{e(\llbracket\varphi\rrbracket_V)}) & (\text{IH}) \\
 & = & e(f(\overline{\llbracket\varphi\rrbracket_V})) & (\text{assump. } (5.5)) \\
 & = & e(\llbracket f(\overline{\varphi})\rrbracket_V). & (\text{def. of } \llbracket\cdot\rrbracket_V)
\end{array}
$$

For the sake of readability, the polarity bookkeeping $\overline{\psi}^\eta$ (cf. page 92) has been suppressed in the computation above. The remaining cases are analogous and are omitted. □

As a consequence of the proposition above, Proposition 5.2.3 applies to τ_ε for any order-type ε on X. Hence:

5.3.11. Corollary. *Let $\mathbb{A}$ be a $\mathcal{L}_{\mathrm{DLE}}$-algebra. If an embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists into a $\mathcal{L}^*_{\mathrm{BAE}}$-algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, and e has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ satisfying the commutativity of the diagrams* (5.5) *for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, then for any $\mathcal{L}_{\mathrm{DLE}}$-inequality $\varphi \leq \psi$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

We finish this subsection by showing that every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ embeds into a perfect Boolean algebra $\mathbb{B}$ in the way described in Corollary 5.3.11:

5.3.12. Proposition. *For every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$, there exists a perfect $\mathcal{L}^*_{\mathrm{BAE}}$-algebra $\mathbb{B}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$ via some order-embedding $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ which is also a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ and a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ satisfying the commutativity of the diagrams* (5.5).

Proof:
Via Birkhoff duality, the perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ can be identified with the algebra of up-sets of its associated prime element structure $\mathbb{X}_{\mathbb{A}}$, which is based on a poset. Let $\mathbb{B}$ be the powerset algebra of the universe of $\mathbb{X}_{\mathbb{A}}$. Since any up-set is in particular a subset, a natural order embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ exists, which is also a complete lattice homomorphism between $\mathbb{A}$ and $\mathbb{B}$. This shows the first part of the claim.

As to the second part, notice that the algebras of upsets of a given poset are naturally endowed with a structure of bi-Heyting algebras. Hence we can define the maps $c\colon \mathbb{B} \to \mathbb{A}$ and $\iota\colon \mathbb{B} \to \mathbb{A}$ by the assignments $b \mapsto b{\uparrow}$ and $b \mapsto \neg((\neg b){\downarrow})$ respectively, where b is identified with its corresponding subset in $\mathbb{X}_{\mathbb{A}}$, $\neg b$ is defined as the relative complement of b in $\mathbb{X}_{\mathbb{A}}$, and $b{\uparrow}$ and $(\neg b){\downarrow}$ are defined using the order in $\mathbb{X}_{\mathbb{A}}$. It can be readily verified that c and ι are the left and right adjoints of e respectively.

Finally, notice that any DLE-frame $\mathbb{F}$ is also an $\overline{\mathcal{L}}^*_{\mathrm{BAE}}$-frame by interpreting the f-type connective $\Diamond_{\geq}$ by means of the binary relation $\geq$, the g-type connective $\Box_{\leq}$ by means of the binary relation $\leq$, each $\overline{f} \in \overline{\mathcal{F}}$ by means of R_f and each $\overline{g} \in \overline{\mathcal{G}}$ by means of R_g. Moreover, the additional properties (2.1) and (2.2) of the relations R_f and R_g guarantee that the diagrams (5.5) commute. □

Notice that Proposition 5.3.12 has a more restricted scope than analogous propositions such as Propositions 5.3.9 or 5.2.7. Indeed, any DLE $\mathbb{A}$ is isomorphic via Priestley-type duality to the algebra of clopen up-sets of its dual Priestley space $\mathbb{X}_{\mathbb{A}}$, which is a Stone space in particular, and this yields a natural embedding of $\mathbb{A}$ into the Boolean algebra of the clopen subsets of $\mathbb{X}_{\mathbb{A}}$. However, this embedding has in general neither a right nor a left adjoint. In the next section, we will see that Proposition 5.3.12 is enough to obtain the Sahlqvist-type correspondence theory for inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities via translation from Sahlqvist-type correspondence theory for inductive $\mathcal{L}_{\mathrm{BAE}}$-inequalities. However, we will see in Section 5.5 that canonicity cannot be straightforwardly obtained in the same way, precisely due to the restriction on Proposition 5.3.12.

5.4 Correspondence via translation

The theory developed so far is ready to be applied to the correspondence of inductive DLE-inequalities (hence also intuitionistic, co-intuitionistic and bi-intuitionistic inductive inequalities). In what follows, we let $\mathcal{L}$ denote any language in $\{\mathcal{L}_I, \mathcal{L}_C, \mathcal{L}_B, \mathcal{L}_{\mathrm{DLE}}\}$, and $\mathcal{L}^*$ its associated target language in $\{\mathcal{L}_{S4}, \mathcal{L}_{S4\Diamond}, \mathcal{L}_{S4B}, \mathcal{L}^*_{\mathrm{BAE}}\}$. The general definition of inductive inequalities (cf. Definition 2.5.3) applies to each of these languages. In particular, the Boolean negation in any $\mathcal{L}^*$ enjoys both the order-theoretic properties of a unary f-type connective and of a unary g-type connective. Hence, Boolean negation occurs unrestricted in inductive $\mathcal{L}^*$-inequalities. Moreover, the algebraic interpretations of the S4-connectives $\Box_{\leq}$ and $\Diamond_{\geq}$ enjoy the order-theoretic properties of normal unary f-type and g-type connectives respectively. Hence, the occurrence of $\Box_{\leq}$ and $\Diamond_{\geq}$ in inductive $\mathcal{L}^*$-inequalities is subject to the same restrictions applied to any connective pertaining to the same class to which they belong.

The following correspondence theorem is a straightforward extension to the $\mathcal{L}^*$-setting of the correspondence result for classical normal modal logic in [51]:

5.4.1. Proposition. *Every inductive $\mathcal{L}^*$-inequality has a first-order correspondent over its class of $\mathcal{L}^*$-frames.*

In what follows, we aim at obtaining the correspondence theorem for inductive $\mathcal{L}$-inequalities from the correspondence theorem for inductive $\mathcal{L}^*$-inequalities as stated in the proposition above. Towards this goal, we need the following

5.4.2. Proposition. *The following are equivalent for any order-type ε on X, and any $\mathcal{L}$-inequality $\varphi \leq \psi$:*

(1) $\varphi \leq \psi$ is an (Ω, ε)-inductive $\mathcal{L}$-inequality;

(2) $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^*$-inequality.

Proof:
By induction on the shape of $\varphi \leq \psi$. In a nutshell: the definitions involved guarantee that: (1) PIA nodes are introduced immediately above ε-critical occurrences of proposition variables; (2) Skeleton nodes are translated as (one or more) Skeleton nodes; (3) PIA nodes are translated as (one or more) PIA nodes. Moreover, this translation does not disturb the dependency order Ω. Hence, from item 1 to item 2, the translation does not introduce any violation on ε-critical branches, and, from item 2 to item 1, the translation does not amend any violation. □

5.4.3. Theorem (Correspondence via translation). *Every inductive $\mathcal{L}$-inequality has a first-order correspondent on $\mathcal{L}$-frames.*

Proof:
Let $\varphi \leq \psi$ be an (Ω, ε)-inductive $\mathcal{L}$-inequality, and $\mathbb{F}$ be an $\mathcal{L}$-frame such that $\mathbb{F} \Vdash \varphi \leq \psi$. By the discrete duality between $\mathcal{L}$-algebras and $\mathcal{L}$-frames, this assumption is equivalent to $\mathbb{A} \models \varphi \leq \psi$, where $\mathbb{A}$ denotes the complex $\mathcal{L}$-algebra of $\mathbb{F}$. Since $\mathbb{A}$ is a perfect $\mathcal{L}$-algebra, it is naturally endowed with the structure of a bi-Heyting algebra. By Propositions 5.2.7, 5.3.4, 5.3.9, 5.3.12, a perfect $\mathcal{L}^*$-algebra $\mathbb{B}$ exists with a natural embedding $e : \mathbb{A} \to \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a right adjoint $\iota \colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c \colon \mathbb{B} \to \mathbb{A}$ such that conditions (5.3) and (5.4) hold, and diagrams (5.5) commute. By Corollaries 5.3.8 and 5.3.11, $\mathbb{A} \models \varphi \leq \psi$ iff $\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, which, by the discrete duality between perfect $\mathcal{L}^*$-algebras and $\mathcal{L}^*$-frames, is equivalent to $\mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$.

By Proposition 5.4.2, $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}^*$-inequality, and hence, by Proposition 5.4.1, $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ has a first-order correspondent $\mathsf{FO}(\varphi)$ on $\mathcal{L}^*$-frames. Since the first-order theory of $\mathbb{F}$ as an $\mathcal{L}$-frame coincides with the first-order theory of $\mathbb{F}$ as an $\mathcal{L}^*$-frame, $\mathsf{FO}(\varphi)$ is also the first-order correspondent of $\varphi \leq \psi$. The steps of this argument are summarized in the following chain of equivalences:

	$\mathbb{F} \Vdash \varphi \leq \psi$	
iff	$\mathbb{A} \models \varphi \leq \psi$	(discrete duality for $\mathcal{L}$-frames)
iff	$\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$	(Corollaries 5.3.8 and 5.3.11)
iff	$\mathbb{F} \Vdash^* \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$	(discrete duality for $\mathcal{L}^*$-frames)
iff	$\mathbb{F} \models \mathsf{FO}(\varphi)$	(Proposition 5.4.1)

□

5.4.4. Remark. Theorem 5.4.3 provides a very concise and uniform route to correspondence for the class of inductive inequalities in any $\mathcal{L}_{\mathrm{DLE}}$-signature. This route bypasses one of the two main tools of algorithmic correspondence theory for logics on a weaker than classical propositional base (the algorithm ALBA [55]). Elsewhere [59, 56, 109, 152], evidence was provided to the effect that the scope of applicability of the algorithm ALBA is in fact wider than just the computation of first-order correspondents. Theorem 5.4.3 shows that in fact, as far as the computation of first-order correspondents in concerned, the algorithm SQEMA, or a suitable generalization of it, is already enough, and ALBA can be actually bypassed when we are only interested in correspondence.

On the other hand, to be able to implement the correspondence-via-translation strategy in a way which is both conceptually significant, and as uniform as in the statement and proof of Theorem 5.4.3, it is key to employ a definition of Sahlqvist-type formulas or inequalities holding uniformly across signatures, and formulated *independently* of the translations. Such a uniform definition (cf. Definition 2.5.3) is the second main tool of unified correspondence theory. Summing up, the translation route to correspondence does not give rise to an alternative 'unified correspondence theory' built on independent bases, but is rather facilitated by the notions and insights pertaining to unified correspondence theory.

5.5 Canonicity via translation

Recall that $\mathcal{L}$ denotes any language in $\{\mathcal{L}_I, \mathcal{L}_C, \mathcal{L}_B, \mathcal{L}_{\mathrm{DLE}}\}$, and $\mathcal{L}^*$ its associated target language in $\{\mathcal{L}_{S4}, \mathcal{L}_{S4\Diamond}, \mathcal{L}_{S4B}, \mathcal{L}^*_{\mathrm{BAE}}\}$.

The following canonicity theorem is a straightforward reformulation and extension to each $\mathcal{L}^*$-setting of the canonicity result for classical normal modal logic in [51]:

5.5.1. Proposition. *For every inductive $\mathcal{L}^*$-inequality $\alpha \leq \beta$ and every $\mathcal{L}^*$-algebra $\mathbb{B}$,*

$$\text{if } \mathbb{B} \models \alpha \leq \beta \ \text{ then } \mathbb{B}^\delta \models \alpha \leq \beta.$$

In what follows, we aim at obtaining the canonicity theorem for inductive $\mathcal{L}$-inequalities from the canonicity theorem for inductive $\mathcal{L}^*$-inequalities as stated in the proposition above. While the correspondence-via-translation strategy works uniformly on each $\mathcal{L}$-setting, the same is not true for canonicity. In the next subsection we start with the most amenable setting.

5.5.1 Canonicity of inductive inequalities in the bi-intuitionistic setting

In what follows, we aim at obtaining the canonicity theorem for inductive $\mathcal{L}_B$-inequalities from the canonicity theorem for inductive $\mathcal{L}_{S4B}$-inequalities as stated in Proposition 5.5.1.

5.5.2. Theorem (Canonicity via translation). *For every inductive $\mathcal{L}_B$-inequality $\varphi \leq \psi$ and every bi-Heyting algebra $\mathbb{A}$,*

$$\textit{if } \mathbb{A} \models \varphi \leq \psi \ \textit{ then } \mathbb{A}^\delta \models \varphi \leq \psi.$$

Proof:
By Proposition 5.3.9, an $\mathcal{L}_{S4B}$-algebra $\mathbb{B}$ exists with a natural embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$ and has both a right adjoint $\iota\colon \mathbb{B} \to \mathbb{A}$ and a left adjoint $c\colon \mathbb{B} \to \mathbb{A}$ such that conditions (5.3) and (5.4) hold. By Corollary 5.3.8, $\mathbb{A} \models \varphi \leq \psi$ iff $\mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$.

By Proposition 5.4.2, $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ is an (Ω, ε)-inductive $\mathcal{L}_{S4B}$-inequality, and hence, by Proposition 5.5.1, $\mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$. By the last part of the statement of Proposition 5.3.9, Corollary 5.3.8 applies also to $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$, and thus $\mathbb{A}^\delta \models \varphi \leq \psi$, as required. The steps of this argument are summarized in the following U-shaped diagram:

$$\begin{array}{lcl} \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \quad \Updownarrow \text{(Cor 5.3.8)} & & \quad \Updownarrow \text{(Cor 5.3.8)} \\ \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \end{array}$$

□

The argument above can be generalized so as to obtain canonicity of inductive inequalities for logics algebraically captured by classes of normal bi-Heyting algebra expansions.

5.5.2 Generalizing the canonicity-via-translation argument

In the present subsection, we discuss the extent to which the proof pattern described in the previous subsection can be applied to the settings of Heyting and co-Heyting algebras, and to normal DLEs. In the case of bi-Heyting algebras, the order embedding e, the existence of which is shown in Proposition 5.3.9, has both a left and a right adjoint. This is a major difference with the cases of Heyting and co-Heyting algebras and normal DLEs, in which at most one of the two adjoints exists in general (cf. Propositions 5.2.7 and 5.3.4), and both adjoints exist if the algebra is perfect.

This implies that the the U-shaped argument discussed in the proof of Theorem 5.5.2, which employed Corollary 5.3.8 on both legs as shown in the diagram below, is not available for Heyting/co-Heyting algebras or DLEs. Indeed, in each of these settings, it can still be applied on the side of the perfect algebras, since any such perfect algebra is also a bi-Heyting algebra, but not on general algebras (left-hand side of the diagram).

$$\begin{array}{lcl} \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \quad \Updownarrow \text{?} & & \quad \Updownarrow \text{(Cor 5.3.8)} \\ \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \end{array}$$

In what follows, we aim at giving a refinement of Corollary 5.3.8 which can replace the question mark in the U-shaped diagram above. We work in the setting of Heyting algebras (similar statements can be obtained straightforwardly for the other settings as well). Recall that the canonical extension $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$ of the embedding $e : \mathbb{A} \hookrightarrow \mathbb{B}$ is a complete homomorphism, and hence both its left and right adjoints exist. Let $c : \mathbb{B}^\delta \to \mathbb{A}^\delta$ denote the left adjoint of $e^\delta : \mathbb{A}^\delta \to \mathbb{B}^\delta$. Then $c(b) \in K(\mathbb{A}^\delta)$ for every $b \in \mathbb{B}$.[8]

Hence, it is immediate to verify that, if $r_\varepsilon : (\mathbb{B}^\delta)^X \to (\mathbb{A}^\delta)^X$ is the map defined for any $U \in (\mathbb{B}^\delta)^X$ and $p \in X$ by:

$$r_\varepsilon(U)(p) = \begin{cases} (\iota^\pi \circ U)(p) & \text{if } \varepsilon(p) = 1 \\ (c \circ U)(p) & \text{if } \varepsilon(p) = \partial \end{cases}$$

then, $(r_\varepsilon(U))(p) \in K(\mathbb{A}^\delta)$ for any 'admissible valuation' $U \in \mathbb{B}^X$ and $p \in X$. Moreover, since the connective $\succ\!\!-$ is not part of the intuitionistic language considered here, and since, as discussed in the proof of Proposition 5.2.7, condition (5.3) lifts from e and ι to e^δ and ι^π, this is enough to show, by induction on the complexity of $\mathcal{L}_I$-formulas, that conditions (a) and (b) of Proposition 5.2.3 hold relative to τ_ε and r_ε defined above.

The following proposition is the required refinement of Corollary 5.3.8 which can replace the question mark in the U-shaped diagram above.

5.5.3. Proposition. *Let $\mathbb{A}$ be a Heyting algebra, and $e : \mathbb{A} \hookrightarrow \mathbb{B}$ be an embedding of $\mathbb{A}$ into a Boolean algebra $\mathbb{B}$ which is a homomorphism of the lattice reducts of $\mathbb{A}$ and $\mathbb{B}$, endowed with its right adjoint $\iota : \mathbb{B} \to \mathbb{A}$ so that condition* (5.3) *holds. Then, for every (Ω, ε)-inductive $\mathcal{L}_I$-inequality $\varphi \leq \psi$,*

$$\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi \quad \textit{iff} \quad \mathbb{B}^\delta \models_{\mathbb{B}} \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi).$$

Proof:
[Sketch of proof] From right to left, if $(\mathbb{A}^\delta, V) \not\models \varphi \leq \psi$ for some $V \in \mathbb{A}^X$, then $[\![\varphi]\!]_V \not\leq [\![\psi]\!]_V$. Since $e\colon \mathbb{A} \hookrightarrow \mathbb{B}$ is an order-embedding, and as discussed above, conditions (a) and (b) of Proposition 5.2.3 hold relative to τ_ε and $r_\varepsilon : (\mathbb{B}^\delta)^X \to (\mathbb{A}^\delta)^X$, this implies that $(\!|\tau_\varepsilon(\varphi)|\!)_{\overline{e}(V)} = e([\![\varphi]\!]_V) \not\leq e([\![\psi]\!]_V) = (\!|\tau_\varepsilon(\psi)|\!)_{\overline{e}(V)}$, that is $(\mathbb{B}^\delta, \overline{e}(V)) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, as required.

Conversely, assume contrapositively that $(\mathbb{B}^\delta, U) \not\models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ for some $U \in \mathbb{B}^X$, that is, $(\!|\tau_\varepsilon(\varphi)|\!)_U \not\leq (\!|\tau_\varepsilon(\psi)|\!)_U$. By applying condition (b) of Proposition 5.2.3, this is equivalent to $e([\![\varphi]\!]_{r_\varepsilon(U)}) \not\leq e([\![\psi]\!]_{r_\varepsilon(U)})$, which, by the monotonicity of e, implies that $[\![\varphi]\!]_{r_\varepsilon(U)} \not\leq [\![\psi]\!]_{r_\varepsilon(U)}$, that is, $(\mathbb{A}, r_\varepsilon(U)) \not\models \varphi \leq \psi$. This is not enough to finish the proof, since $r_\varepsilon(U)$ is not guaranteed to belong in $\mathbb{A}^X$; however, as observed above, $r_\varepsilon(U)(p) \in K(\mathbb{A}^\delta)$ for each proposition variable p. To finish the proof, we need to show

[8] Indeed, e^δ, being a complete homomorphism, is in particular a box-type map, of which its left adjoint c is then the 'black-diamond' (in the notation of [57]), and it is well-known from the theory of canonical extensions of box-type operators that their left adjoints send closed elements to closed elements.

that an admissible valuation $V' \in \mathbb{A}^X$ can be manufactured from $r_\varepsilon(U)$ and $\varphi \leq \psi$ in such a way that $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$. In what follows, we provide a sketch of the proof of the existence of the required V'. Clearly, if $\varepsilon(p) = 1$ for every proposition variable p occurring in $\varphi \leq \psi$, then $r_\varepsilon(U)(p) = \iota^\pi(\llbracket p \rrbracket_U) = \iota(\llbracket p \rrbracket_U) \in \mathbb{A}$, and then any $V' \in \mathbb{A}^X$ which agrees with $r_\varepsilon(U)$ on all variables occurring in $\varphi \leq \psi$ would be enough to finish the proof. Assume that $\varepsilon(q) = \partial$ for some proposition variable q occurring in $\varphi \leq \psi$. Then we define $V'(q) \in \mathbb{A}$ as follows. We run ALBA on $\varphi \leq \psi$ according to the solving order Ω, up to the point when we solve for the negative occurrences of q, which by assumption are ε-critical. Notice that ALBA preserves truth under assignments.[9] Then the inequality providing the minimal valuation of q is of the form $q \leq \alpha$, where α is *pure* (i.e. no proposition variables occur in α). By Lemma 9.5 in [55], every inequality in the antecedent of the quasi-inequality obtained by applying first approximation to an inductive inequality is of the form $\gamma \leq \delta$ with γ syntactically closed and δ syntactically open. Hence, α is pure and syntactically open, which means that the interpretation of α is an element in $O(\mathbb{A}^\delta)$. Therefore, by compactness, there exists some $a \in \mathbb{A}$ such that $r_\varepsilon(U)(q) \leq a \leq \alpha$. Then we define $V'(q) = a$. Finally, it remains to be shown that $(\mathbb{A}^\delta, V') \not\models \varphi \leq \psi$. This immediately follows from the fact that ALBA steps preserves truth under assignments, and that all the inequalities in the system are preserved in the change from $r_\varepsilon(U)$ to V'. □

However, having replaced Corollary 5.3.8 with Proposition 5.5.3 is still not enough for the U-shaped argument above to go through. Indeed, notice that, if $\varphi \leq \psi$ contains some q with $\varepsilon(q) = \partial$, then $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$ contains occurrences of the connective $\Diamond_\geq$, the algebraic interpretation of which in $\mathbb{B}^\delta$ is based on the left adjoint c of e^δ, which, as discussed above, maps elements in $\mathbb{B}$ to elements in $K(\mathbb{B}^\delta)$. Hence, the canonicity of $\tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi)$, understood as the preservation of its validity from $\mathbb{B}$ to $\mathbb{B}^\delta$, cannot be argued by appealing to Proposition 5.5.1: indeed, Proposition 5.5.1 holds under the assumption that $\mathbb{B}$ is an $\mathcal{L}^*$-subalgebra of $\mathbb{B}^\delta$, while, as discussed above, $\mathbb{B}$ is not in general closed under $\Diamond_\geq$.

In order to be able to adapt the canonicity-via-translation argument to the case of Heyting algebras (or co-Heyting algebras, or normal DLEs), we would need to strengthen Proposition 5.5.1 so as to obtain the following equivalence for any inductive $\mathcal{L}^*$-inequality $\alpha \leq \beta$:

$$\mathbb{B}^\delta \models_{\mathbb{B}} \alpha \leq \beta \text{ iff } \mathbb{B}^\delta \models \alpha \leq \beta \tag{5.6}$$

in a setting in which the subalgebra $\mathbb{B}$ is not required to be an $\mathcal{L}^*$-subalgebra of $\mathbb{B}^\delta$, and $f(\overline{b}) \in K(\mathbb{B}^\delta)$ for every f-type connective in $\mathcal{L}^*$ and $\overline{b} \in \mathbb{B}^{n_f}$, and $g(\overline{b}) \in O(\mathbb{B}^\delta)$ for every g-type connective in $\mathcal{L}^*$ and $\overline{b} \in \mathbb{B}^{n_g}$.

[9] In [55] it is proved that ALBA steps preserve validity of quasi-inequalities. In fact, it ensures something stronger, namely that truth under assignments is preserved, modulo the values of introduced and eliminated variables. This notion of equivalence is studied in e.g. [49]. We are therefore justified in our assumption that the value of q is held constant as are the values of all variables occurring in $\varphi \leq \psi$ which have not yet been eliminated up to the point where q is solved for.

Such a strengthening cannot be straightforwardly obtained with the tools provided by the present state-of-the-art in canonicity theory. To see where the problem lies, let us try and apply ALBA/SQEMA in an attempt to prove the left-to-right direction of (5.6) for the 'Sahlqvist' inequality $\Box_{\leq} p \leq \Diamond_{\geq}\Box_{\leq} p$, assuming that $\Diamond_{\geq}$ is left adjoint to $\Box_{\leq}$, and $(\![\Box_{\leq} p]\!)_U \in O(\mathbb{B}^\delta)$ and $(\![\Diamond_{\geq} p]\!)_U \in K(\mathbb{B}^\delta)$ for any admissible valuation $U \in \mathbb{B}^X$:

$$\begin{array}{ll} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p[\Box_{\leq} p \leq \Diamond_{\geq}\Box_{\leq} p] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box_{\leq} p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\Diamond_{\geq}\mathbf{i} \leq p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{array}$$

The minimal valuation term $\Diamond_{\geq}\mathbf{j}$, computed by ALBA/SQEMA when solving for the negative occurrence of p, is closed. However, substituting this minimal valuation into $\Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}$ would get us $\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{j} \leq \mathbf{m}$ with $\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{j}$ neither closed nor open. Hence, we cannot anymore appeal to the Esakia lemma in order to prove the following equivalence:[10]

$$\begin{array}{ll} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall p \forall \mathbf{i} \forall \mathbf{m}[(\Diamond_{\geq}\mathbf{i} \leq p \ \&\ \Diamond_{\geq}\Box_{\leq} p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \mathbb{B}^\delta \models_{\mathbb{B}} \forall \mathbf{i} \forall \mathbf{m}[\Diamond_{\geq}\Box_{\leq}\Diamond_{\geq}\mathbf{i} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{array}$$

An analogous situation arises when solving for the positive occurrence of p. Other techniques for proving canonicity, such as Jónsson-style canonicity [123, 152], display the same problem, since they also rely on an Esakia lemma which is not available if $\mathbb{B}$ is not closed under $\Box_{\leq}$ and $\Diamond_{\geq}$.

5.6 Conclusions

From the results of this chapter, bi-intuitionistic logic stands out as a particularly well behaved setting, and its performance compares favourably to that of the better known intuitionistic logic. Another advantage of bi-intuitionistic logic over intuitionistic logic is that for each additional $\Box$-type connective it possible to define a dual *normal* diamond along the usual Boolean pattern as $\Diamond\varphi := \top \succ\!\!- \Box(\varphi \to \bot)$, and likewise for each additional $\Diamond$-type connective a dual *normal* box as $\Box\varphi := \Diamond(\top \succ\!\!- \varphi) \to \bot$. Following this pattern in the intuitionistic case, using only the intuitionistic negation, gives rise to connectives which are monotone but neither regular nor normal. Together with the fact that bi-intuitionistic logic is sound and complete w.r.t. partial orders, this makes bi-intuitionistic logic a particularly attractive basic framework.

We saw that, where applicable, the translation method has an extraordinary synthesizing power. However, as already mentioned at various points early on, we do not believe that the translation approach can provide autonomous foundations to correspondence theory for non-classical logics, and this for two reasons. First, in Section 5.5

[10] In other words, if $\mathbb{B}$ is not closed under $\Diamond_{\geq}$ or $\Box_{\leq}$, the soundness of the application of the Ackermann rule under admissible assignments cannot be argued anymore by appealing to the Esakia lemma, and hence, to the topological Ackermann lemma.

we saw that, outside of the bi-intuitionistic setting, it is not clear how the canonicity-via-translation argument could be made to work. To make it work, one would likely need techniques which import novel order-theoretic and topological insights which go well beyond the scope of the translation method itself. The existing canonicity techniques therefore remain the most straightforward route toward the result. Second, as argued in Remark 5.4.4, unified correspondence is also needed to provide the right background framework in which correspondence-via-translation can be meaningfully investigated.

The progress we made over [88] (namely the canonicity-via-translation for the bi-intuitionistic setting) comes from embracing the full extent of the algebraic analysis. Specifically, canonicity-via-translation hinges upon the fact that the interplay of persistent and non-persistent valuations on frames can be understood and reformulated in terms of an adjunction situation between two complex algebras of the same frame. In its turn, this adjunction situation generalizes to arbitrary algebras. The same modus operandi, which achieves generalization through algebras via duality, has been fruitfully employed by some of the authors also for very different purposes, such as the definition of the non-classical counterpart of a given logical framework (cf. [136, 144, 47]).

Chapter 6

Unified correspondence and proof theory for strict implication

In the present chapter, which is a revised version of the paper [145][1], we specialize unified correspondence theory to strict implication logics, show that strict implication logics can be conservatively extended to suitable axiomatic extensions of the bounded distributive lattice full non-associative Lambek calculus (BDFNL), transform many strict implication sequents into analytic rules, and develop Gentzen-style cut-free sequent calculi for BDFNL and its analytic rule extensions.

6.1 Introduction

Strict implication is an intensional implication which is semantically interpreted on Kripke binary relational models in the same fashion as intuitionistic implication. Kripke frames for intuitionistic logic are partially ordered sets, and valuations are required to be persistent, i.e., to map propositional variables to upsets. The intuitionistic implication is already an example of strict implication. Subintuitionistic logics, which are prime examples of strict implication logics (cf. [62, 193, 9, 28, 23, 156, 77, 74, 7, 128]), arise semantically by dropping some conditions from the intuitionistic models outlined above, such as the requirement that the accessibility relation to be reflexive or transitive, and the persistency of valuations. For example, Visser's basic propositional logic BPL [193] is a subintuitionistic logic characterized by the class of all transitive frames under the semantics by dropping only the reflexivity condition on frames from the intuitionistic case, and it is embedded into the normal modal logic K4 via the Gödel-McKinsey-Tarski translation. Another example is the least subintuitionistic logic F introduced by Corsi [62] which is characterized by the class of all Kripke frames under

[1] My specific contribution to this research has been the development of the underlying algebraic results, the connection with unified correspondence and the write-up of a preliminary version of the paper.

the semantics by dropping all conditions on frames or models. Naturally, F is embeddable into the least normal modal logic K.

The present chapter proposes a uniform approach to the proof theory of the family of strict implication logics. Cut-free sequent calculi exist in the literature for some members of this family [120], for instance, for Visser's propositional logics [121]. These calculi lack a left- and a right-introduction rule for $\rightarrow$. Instead, there is only one rule in which 2^n premisses are needed when the conclusion has n implication formulas as the antecedent of the sequent. In contrast with this, in the present chapter, we provide modular cut-free calculi for a wide class of strict implication logics, each of which has the standard left- and right-introduction rules. Our methodology uses unified correspondence theory. It takes the move from some general semantic conservativity results which naturally arise from the semantic environment of unified correspondence. Specifically, we use the fact that certain strict implication logics can be conservatively extended to suitable axiomatic extensions of the bounded distributive lattice full non-associative Lambek calculus (BDFNL)[2] , and develop Gentzen-style cut-free sequent calculi for these axiomatic extensions, using the tools of unified correspondence.

From the point of view of unified correspondence, the family of strict implication logics is a very interesting subclass of normal DLE-logics (i.e., logics algebraically identified by varieties of bounded distributive lattice expansions), not only because they are very well-known and very intensely investigated, but also because they are enjoying two different and equally natural relational semantics, namely, the one described above, interpreting the binary implication by means of a *binary* relation [28], and another, arising from the standard treatment of binary modal operators, interpreting the binary implication by means of a *ternary* relation [135]. The existence of these two different semantics makes unified correspondence a very appropriate tool to study the Sahlqvist-type theory of these logics, because of one of the features specific to unified correspondence theory, namely the possibility of developing Sahlqvist-type theory for the logics of strict implication in a modular and simultaneous way for their two types of relational semantics.

In the present chapter we specialize the tools of unified correspondence theory from the general setting of normal DLE-logics to the setting of strict implication logics. The semantic environment of unified correspondence theory allows for a general semantic conservativity result for normal DLE-logics, which has been briefly outlined in [109] and is further clarified in the present chapter (cf. Theorem 6.2.11), and specialized to the setting of strict implication logics.

A second reason for exploring strict implication logics with the tools of unified correspondence is given by the recent developments mentioned above, establishing systematic connections between correspondence results for normal DLE-logics and the characterization of the axiomatic extensions of basic normal DLE-logics which admit display calculi with cut elimination. In particular, in [109], the tool (a) of uni-

[2] Non-associative Lambek calculus was first developed by Lambek [138, 139]. For details about Lambek calculi and substructural logics, we refer to [85, 25, 26, 148].

fied correspondence theory has been used to provide the syntactic characterization of those axioms which correspond to analytic rules, and tool (b) has been used to provide an effective computation of the rules corresponding to each analytic axiom. This work provides an exhaustive answer, relative to the setting of display calculi, to a key question in structural proof theory which has been intensely investigated in various proof-theoretic settings (cf. [150, 35, 37, 105, 36, 142, 137, 146, 141, 134]).

In fact, a major conceptual motivation of the present chapter is provided by the insight that the unified correspondence methodology can be applied to the analyticity issue also in proof-theoretic settings different from display calculi. Following this insight, in the present chapter, we use the tools of unified correspondence in two different ways. Firstly, we present a modified version of the algorithm ALBA which is specific to the task of the direct computation of analytic rules of a Gentzen-style calculus for certain logics of strict implication. Secondly, we use this algorithm as a calculus not only to compute analytic rules, but also to establish semantic (algebraic), hence logical equivalences between axioms of different but related logical signatures. This latter one is a novel application of unified correspondence.

The structure of the chapter is organized as follows. In Section 6.2, we will summarize unified correspondence theory for strict implication logics. Specifically, a general theorem on semantic conservativity will be given, and an ALBA algorithm and a first-order correspondence result will be specialized. In Section 6.3, we will introduce the Ackermann lemma based calculus ALC for calculating correspondence on over algebras between the strict implication language $\mathcal{L}_{\mathrm{SI}}$ and the language $\mathcal{L}_{\bullet}$. More conservativity results will be obtained by using ALC. In Section 6.4, we will develop cut-free Gentzen-style sequent calculus for BDFNL, and then extend it with analytic rules to obtain cut-free sequent calculi.

6.2 Preliminaries

In this section, we will summarize the unified correspondence theory for strict implication logics. For most of the auxiliary definitions, see Chapter 2.

6.2.1 Syntax and semantics for strict implication logics

We will now specialize normal DLE-logics (cf. Chapter 2) to strict implication logics. The *strict implication language* $\mathcal{L}_{\mathrm{SI}}$ is identified with the DLE-language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ where $\mathcal{F} = \varnothing$ and $\mathcal{G} = \{\rightarrow\}$. The order-type of $\rightarrow$ is $(\partial, 1)$.

6.2.1. Definition. The *terms* (*formulas*) of $\mathcal{L}_{\mathrm{SI}}$ are recursively defined as follows:

$$\varphi ::= p \mid \top \mid \bot \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi)$$

where $p \in \mathsf{AtProp}$. Terms (formulas) in $\mathcal{L}_{\mathrm{SI}}$ are denoted by lowercase Greek letters e.g. φ, ψ, γ, or by lower case Latin letters e.g. s, t. An $\mathcal{L}_{\mathrm{SI}}$-sequent is an expression of the form $\varphi \vdash \psi$.

The definition of normal DLE-algebra (cf. Definition 2.1.2) is specialized into the following definition on which the language $\mathcal{L}_{\mathrm{SI}}$ is interpreted.

6.2.2. Definition. An algebra $\mathbb{A} = (A, \wedge, \vee, \bot, \top, \to)$ is called a *bounded distributive lattice with strict implication* (BDI) if its $(\wedge, \vee, \bot, \top)$-reduct is a bounded distributive lattice and $\to$ is a binary operation on A satisfying the following conditions for all $a, b, c \in A$:

(C1) $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$,

(C2) $(a \to c) \wedge (b \to c) = (a \vee b) \to c$,

(C3) $a \to \top = \top = \bot \to a$.

Let $\mathbb{BDI}$[3] be the class of all BDIs. Henceforth, we also write a BDI as $(A, \to)$ where A is supposed to be a bounded distributive lattice.

In BDIs, the turnstile $\vdash$ is interpreted as their order $\leq$. $\varphi \vdash \psi$ is valid in $\mathbb{A}$ if $\mu(\varphi) \leq \mu(\psi)$ for every assignment μ over PROP to $\mathbb{A}$. The notation $\mathbb{BDI} \models \varphi \vdash \psi$ denotes that $\varphi \vdash \psi$ is valid in all BDIs.

6.2.3. Definition. The algebraic sequent system $\mathbf{S}_{\mathbb{BDI}}$ consists of the following axioms and rules:

- Axioms:

$$\begin{array}{l}
(\mathrm{Id})\ \varphi \vdash \varphi, \quad (\mathrm{D})\ \varphi \wedge (\psi \vee \gamma) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \gamma),\\
(\top)\ \varphi \vdash \top, \quad (\bot)\ \bot \vdash \varphi, \quad (\mathrm{N}_\top)\ \top \vdash \varphi \to \top, \quad (\mathrm{N}_\bot)\ \top \vdash \bot \to \varphi,\\
(\mathrm{M}_1)\ (\varphi \to \psi) \wedge (\varphi \to \gamma) \vdash \varphi \to (\psi \wedge \gamma),\\
(\mathrm{M}_2)\ (\varphi \to \gamma) \wedge (\psi \to \gamma) \vdash (\varphi \vee \psi) \to \gamma,
\end{array}$$

- Rules:

$$(\mathrm{M}_3)\ \frac{\varphi \vdash \psi}{\chi \to \varphi \vdash \chi \to \psi}, \quad (\mathrm{M}_4)\ \frac{\varphi \vdash \psi}{\psi \to \chi \vdash \varphi \to \chi},$$

$$(\wedge\mathrm{L})\ \frac{\varphi_i \vdash \psi}{\varphi_1 \wedge \varphi_2 \vdash \psi}(i = 1, 2), \quad (\wedge\mathrm{R})\ \frac{\gamma \vdash \varphi \quad \gamma \vdash \psi}{\gamma \vdash \varphi \wedge \psi},$$

$$(\vee\mathrm{L})\ \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \gamma}, \quad (\vee\mathrm{R})\ \frac{\psi \vdash \varphi_i}{\psi \vdash \varphi_1 \vee \varphi_2}(i = 1, 2),$$

$$(\mathrm{cut})\ \frac{\varphi \vdash \psi \quad \psi \vdash \gamma}{\varphi \vdash \gamma},$$

[3]Here we abuse notation to use mathbb font to denote both a single algebra/frame/model and a class of algebras, which is clear from context.

It is easy to see that $\mathbf{S}_{\mathbb{BDI}}$ is a specialization of $\mathbf{L}_{\mathbb{DLE}}$ (cf. Definition 2.1.4), and that by standard algebraic completeness, $\varphi \vdash \psi$ is provable in $\mathbf{S}_{\mathbb{BDI}}$ iff $\mathbb{BDI} \models \varphi \vdash \psi$. Some extensions of $\mathbf{S}_{\mathbb{BDI}}$, strict implication logics extending it, can be obtained by adding 'characteristic' sequents. Table 6.1 list some characteristic sequents that are considered in literature.[4] For any sequent system $\mathbf{S}$ and a set of sequents Σ, the notation $\mathbf{S} + \Sigma$

Table 6.1: Some Characteristic Sequents

Name	Sequent	Literature
(I)	$q \vdash p \to p$	[28]
(Tr)	$(p \to q) \wedge (q \to r) \vdash p \to r$	[28][157, p.44]
(MP)	$p \wedge (p \to q) \vdash q$	[28, 120]
(W)	$p \vdash q \to p$	[28][157, p.34]
(RT)	$p \to q \vdash r \to (p \to q)$	[28, 120]
(B)	$p \to q \vdash (r \to p) \to (r \to q)$	[157, p.32]
(B′)	$p \to q \vdash (q \to r) \to (p \to r)$	[157, p.32]
(C)	$p \to (q \to r) \vdash q \to (p \to r)$	[157, p.32]
(Fr)	$p \to (q \to p) \vdash (p \to q) \to (p \to r)$	[157, p.44]
(W′)	$p \to (p \to q) \vdash p \to q$	[157, p.44]
(Sym)	$p \vdash ((p \to q) \to r) \vee q$	[120]
(Euc)	$\top \vdash ((p \to q) \to r) \vee (p \to q)$	[120]
(D)	$\top \to \bot \vdash \bot$	[120]

stands for the system obtained from $\mathbf{S}$ by adding all instances of sequents in Σ as new axioms. Strict implication logics in Table 6.2 can be obtained using these characteristic sequents. Some of them are considered in literature.[5]

Each sequent $\varphi \vdash \psi$ defines a class of BDIs. Each strict implication logic $\mathbf{S}_{\mathbb{BDI}} + \Sigma$ defines a class of BDIs denoted by $\mathsf{Alg}(\Sigma)$. For example, some subvarieties are considered in [29]. A BDI $(A, \to)$ is called a *weak Heyting algebra* (WH-algebra) if the following conditions are satisfied for all $a, b, c \in A$:

(C4) $b \leq a \to a$.

(C5) $(a \to b) \wedge (b \to c) \leq (a \to c)$.

Let $\mathbb{WH}$ be the class of all WH-algebras. A wKT_σ-algebra is a WH-algebra $(A, \to)$ satisfying the condition $a \wedge (a \to b) \leq b$ for all $a, b \in A$. A *basic algebra* is a WH-algebra $(A, \to)$ satisfying the condition $a \leq b \to a$ for all $a, b \in A$. Let $\mathbb{T}$ and $\mathbb{BCA}$

[4] These characteristic sequents may have different names or forms in literature. For example, (MP) is written as $p, p \to q \vdash q$ where the comma means conjunction. The sequent (Fr) is named by the Frege axiom $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

[5] These logics are presented in various ways in literature as Hilbert-style systems, natural deduction systems or sequent systems. The name GK^I [120] stands for the Gentzen-style sequent calculus for the minimal strict implication logic under binary relational semantics which can be embedded into the minimal normal modal logic K.

Table 6.2: Some Strict Implication Logics

Name	System	Literature
$\mathbf{S}_{\mathbb{WH}}$, GK^I	$\mathbf{S}_{\mathbb{BDI}}$ + (I) + (Tr)	[28, 120, 62, 68, 197]
$\mathbf{S}_{\mathbb{T}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (MP)	
$\mathbf{S}_{\mathbb{W}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (W)	
$\mathbf{S}_{\mathbb{RT}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (RT)	
$\mathbf{S}_{\mathbb{B}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (B)	
$\mathbf{S}_{\mathbb{B}'}$	$\mathbf{S}_{\mathbb{BDI}}$ + (B′)	
$\mathbf{S}_{\mathbb{C}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (C)	
$\mathbf{S}_{\mathbb{FR}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (Fr)	
$\mathbf{S}_{\mathbb{W}'}$	$\mathbf{S}_{\mathbb{BDI}}$ + (W′)	
$\mathbf{S}_{\mathbb{SYM}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (Sym)	
$\mathbf{S}_{\mathbb{EUC}}$	$\mathbf{S}_{\mathbb{BDI}}$ + (Euc)	
$\mathbf{S}_{\mathbb{BCA}}$	$\mathbf{S}_{\mathbb{T}}$ + (W)	[171, 28, 120, 193, 8, 9, 121]
GKT^I	GK^I + (MP)	[62, 120]
$\mathsf{GK4}^I$	GK^I + (RT)	[62, 120]
$\mathsf{GS4}^I$	GKT^I + (RT)	[62, 120]
GKB^I	GK^I + (Sym)	[62, 120]
$\mathsf{GK5}^I$	GK^I + (Euc)	[120]
$\mathsf{GK45}^I$	$\mathsf{GK5}^I$ + (RT)	[120]
$\mathsf{GKS5}^I$	$\mathsf{GK45}^I$ + (W)	[120]
$\mathsf{GK4}^{I+}$	GK^I + (W)	[120]
GKD^I	GK^I + (D)	[62, 120]

be the classes of all wKT_σ-algebras and basic algebras respectively. The variety of Heyting algebras is a subvariety of $\mathbb{BCA}$, i.e., it is the class of all basic algebras $(A, \to)$ satisfying the condition $\top \to a \leq a$ for all $a \in A$ (cf. e.g. [9, 6]).

As a corollary of the soundness and completeness of DLE-logics with respect to their $\mathcal{L}_{\mathrm{DLE}}$-algebras, one gets the following theorem immediately:

6.2.4. Theorem. *For any strict implication logic $\mathbf{S}_{\mathbb{BDI}} + \Sigma$, an $\mathcal{L}_{\mathrm{SI}}$-sequent $\varphi \vdash \psi$ is derivable in $\mathbf{S}_{\mathbb{BDI}} + \Sigma$ if and only if $\mathsf{Alg}(\Sigma) \models \varphi \vdash \psi$.*

6.2.2 The expanded language for strict implication logics

There two ways to specialize the language $\mathcal{L}^*_{\mathbb{DLE}}$ (cf. Section 2.2) and hence the logic $\mathbf{L}_{\mathbb{DLE}}$ to the strict implication language: a full and a partial specialization. The full specialization results a language of bi-intuitionsitic Lambek calculus $\mathcal{L}^*_{\mathrm{SI}}$ which will not be explored in this chapter. The partial specialization is to add the connectives $\{\bullet, \to, \leftarrow\}$ to $\mathcal{L}_{\mathrm{SI}}$ and get the language of full Lambek calculus, as we mentioned in the introduction, denoted by $\mathcal{L}_{\mathrm{LC}}$, where the order-types of $\bullet, \to, \leftarrow$ are $(1, 1)$, $(\partial, 1)$, $(1, \partial)$ respectively. Clearly $\mathcal{L}_{\mathrm{SI}} \subsetneq \mathcal{L}_{\mathrm{LC}} \subsetneq \mathcal{L}^*_{\mathrm{SI}}$. The partial specialization of $\mathcal{L}^*_{\mathrm{DLE}}$-algebras to the language $\mathcal{L}_{\mathrm{LC}}$ is given in the following definition:

6.2.5. Definition. An algebra $\mathbb{A} = (A, \wedge, \vee, \top, \bot, \to, \bullet, \leftarrow)$ is called a *bounded distributive lattice-ordered residuated groupoid* (BDRG), if $(A, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice, and $\bullet, \to, \leftarrow$ are binary operations on A satisfying the following residuation law for all $a, b, c \in A$:

$$\text{(RES) } a \bullet b \leq c \text{ iff } b \leq a \to c \text{ iff } a \leq c \leftarrow b.$$

Let $\mathbb{BDRG}$ be the class of all BDRGs.

6.2.6. Definition. The algebraic sequent calculus BDFNL consists of the following axioms and rules:

- Axioms:

$$\text{(Id) } \varphi \vdash \varphi, \quad (\top)\ \varphi \vdash \top, \quad (\bot)\ \bot \vdash \varphi,$$

$$\text{(D) } \varphi \wedge (\psi \vee \gamma) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \gamma),$$

- Rules:

$$(\wedge\text{L})\ \frac{\varphi_i \vdash \psi}{\varphi_1 \wedge \varphi_2 \vdash \psi}\ (i = 1, 2), \quad (\wedge\text{R})\ \frac{\gamma \vdash \varphi \quad \gamma \vdash \psi}{\gamma \vdash \varphi \wedge \psi},$$

$$(\vee\text{L})\ \frac{\varphi \vdash \gamma \quad \psi \vdash \gamma}{\varphi \vee \psi \vdash \gamma}, \quad (\vee\text{R})\ \frac{\psi \vdash \varphi_i}{\psi \vdash \varphi_1 \vee \varphi_2}\ (i = 1, 2),$$

$$(\text{Res1})\ \frac{\varphi \bullet \psi \vdash \gamma}{\psi \vdash \varphi \to \gamma}, \quad (\text{Res2})\ \frac{\psi \vdash \varphi \to \gamma}{\varphi \bullet \psi \vdash \gamma},$$

$$\text{(Res3)}\ \frac{\varphi \bullet \psi \vdash \gamma}{\varphi \vdash \gamma \leftarrow \psi}, \quad \text{(Res4)}\ \frac{\varphi \vdash \gamma \leftarrow \psi}{\varphi \bullet \psi \vdash \gamma},$$

$$\text{(cut)}\ \frac{\varphi \vdash \psi \quad \psi \vdash \gamma}{\varphi \vdash \gamma}.$$

6.2.7. FACT. The following monotonicity rules are derivable in BDFNL:

$$(1)\frac{\varphi \vdash \psi}{\varphi \bullet \chi \vdash \psi \bullet \chi}, \quad (2)\frac{\varphi \vdash \psi}{\chi \bullet \varphi \vdash \chi \bullet \psi},$$

$$(3)\frac{\varphi \vdash \psi}{\chi \to \varphi \vdash \chi \to \psi}, \quad (4)\frac{\varphi \vdash \psi}{\psi \to \chi \vdash \varphi \to \chi}.$$

Proof:
Here we derive only (1) and (3). The remaining rules are derived similarly.

$$\dfrac{\dfrac{\varphi \vdash \psi \qquad \dfrac{\psi \bullet \chi \vdash \psi \bullet \chi}{\psi \vdash (\psi \bullet \chi) \leftarrow \chi}\,\text{(Res3)}}{\varphi \vdash (\psi \bullet \chi) \leftarrow \chi}\,\text{(cut)}}{\varphi \bullet \chi \vdash \psi \bullet \chi}\,\text{(Res4)} \qquad \dfrac{\dfrac{\dfrac{\chi \to \varphi \vdash \chi \to \varphi}{\chi \bullet (\chi \to \varphi) \vdash \varphi}\,\text{(Res2)} \qquad \varphi \vdash \psi}{\chi \bullet (\chi \to \varphi) \vdash \psi}\,\text{(cut)}}{\chi \to \varphi \vdash \chi \to \psi}\,\text{(Res1)}$$

This completes the proof.

□

The interpretation of $\mathcal{L}_{\mathrm{LC}}$-sequents in BDRGs is standard, i.e., $\vdash$ is interpreted as the lattice order $\leq$. By $\mathbb{BDRG} \models \varphi \vdash \psi$ we mean that $\varphi \vdash \psi$ is valid in all BDRGs. An $\mathcal{L}_{\mathrm{LC}}$-*supersequent* is an expression of the form $\Phi \Rightarrow \chi \vdash \delta$ where Φ is a set of $\mathcal{L}_{\mathrm{LC}}$-sequents. We say that $\Phi \Rightarrow \chi \vdash \delta$ is *derivable* in BDFNL if there exists a derivation of $\chi \vdash \delta$ from assumptions in Φ. We say that $\Phi \Rightarrow \chi \vdash \delta$ is *valid* in a BDRG $\mathbb{A}$ if $\mathbb{A} \models \Phi$ implies $\mathbb{A} \models \chi \vdash \psi$. We use $\mathbb{BDRG} \models \Phi \Rightarrow \chi \vdash \delta$ to denote that $\chi \vdash \delta$ is valid in all BDRGs. By the Lindenbaum-Tarski construction, one gets the following result (cf. [25]):

6.2.8. THEOREM (STRONG COMPLETENESS). *For every $\mathcal{L}_{\mathrm{LC}}$-supersequent $\Phi \Rightarrow \chi \vdash \delta$, $\Phi \Rightarrow \chi \vdash \delta$ is derivable in* BDFNL *if and only if $\mathbb{BDRG} \models \Phi \Rightarrow \chi \vdash \delta$.*

6.2.3 Semantic conservativity via canonical extension

In this subsection, we will present general results on the semantic conservativity of $\mathcal{L}^*_{\mathrm{DLE}}$-logics over $\mathcal{L}_{\mathrm{DLE}}$-logics. The proofs of the conservativity is by canonical extensions of DLEs, which has been briefly outlined in [109]. As a special case, the Lambek calculus BDFNL is a conservative extension of the strict implication logic $\mathbf{S}_{\mathbb{BDI}}$. In this subsection we will make use of definitions from Chapter 2.

6.2.9. LEMMA. *For every $\mathcal{L}^*_{\mathrm{DLE}}$-algebra $(H, \wedge, \vee, \mathcal{F}^*, \mathcal{G}^*)$, its $(\wedge, \vee, \top, \bot, \mathcal{F}, \mathcal{G})$-reduct is a normal DLE.*

Proof:
Straightforward consequence of the fact that left adjoints (resp. right adjoints) preserve existing joins (resp. meets). See [64, Proposition 7.31]. □

How can an $\mathcal{L}_{\mathrm{DLE}}$-algebra be extended to an $\mathcal{L}^*_{\mathrm{DLE}}$-algebra? This can be done in the canonical extension $\mathbb{A}^\delta = (A^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ of $\mathbb{A}$. The canonical extension $\mathbb{A}^\delta$ of the bounded distributive lattice $\mathbb{A}$ is a perfect lattice which allows for defining adjoints. For each $f \in F^{\mathbb{A}}$ and $1 \leq i \leq n_f$, define

$$f_i^\sharp[u_i] = \begin{cases} \bigvee\{w \in \mathbb{A}^\delta \mid f_i[w] \leq u_i\}, & \text{if } \varepsilon_f(i) = 1. \\ \bigwedge\{w \in \mathbb{A}^\delta \mid f_i[w] \leq u_i\}, & \text{if } \varepsilon_f(i) = \partial. \end{cases}$$

For each $g \in G^{\mathbb{A}}$ and $1 \leq g \leq n_g$, define

$$g_j^\flat[u_j] = \begin{cases} \bigwedge\{w \in \mathbb{A}^\delta \mid u_j \leq g_j[w]\}, & \text{if } \varepsilon_g(j) = 1. \\ \bigvee\{w \in \mathbb{A}^\delta \mid u_j \leq g_j[w]\}, & \text{if } \varepsilon_g(j) = \partial. \end{cases}$$

Let $\mathcal{F}^{\mathbb{A}^{\delta *}}$ and $\mathcal{G}^{\mathbb{A}^{\delta *}}$ be extensions of $\mathcal{F}^{\mathbb{A}^\delta}$ and $\mathcal{G}^{\mathbb{A}^\delta}$ by adding all operators defined in the above way.

6.2.10. Lemma. *The algebra $\mathbb{A}^{\delta E} = (A^\delta, \mathcal{F}^{\mathbb{A}^{\delta *}}, \mathcal{G}^{\mathbb{A}^{\delta *}})$ is a perfect $\mathcal{L}^*_{\mathrm{DLE}}$-algebra.*

Proof:
It suffices to show the residuation laws. We prove only the case for $f \in \mathcal{F}$ and $\varepsilon_f(i) = 1$. The remaining cases are similar. By definition, our goal is to show

$$f_i[u_i] \leq w \text{ iff } u_i \leq \bigvee\{v \in \mathbb{A}^\delta \mid f_i[v] \leq w\}.$$

The 'only if' part is obvious. For the 'if' part, assume $u_i \leq \bigvee\{v \in \mathbb{A}^\delta \mid f_i[v] \leq w\}$. Then $f_i[u_i] \leq f_i[\bigvee\{v \in \mathbb{A}^\delta \mid f_i[v] \leq w\}]$. By distributivity, one gets $f_i[u_i] \leq \bigvee\{f_i[v] \mid f_i[v] \leq w\} \leq w$. □

6.2.11. Theorem. *The logic $\mathbf{L}^*_{\mathbb{DLE}}$ is a conservative extension of $\mathbf{L}_{\mathbb{DLE}}$, i.e., for every $\mathcal{L}_{\mathrm{DLE}}$-sequent $\varphi \vdash \psi$, $\varphi \vdash \psi$ is derivable in $\mathbf{L}_{\mathbb{DLE}}$ if and only if $\varphi \vdash \psi$ is derivable in $\mathbf{L}^*_{\mathbb{DLE}}$.*

Proof:
Assume that $\varphi \vdash \psi$ is derivable in $\mathbf{L}_{\mathbb{DLE}}$. By the completeness of $\mathbf{L}_{\mathbb{DLE}}$, $\varphi \vdash \psi$ is valid in all DLEs. By Lemma 6.2.9, $\varphi \vdash \psi$ is also valid in all $\mathcal{L}^*_{\mathrm{DLE}}$-algebras. Hence by the completeness of $\mathbf{L}^*_{\mathbb{DLE}}$, $\varphi \vdash \psi$ is derivable in it. Conversely, assume that the $\mathcal{L}_{\mathrm{DLE}}$-sequent $\varphi \vdash \psi$ is not derivable in $\mathbf{L}_{\mathbb{DLE}}$. Then by the completeness of $\mathbf{L}_{\mathbb{DLE}}$ with respect to. the class of DLEs, there exists a DLE $\mathbb{A}$ and a variable assignment under which $\varphi^{\mathbb{A}} \nleq \psi^{\mathbb{A}}$, where $\varphi^{\mathbb{A}}$ and $\psi^{\mathbb{A}}$ are values of φ and ψ in $\mathbb{A}$ under that assignment

respectively. Consider the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$. Since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$, the sequent $\varphi \vdash \psi$ is not satisfied in $\mathbb{A}^\delta$ under the variable assignment $\iota \circ v$ (ι denoting the canonical embedding $\mathbb{A} \hookrightarrow \mathbb{A}^\delta$). By Lemma 6.2.10, one gets an $\mathcal{L}^*_{\mathrm{DLE}}$-algebra $\mathbb{A}^{\delta^E}$ which refutes $\varphi \vdash \psi$. By the completeness of $\mathbf{L}^*_{\mathbb{DLE}}$, $\varphi \vdash \psi$ is not derivable in $\mathbf{L}^*_{\mathbb{DLE}}$. □

The minimal logics $\mathbf{L}^*_{\mathbb{DLE}}$ is in the full language $\mathcal{L}^*_{\mathrm{DLE}}$ with all adjoints. If the language $\mathcal{L}_{\mathrm{DLE}}$ is expanded partially, i.e., with a portion of adjoint pairs, one can also obtain more general semantic conservativity results the proofs of which are the same as the proof of Theorem 6.2.11. Consider the language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F},\mathcal{G})$. Let $\mathcal{X} \subseteq \mathcal{F}$ and $\mathcal{Y} \subseteq \mathcal{G}$. Define $\mathcal{X}^\sharp$ as the extension of $\mathcal{X}$ with right adjoints, and $\mathcal{Y}^\flat$ as the extension of $\mathcal{Y}$ with left adjoints.

6.2.12. Theorem. *Let $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F},\mathcal{G})$ be a DLE-language, $\mathcal{X} \subseteq \mathcal{F}$ and $\mathcal{Y} \subseteq \mathcal{G}$. The minimal logic $\mathbf{L}^*_{\mathbb{DLE}}(\mathcal{F}^*,\mathcal{G}^*)$ is a conservative extension of the minimal logic $\mathbf{L}_{\mathbb{DLE}}(\mathcal{F},\mathcal{G},\mathcal{X}^\sharp,\mathcal{Y}^\flat)$ which is also a conservative extension of the logic $\mathbf{L}_{\mathbb{DLE}}(\mathcal{F},\mathcal{G})$.*

Let us consider the specialization of Theorem 6.2.12 to the strict implication logic $\mathbf{S}_{\mathbb{BDI}}$ and the Lambek calculus BDFNL. First, as a corollary of Lemma 6.2.9, the $(\wedge,\vee,\bot,\top,\to)$-reduct of a BDRG is a BDI. Second, the canonical extension of a BDI $(A,\to)$ is the π-extension $(A^\delta,\to^\pi)$ which is also a BDI (cf. [87, 86]), and we can define binary operators $\bullet$ and $\leftarrow$ on $\mathbb{A}^\delta$ by setting $u \bullet v = \bigwedge\{w \in \mathbb{A}^\delta \mid v \leq u \to^\pi w\}$ and $u \leftarrow v = \bigvee\{w \in \mathbb{A}^\delta \mid w \bullet v \leq u\}$. As a corollary of Lemma 6.2.10, one gets the residuation law: for all $u,v,w \in \mathbb{A}^\delta$, $u \bullet v \leq w$ iff $v \leq u \to^\pi w$. Then one can apply Theorem 6.2.12 immediately to get the following corollary:

6.2.13. Corollary. BDFNL *is a conservative extension of* $\mathbf{S}_{\mathbb{BDI}}$.

6.2.4 Inductive inequalities of strict implication logics

In this subsection, we will specialize the definition of *inductive* $\mathcal{L}_{\mathrm{DLE}}$-inequalities (cf. [109] and Section 2.5) to the language of strict implication logic. Some terminologies in this subsection are defined in Section 2.5.

6.2.14. Definition. (cf. Definition 2.5.2) In any signed generation tree, nodes will be respectively called *syntactically right adjoint (SRA)*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)* and Δ-*adjoints*, according to Table 6.3.

6.2.15. Definition. [Inductive inequalities] (cf. Definition 2.5.3) For any dependency order $<_\Omega$ (cf. Definition 2.5.3) on variables $p_1,\dots p_n$ and any order-type $\varepsilon \in \{1,\partial\}^n$ (cf. page 7), the signed generation tree $*\varphi(p_1,\dots p_n)$ (where $* \in \{-,+\}$) is (Ω,ε)-*inductive* if

(1) each ε-critical branch (cf. page 15) ending with leaf node p_i is good (cf. Definition 2.5.2) for each $1 \leq i \leq n$;

(2) in each ε-critical branch ending with leaf node p_i, every binary SRR-node occurring is of the form $h(\gamma, \beta)$ or $h(\beta, \gamma)$, where $h \in \{\wedge, \vee, \rightarrow\}$, and:

(a) $\varepsilon^\partial(\gamma) \prec *\varphi$, and

(b) $p_k <_\Omega p_i$ for every p_k occurring in γ and for every $1 \leq k \leq n$.

The inequality $\varphi \leq \psi$ is called (Ω, ε)-*inductive* if $+\varphi$ and $-\psi$ are both (Ω, ε)-inductive. The inequality $\varphi \leq \psi$ is called *inductive* if it is (Ω, ε)-inductive for some dependency order Ω and order-type ε.

Table 6.3: Skeleton nodes and PIA nodes for $\mathcal{L}_{\mathrm{SI}}$.

Skeleton	PIA
Δ-adjoints	SRA
$+$ $\vee$ $\wedge$	$+$ $\wedge$
$-$ $\wedge$ $\vee$	$-$ $\vee$
SLR	SRR
$+$ $\wedge$	$+$ $\vee$ $\rightarrow$
$-$ $\vee$ $\rightarrow$	$-$ $\wedge$

6.2.16. Example. Every sequent $\varphi \vdash \psi$ can be presented as an inequality when $\vdash$ is replaced with $\leq$ due to the algebraic interpretation of $\vdash$. The inequalities obtained from Table 6.1 are inductive. For instance, (Fr) is inductive for $\varepsilon_p = \varepsilon_p = \varepsilon_r = 1$ and $p <_\Omega q <_\Omega r$. Henceforth we do not distinguish "sequent" and "inequality" if no confusion will arise.

6.2.5 The algorithm ALBA for strict implication logics

In what follows we will specify the algorithm ALBA introduced in Section 2.6 to the language $\mathcal{L}_{\mathrm{SI}}$ of strict implication logics. The language $\mathcal{L}^+_{\mathrm{LC}}$ of the algorithm is defined as follows:

$$\varphi ::= p \mid \top \mid \bot \mid \mathbf{i} \mid \mathbf{m} \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \bullet \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftarrow \varphi)$$

where $p \in \mathsf{AtProp}$, $\mathbf{i} \in \mathsf{NOM}$ is called *nominal*, $\mathbf{m} \in \mathsf{CONOM}$ is called *conomimal*. This language is interpreted in perfect BDRGs A, where nominals (resp. conominals) are interpreted as completely join-irreducibles (resp. completely meet-irreducibles) of A.

An $\mathcal{L}^+_{\mathrm{LC}}$-inequality is $\varphi \leq \psi$ such that φ and ψ are $\mathcal{L}^+_{\mathrm{LC}}$-formulas. An $\mathcal{L}^+_{\mathrm{LC}}$-quasi-inequality is $\varphi_1 \leq \psi_1 \ \& \ \ldots \ \& \ \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ where each $\varphi_i \leq \psi_i$ for $1 \leq i \leq n$ and $\varphi \leq \psi$ are $\mathcal{L}^+_{\mathrm{LC}}$-inequalities.

The algorithm ALBA is specialized to the language $\mathcal{L}_{\mathrm{SI}}$ from the general version in [55, 109]. The algorithm transforms the input $\mathcal{L}_{\mathrm{SI}}$-inequalities into equivalent $\mathcal{L}^+_{\mathrm{LC}}$

quasi-inequalities with nominals and conominals only, where propositional variables are eliminated by the Ackermann rules. The proof of the soundness of ALBA rules in the language $\mathcal{L}_{\mathrm{SI}}$ is similar to [55, 48] and hence omitted. ALBA receives the input inequality $\varphi \leq \psi$ and runs in three stages:

First stage: preprocessing and first approximation stage. ALBA preprocesses $\varphi \leq \psi$ by applying the following rules exhaustively in $+\varphi$ and $-\psi$:

(1) (a) Push down $+\wedge$ towards variables by distributing over children node labelled with $+\vee$ which are Skeleton nodes;

(b) Push down $-\vee$ towards variables by distributing over children node labelled with $-\wedge$ which are Skeleton nodes;

(c) Push down $- \rightarrow$ towards variables by distributing over its second (resp. first) child node labelled with $-\wedge$ (resp. $+\vee$) which are Skeleton nodes.

(2) Splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

(3) Monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

where $\beta(p)$ is positive in p and $\alpha(p)$ is negative in p.

Let $\mathsf{Preprocess}(\varphi \leq \psi) := \{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ be the set of inequalities obtained by applying the above rules exhaustively. Then the following rule (which is called the *first approximation rule*) is applied to each $\varphi_i \leq \psi_i$ in $\mathsf{Preprocess}(\varphi \leq \psi)$:

$$\frac{\varphi \leq \psi}{\mathbf{i}_0 \leq \varphi \quad \psi \leq \mathbf{m}_0}$$

where $\mathbf{i}_0$ is a nominal and $\mathbf{m}_0$ is a conominal. After the first approximation rule, for each inequality $\varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)$, the algorithm produces a system of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$.

Second stage: reduction and elimination stage. The present stage aims at eliminating all propositional variables from each system obtained in the previous stage. The variables are eliminated by the so called Ackermann rules, and the other rules in this stage are applied in order to reach the shape to apply the Ackermann rule.

Splitting rules.

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

Residuation rule.

$$\frac{\psi \leq \varphi \rightarrow \gamma}{\varphi \bullet \psi \leq \gamma}$$

Approximation rules.

$$\frac{\varphi \to \psi \leq \mathbf{m}}{\mathbf{i} \leq \varphi \quad \mathbf{i} \to \psi \leq \mathbf{m}} \qquad \frac{\varphi \to \psi \leq \mathbf{m}}{\psi \leq \mathbf{n} \quad \varphi \to \mathbf{n} \leq \mathbf{m}}$$

$$\frac{\mathbf{i} \leq \varphi \bullet \psi}{\mathbf{j} \leq \varphi \quad \mathbf{i} \leq \mathbf{j} \bullet \psi} \qquad \frac{\mathbf{i} \leq \varphi \bullet \psi}{\mathbf{j} \leq \psi \quad \mathbf{i} \leq \varphi \bullet \mathbf{j}}$$

where the variables $\mathbf{i}, \mathbf{j}$ (resp. $\mathbf{m}, \mathbf{n}$) are nominals (resp. conominals). The nominal $\mathbf{j}$ and conominal $\mathbf{n}$ must be *fresh* (cf. page 19).

Ackermann rules. These two rules aims at eliminating propositional variables, which operates on the whole system rather than a single inequality.

$$\frac{\&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \& \&\{\alpha_i \leq p \mid 1 \leq i \leq n\} \;\Rightarrow\; \mathbf{i}_0 \leq \mathbf{m}_0}{\&\{\beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i}_0 \leq \mathbf{m}_0} \text{ (RAR)}$$

where $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p, $\beta_1(p), \ldots, \beta_m(p)$ are positive in p and p does not occur in $\alpha_1, \ldots, \alpha_n$.

$$\frac{\&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \& \&\{p \leq \alpha_i \mid 1 \leq i \leq n\} \;\Rightarrow\; \mathbf{i}_0 \leq \mathbf{m}_0}{\&\{\beta_j(\bigwedge_{i=1}^n \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i}_0 \leq \mathbf{m}_0} \text{ (LAR)}$$

where $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p, $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and p does not occur in $\alpha_1, \ldots, \alpha_n$.

Third stage: output stage. If for some systems, some variables cannot be eliminated, then ALBA halts and reports failure. Otherwise, every system $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ has been reduced to a system $\mathsf{Reduce}(\varphi_i \leq \psi_i)$ with no propositional variables. Let $\mathsf{ALBA}(\varphi \leq \psi) := \{\&[\mathsf{Reduce}(\varphi_i \leq \psi_i)] \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0 \mid \varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)\}$, which contains no propositional variables. ALBA outputs $\mathsf{ALBA}(\varphi \leq \psi)$ and terminates. The following theorem is a generalization of [55, Theorem 10.11], and hence the proof of it is omitted.

6.2.17. Theorem. *For the language $\mathcal{L}_{\mathrm{SI}}$, its corresponding version of* ALBA *succeeds on all inductive $\mathcal{L}_{\mathrm{SI}}$-inequalities, which are hence canonical, and their corresponding logics are complete with respect to the elementary classes of relational structures defined by their first-order correspondents.*

6.2.18. Example. The running of ALBA on the inductive $\mathcal{L}_{\mathrm{SI}}$-sequents (inequalities) in

Table 6.1 will produce pure inequalities as below:

Sequent	Output
(I)	$\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{j})$
(Tr)	$\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq (\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i})$
(MP)	$\forall \mathbf{i}(\mathbf{i} \leq \mathbf{i} \bullet \mathbf{i})$
(W)	$\forall \mathbf{ij}(\mathbf{i} \bullet \mathbf{j} \leq \mathbf{j})$
(RT)	$\forall \mathbf{ijk}(\mathbf{i} \bullet (\mathbf{j} \bullet \mathbf{k}) \leq \mathbf{i} \bullet \mathbf{k})$
(B)	$\forall \mathbf{ijk}(\mathbf{i} \bullet (\mathbf{j} \bullet \mathbf{k}) \leq (\mathbf{i} \bullet \mathbf{j}) \bullet \mathbf{k})$
(B′)	$\forall \mathbf{ijk}(\mathbf{i} \bullet (\mathbf{j} \bullet \mathbf{k}) \leq (\mathbf{i} \bullet \mathbf{k}) \bullet \mathbf{j})$
(C)	$\forall \mathbf{ijk}(\mathbf{i} \bullet (\mathbf{j} \bullet \mathbf{k}) \leq \mathbf{j} \bullet (\mathbf{i} \bullet \mathbf{k}))$
(Fr)	$\forall \mathbf{ijk}(\mathbf{i} \bullet (\mathbf{j} \bullet \mathbf{k}) \leq (\mathbf{i} \bullet \mathbf{j}) \bullet (\mathbf{i} \bullet \mathbf{k}))$
(W′)	$\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{j} \bullet (\mathbf{j} \bullet \mathbf{i}))$
(Sym)	$\forall \mathbf{ij} \forall \mathbf{mn}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m}\ \&\ \mathbf{i} \rightarrow \mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m})$
(Euc)	$\forall \mathbf{ij} \forall \mathbf{mn}_0\mathbf{n}_1(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{n}_0\ \&\ \mathbf{i} \rightarrow \mathbf{n}_1 \leq \mathbf{m}\ \&\ \mathbf{j} \rightarrow \mathbf{n}_0 \leq \mathbf{m} \Rightarrow \top \leq \mathbf{m})$
(D)	$\top \rightarrow \bot \leq \bot$

Here we show only the running of ALBA on $(p \rightarrow q) \wedge (q \rightarrow r) \leq p \rightarrow r$ which proceeds as below:

$$
\begin{array}{ll}
 & (p \rightarrow q) \wedge (q \rightarrow r) \leq p \rightarrow r \text{ (First Approximation)} \\
\Leftrightarrow & \forall \mathbf{i} \forall \mathbf{m}(\mathbf{i} \leq (p \rightarrow q) \wedge (q \rightarrow r)\ \&\ p \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (Spliting)} \\
\Leftrightarrow & \forall \mathbf{i} \forall \mathbf{m}(\mathbf{i} \leq p \rightarrow q\ \&\ \mathbf{i} \leq q \rightarrow r\ \&\ p \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (Residuation)} \\
\Leftrightarrow & \forall \mathbf{i} \forall \mathbf{m}(p \bullet \mathbf{i} \leq q\ \&\ q \bullet \mathbf{i} \leq r\ \&\ p \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (Approximation)} \\
\Leftrightarrow & \forall \mathbf{ij} \forall \mathbf{m}(p \bullet \mathbf{i} \leq q\ \&\ q \bullet \mathbf{i} \leq r\ \&\ \mathbf{j} \leq p\ \&\ \mathbf{j} \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (RAR)} \\
\Leftrightarrow & \forall \mathbf{ij} \forall \mathbf{m}(\mathbf{j} \bullet \mathbf{i} \leq q\ \&\ q \bullet \mathbf{i} \leq r\ \&\ \mathbf{j} \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (RAR)} \\
\Leftrightarrow & \forall \mathbf{ij} \forall \mathbf{m}((\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i} \leq r\ \&\ \mathbf{j} \rightarrow r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}) \text{ (RAR)} \\
\Leftrightarrow & \forall \mathbf{ij} \forall \mathbf{m}(\mathbf{j} \rightarrow ((\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m})
\end{array}
$$

The output pure quasi-inequality is equivalent to $\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq (\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i})$.

The algorithm ALBA does not only work for the distributive setting but also in general work for non-distributive lattice setting [57]. Hence the algorithm ALBA can be specialized to the full Lambek calculus. For the $\{\bullet, \leftarrow, \rightarrow\}$-fragment of full Lambek calculus, Kurtonina [135] presented a set of Sahlqvist formulas from which the first-order correspondents can be calculated by the Sahlqvist-van Benthem quantifier elimination procedure. Kurtonina's definition of Sahlqvist formulas is narrower than inductive inequalities provided by ALBA. For example, The (Fr) inequality is inductive but not Sahlqvist. This remark is also discussed in [57, Example 3.8].

6.2.6 First-order correspondents

Given an inductive $\mathcal{L}_{\mathrm{SI}}$-inequality $\varphi \leq \psi$, the running of ALBA on it will output a pure quasi-inequality, namely, a quasi-inequality in which no propositional variable occurs.

Then the first-order correspondent of $\varphi \leq \psi$ is obtained when the Kripke semantics for $\mathcal{L}^{+}_{\mathrm{LC}}$ is given such that $\mathcal{L}^{+}_{\mathrm{LC}}$-terms are translated into a first-order language. For calculating the first-order correspondents of inductive $\mathcal{L}_{\mathrm{SI}}$-inequalities, there are two kinds of Kripke semantics for the language $\mathcal{L}^{+}_{\mathrm{LC}}$ (i.e., the extension of $\mathcal{L}_{\mathrm{LC}}$ with normals and conominals): *binary* and *ternary* relational semantics.

Binary relational semantics. The binary relational semantics for $\mathcal{L}_{\mathrm{LC}}$ is given in ordinary Kripke structures. A *binary frame* is a pair $\mathbb{F} = (W, R)$ where W is a non-empty set and R is a binary relation on W. A *binary model* is a triple $\mathbb{M} = (W, R, V)$ where (W, R) is a binary frame and $V : \mathsf{Prop} \cup \mathsf{NOM} \cup \mathsf{CONOM} \to \mathcal{P}(W)$ is a valuation such that (i) for each $\mathbf{i} \in \mathsf{NOM}$, $V(\mathbf{i}) = \{w\}$ for some $w \in W$; and (ii) for each $\mathbf{m} \in \mathsf{CONOM}$, $V(\mathbf{m}) = W - \{u\}$ for some $u \in W$. Note that here there are no additional conditions assumed for the binary relation or the valuation. For any $\mathcal{L}_{\mathrm{SI}}$-formula φ, the *satisfiability relation* $\mathbb{M}, w \models \varphi$ under the binary relational semantics is defined inductively as follows:

(1) $\mathbb{M}, w \models p$ iff $w \in V(p)$.

(2) $\mathbb{M}, w \models \mathbf{i}$ iff $V(\mathbf{i}) = \{w\}$.

(3) $\mathbb{M}, w \models \mathbf{m}$ iff $V(\mathbf{m}) = W - \{w\}$.

(4) $\mathbb{M}, w \not\models \bot$.

(5) $\mathbb{M}, w \models \varphi \wedge \psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}, w \models \psi$.

(6) $\mathbb{M}, w \models \varphi \vee \psi$ iff $\mathbb{M}, w \models \varphi$ or $\mathbb{M}, w \models \psi$.

(7) $\mathbb{M}, w \models \varphi \to \psi$ iff $\forall u \in W(wRu \;\&\; \mathbb{M}, u \models \varphi \Rightarrow \mathbb{M}, u \models \psi)$.

(8) $\mathbb{M}, w \models \varphi \leftarrow \psi$ iff $\forall u \in W(uRw \;\&\; \mathbb{M}, u \models \psi \Rightarrow \mathbb{M}, w \models \varphi)$.

(9) $\mathbb{M}, w \models \varphi \bullet \psi$ iff $\exists u \in W(uRw \;\&\; \mathbb{M}, w \models \varphi \;\&\; \mathbb{M}, u \models \psi)$.

Without the semantic clauses for nominals, conominals, $\leftarrow$ and $\bullet$, we get the binary relation semantics for strict implication language [28].[6] The algorithm ALBA provides a general correspondence theory for the issue of the frame definability by sequents raised in [28].

For a binary frame $\mathbb{F} = (W, R)$, the dual algebra of $\mathbb{F}$ is defined as $\mathbb{F}^{+} = (\mathcal{P}(W), \cup, \cap, \varnothing, W, \to^{2}_{R}, \bullet^{2}_{R}, \leftarrow^{2}_{R})$ where $\to^{2}_{R}$, $\leftarrow^{2}_{R}$ and $\bullet^{2}_{R}$ are binary operations defined on $\mathcal{P}(W)$ by setting

(1) $X \to^{2}_{R} Y = \{w \in W \mid R(w) \cap X \subseteq Y\}$;

(2) $X \leftarrow^{2}_{R} Y = \{w \in W \mid \forall u(uRw \;\&\; u \in Y \Rightarrow w \in X)\}$;

[6] In [28], the least weak strict implication logic wK_{σ} is introduced using sequents and shown to be strongly complete with respect to the class of all frames under the binary relational semantics. It is not hard to check that the algebraic sequent system $\mathbf{S}_{\mathbb{WH}}$ is equivalent to wK_{σ}.

(3) $X \bullet_R^2 Y = \{w \in W \mid \exists u(Ruw \,\&\, w \in X \,\&\, u \in Y)\}$;

It is easy to prove that the algebra $\mathbb{F}^+$ is a BDRG. As [55, Theorem 8.1], ALBA is also correct on binary relational frames. Then we can calculate the first-order correspondents of inductive $\mathcal{L}_{\mathrm{SI}}$-sequents under the binary relational semantics.

6.2.19. Example. The outputs of ALBA running on the inductive inequalities in Example 6.2.18 can be transformed into first-order correspondents of the corresponding inductive sequents under the binary relational semantics as below:

Sequent	Binary Relational Correspondent
(I)	$\forall xy(Ryx \supset x = x)$
(Tr)	$\forall xy(Ryx \supset Ryx)$
(MP)	$\forall xRxx$
(W)	$\forall xy(Ryx \supset x = y)$
(RT)	$\forall xyz(Rxy \wedge Ryz \supset Rxz)$
(B)	$\forall xyz(Ryx \wedge Rzy \supset Rzx \wedge Ryx)$
(B′)	$\forall xyz(Ryx \wedge Rzy \supset Ryx \wedge Rzx)$
(C)	$\forall xyz(Ryx \wedge Rzy \supset Rxx \wedge x = y \wedge Rzx)$
(Fr)	$\forall xyz(Ryx \wedge Rzy \supset Rxx \wedge Ryx \wedge Rzx)$
(W′)	$\forall xy(Ryx \supset Rxx)$
(Sym)	$\forall xy(Rxy \supset Ryx)$
(Euc)	$\forall xyz(Rxy \wedge Rxz \supset Ryz)$
(D)	$\forall x\exists yRxy$

Here we calculate only the first-order binary relational correspondents of (Tr) and (Sym).

(1) The output of running ALBA on (Tr) is the pure inequality $\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq (\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i})$. Note that $z \in \{x\} \bullet^2 \{y\}$ if and only if Ryz and $z = x$.

$$
\begin{aligned}
\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq (\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i}) &\Leftrightarrow \forall xy(\{x\} \bullet^2 \{y\} \subseteq (\{x\} \bullet^2 \{y\}) \bullet^2 \{y\}) \\
&\Leftrightarrow \forall xyz(z \in \{x\} \bullet^2 \{y\} \supset z \in (\{x\} \bullet^2 \{y\}) \bullet^2 \{y\}) \\
&\Leftrightarrow \forall xyz(Ryz \wedge z = x \supset \exists u(Ruz \wedge z \in \{x\} \bullet^2 \{y\} \wedge u = y)) \\
&\Leftrightarrow \forall xyz(Ryz \wedge z = x \supset Ryz \wedge Ryz \wedge z = x) \\
&\Leftrightarrow \forall xyz(Ryz \wedge z = x \supset Ryz \wedge z = x)
\end{aligned}
$$

which is a tautology. (Tr) is in fact derivable in $\mathbf{S}_{\mathbb{WH}}$, and the system $\mathbf{S}_{\mathbb{WH}}$ is strongly complete with respect to the class of all binary frames ([28]).

(2) The output of running ALBA on (Sym) is the pure quasi-inequality $\forall \mathbf{ij}\forall \mathbf{mn}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m} \,\&\, \mathbf{i} \to \mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m})$. Let $\mathbf{j}, \mathbf{i}, \mathbf{m}, \mathbf{n}$ be interpreted as $\{x\}, \{y\}, \{u\}^c, \{v\}^c$

respectively where $(.)^c$ is the complement operation. The calculation is as below:

$$
\begin{aligned}
\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m} &\Leftrightarrow \{x\} \bullet^2 \{y\} \subseteq \{u\}^c \\
&\Leftrightarrow \forall z(Ryz \wedge z = x \supset z \neq u) \\
&\Leftrightarrow Ryx \supset x \neq u \\
\mathbf{i} \rightarrow \mathbf{n} \leq \mathbf{m} &\Leftrightarrow \forall w(w \in \{y\} \rightarrow \{v\}^c \supset w \neq u) \\
&\Leftrightarrow \forall w(\forall w_0(Rww_0 \wedge w_0 = y \supset w_0 \neq v) \supset w \neq u) \\
&\Leftrightarrow \forall w((Rwy \supset y \neq v) \supset w \neq u) \\
&\Leftrightarrow \forall w(w = u \supset Rwy \wedge y = v) \\
&\Leftrightarrow Ruy \wedge y = v \\
\forall \mathbf{ij} \forall \mathbf{mn}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m} \;\&\; \mathbf{i} \rightarrow \mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m})& \\
&\Leftrightarrow \forall xyuv((Ryx \supset x \neq u) \wedge Ruy \wedge y = v \supset x \neq u) \\
&\Leftrightarrow \forall xyu((Ryx \supset x \neq u) \wedge Ruy \supset x \neq u) \\
&\Leftrightarrow \forall xyu(x = u \supset (Ruy \supset Ryx \wedge x = u)) \\
&\Leftrightarrow \forall xy(Rxy \supset Ryx)
\end{aligned}
$$

The sequent (Sym) defines the symmetry condition on binary frames.

Ternary relational semantics. The strict implication can be viewed as a binary modal operator added to distributive lattices, and hence there is a ternary relational semantics for it (cf. [21, 70]). A *ternary frame* is a frame $\mathfrak{F} = (W, S)$ where S is a ternary relation on W. A *ternary model* is a ternary frame with a valuation. The satisfiability relation $\mathfrak{M}, w \Vdash \varphi$ for the language $\mathcal{L}_{\mathrm{LC}}$ under the ternary relational semantics is defined as usual. In particular, the semantic clauses for implications and the product are the following (cf. [135]):

(1) $\mathfrak{M}, w \Vdash \varphi \rightarrow \psi$ iff $\forall u, v(Svuw \;\&\; \mathfrak{M}, u \Vdash \varphi \;\Rightarrow\; \mathfrak{M}, v \Vdash \psi)$.

(2) $\mathfrak{M}, w \Vdash \varphi \leftarrow \psi$ iff $\forall u, v(Svwu \;\&\; \mathcal{M}, u \Vdash \psi \;\Rightarrow\; \mathfrak{M}, v \Vdash \varphi)$.

(3) $\mathfrak{M}, w \Vdash \varphi \bullet \psi$ iff $\exists u, v(Swuv \;\&\; \mathfrak{M}, u \Vdash \varphi \;\&\; \mathfrak{M}, v \Vdash \psi)$.

Given a ternary frame $\mathfrak{F} = (W, S)$, the dual of $\mathfrak{F}$ is defined as $\mathfrak{F}^* = (\mathcal{P}(W), \cup, \cap, \varnothing, W, \rightarrow_S^3, \bullet_S^3, \leftarrow_S^3)$ where $\rightarrow_S^3$, $\leftarrow_S^3$ and $\bullet_S^3$ are binary operations defined on $\mathcal{P}(W)$ by

(1) $X \rightarrow_S^3 Y = \{w \in W \mid \forall uv(Svuw \;\&\; u \in X \Rightarrow v \in Y)\}$;

(2) $X \leftarrow_S^3 Y = \{w \in W \mid \forall uv(Svwu \;\&\; u \in Y \Rightarrow v \in X\}$;

(3) $X \bullet_S^3 Y = \{w \in W \mid \exists uv(Swuv \;\&\; u \in X \;\&\; v \in Y)\}$.

It is easy to check that $\mathfrak{F}^*$ is a BDRG. Then under the ternary relational semantics one can calculate the first-order correspondents of inductive sequents.

6.2.20. Example. As Example 6.2.19, we present the first-order correspondents of these inductive sequents under the ternary relational semantics as below:

Sequent	Ternary Relational Correspondent
(I)	$\forall xyz(Szxy \supset z = x)$
(Tr)	$\forall xyz(Szxy \supset \exists u(Szuy \wedge Suxy))$
(MP)	$\forall x Sxxx$
(W)	$\forall xyz(Szyx \supset z = y)$
(RT)	$\forall xyzuv(Suxv \wedge Svyz \supset Suxz)$
(B)	$\forall xyzuw(Suxw \wedge Swyz \supset \exists v(Suvz \wedge Svxy))$
(B′)	$\forall xyzuw(Suxw \wedge Swyz \supset \exists v(Suvy \wedge Svxz))$
(C)	$\forall xyzuw(Suxw \wedge Swyz \supset \exists v(Suyv \wedge Svxz))$
(Fr)	$\forall xyzuw(Suxw \wedge Swyz \supset \exists v_0v_1(Suv_0v_1 \wedge Sv_0xy \wedge Sv_1xz))$
(W′)	$\forall xyz(Suxy \supset \exists u(Szxu \wedge Suxy))$
(Sym)	$\forall xyv(Svyx \supset Sxxy)$
(Euc)	$\forall xyzuv(Suxz \wedge Suyz \supset Svxz)$
(D)	$\forall x\exists yz Szyx$

Here we calculate only the first-order ternary relational correspondents of (Tr) and (Sym). Note that $z \in \{x\} \bullet^3 \{y\}$ if and only if $Szxy$.

$$\begin{aligned}
\forall \mathbf{ij}(\mathbf{j} \bullet \mathbf{i} \leq (\mathbf{j} \bullet \mathbf{i}) \bullet \mathbf{i}) &\Leftrightarrow \forall xy(\{x\} \bullet^3 \{y\} \subseteq (\{x\} \bullet^3 \{y\}) \bullet^3 \{y\}) \\
&\Leftrightarrow \forall xyz(z \in \{x\} \bullet^3 \{y\} \supset z \in (\{x\} \bullet^3 \{y\}) \bullet^3 \{y\}) \\
&\Leftrightarrow \forall xyz(Szxy \supset \exists uv(Szuv \wedge u \in (\{x\} \bullet^3 \{y\}) \wedge v \in \{y\})) \\
&\Leftrightarrow \forall xyz(Szxy \supset \exists uv(Szuv \wedge Suxy \wedge v \in \{y\})) \\
&\Leftrightarrow \forall xyz(Szxy \supset \exists u(Szuy \wedge Suxy)).
\end{aligned}$$

The result is not a tautology. The sequent (Tr) defines a special class of ternary relational frames.

(2) For (Sym), let **j**, **i**, **m**, **n** be interpreted as $\{x\}, \{y\}, \{u\}^c, \{v\}^c$ respectively where $(.)^c$

is the complement operation. The calculation is as below:

$$
\begin{aligned}
\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m} &\Leftrightarrow \{x\} \bullet^3 \{y\} \subseteq \{u\}^c \\
&\Leftrightarrow \forall z(Szxy \supset z \neq u) \\
&\Leftrightarrow \forall z(z = u \supset \sim Szxy) \\
&\Leftrightarrow \ \sim Suxy \\
\mathbf{i} \rightarrow \mathbf{n} \leq \mathbf{m} &\Leftrightarrow \forall w(w \in \{y\} \rightarrow \{v\}^c \supset w \neq u) \\
&\Leftrightarrow \forall w(\forall w_0 w_1(Sw_1 w_0 w \wedge w_0 = y \supset w_1 \neq v) \supset w \neq u) \\
&\Leftrightarrow \forall w(\forall w_1(Sw_1 yw \supset w_1 \neq v) \supset w \neq u) \\
&\Leftrightarrow \forall w(\forall w_1(w_1 = v \supset \sim Sw_1 yw) \supset w \neq u) \\
&\Leftrightarrow \forall w(\sim Svyw \supset w \neq u) \\
&\Leftrightarrow \forall w(w = u \supset Svyw) \\
&\Leftrightarrow Svyu \\
\forall \mathbf{ij} \forall \mathbf{mn}(\mathbf{j} \bullet \mathbf{i} \leq \mathbf{m} \ \& \ \mathbf{i} \rightarrow \mathbf{n} \leq \mathbf{m} &\Rightarrow \mathbf{j} \leq \mathbf{m}) \\
&\Leftrightarrow \forall xyuv(\sim Suxy \wedge Svyu \supset x \neq u) \\
&\Leftrightarrow \forall xyuv(x = u \supset (Suxy \vee \sim Svyu)) \\
&\Leftrightarrow \forall xyv(Svyx \supset Sxxy)
\end{aligned}
$$

The sequent (Sym) defines ternary frames satisfying $\forall xyv(Svyx \supset Sxxy)$.

6.3 Algebraic correspondence: an application of ALBA

The algorithm ALBA [48, 55] is essentially a calculus for correspondence between non-classical logic and first-order logic. It is used for obtaining analytic rules in display calculi for DLE-logics [109]. For the main purpose of the present chapter, we will use ALBA in a modified form, i.e., the Ackermann based calculus ALC based on BDFNL, as a tool for obtaining analytic rules from certain axioms in the strict implication logic such that Gentzen-style cut-free sequent calculi will be constructed in the next section. The calculus ALC is also a calculus designed for correspondence, not correspondence between DLE-language and first-order language over Kripke frames, but correspondence over BDRGs between the language $\mathcal{L}_{\mathrm{SI}}$ and the language $\mathcal{L}_{\bullet}$ built from propositional variables and constants $\top, \bot$ using only the operator $\bullet$ of product. The language $\mathcal{L}_{\bullet}$ is quite natural because many properties of the product, e.g. the associativity, commutativity, contraction and weakening, can be defined in terms of $\mathcal{L}_{\bullet}$-sequents.

Let us start from a motivating example. The logic $\mathbf{S}_{\mathbb{WH}}$ for weak Heyting algebras is obtained from $\mathbf{S}_{\mathbb{BDI}}$ by adding the inductive sequents (Tr) $(p \rightarrow q) \wedge (q \rightarrow r) \vdash p \rightarrow r$ and (I) $q \vdash p \rightarrow p$. Obviously, the logic $\mathbf{S}_{\mathbb{WH}}$ can be conservatively extended to the extension of BDFNL with all instances of (Tr) and (I). From proof-theoretic point of view, we need to know which structural rules the additional axioms can be equivalently transformed into if there exists.[7] In fact, in BDFNL, one can prove that (I) is equivalent to (*wl*) $p \bullet q \vdash p$, and that (Tr) is equivalent to (*tr*) $p \bullet s \vdash (p \bullet s) \bullet s$. Then it is easy to transform the sequents (*wl*) and (*tr*) into analytic rules as we will show in the next section. Here we are in fact saying that two sequents define the same class of BDRGS. Formally, we say that a sequent $\varphi \vdash \psi$ *algebraically corresponds* to $\varphi' \vdash \psi'$ over BDRGs when they define the same class of BDRGs.

6.3.1. Example. The fact that the sequent (I) algebraically corresponds to (*wl*) is follows immediately from the residuation law. Now we prove that the sequent (Tr) algebraically corresponds to (*tr*). Let $\mathbb{A} = (A, \to, \bullet, \leftarrow)$ be any BDRG. We need to show $\forall abd \in A[(a \to b) \wedge (b \to d) \leq a \to d]$ iff $\forall ac \in A[a \bullet c \leq (a \bullet c) \bullet c]$. One proof is as follows:

$$
\begin{array}{rcl}
 & & \forall abd[(a \to b) \wedge (b \to d) \leq a \to d] \\
\text{(I)} & \Leftrightarrow & \forall abcd[c \leq a \to b \;\&\; c \leq b \to d \Rightarrow c \leq a \to d] \\
\text{(II)} & \Leftrightarrow & \forall abcd[a \bullet c \leq b \;\&\; b \bullet c \leq d \Rightarrow a \bullet c \leq d] \\
\text{(III)} & \Leftrightarrow & \forall acb[a \bullet c \leq b \Rightarrow a \bullet c \leq b \bullet c] \\
\text{(IV)} & \Leftrightarrow & \forall ac[a \bullet c \leq (a \bullet c) \bullet c].
\end{array}
$$

The steps (I) and (III) are obvious. The step (II) is by residuation in BDRGs. For the 'if' part of step (IV), assume that $\forall ac[a \bullet c \leq (a \bullet c) \bullet c]$. Let $b \in A$ and $a \bullet c \leq b$. Then one gets $(a \bullet c) \bullet c \leq b \bullet c$. By the assumption, one gets $a \bullet c \leq b \bullet c$. The 'only if' part is the instantiation of the universal quantifier.

For the algebraic correspondence, we will not take first-order language but $\mathcal{L}_\bullet$ as the corresponding language of $\mathcal{L}_{\mathrm{SI}}$. Nominals and conominals will not be needed. Instead, we introduce a calculus ALC in which propositional variables will play the role of nominals or comonimals in ALBA. The calculus ALC will be defined using supersequent rules of the form

$$
\frac{\Phi \Rightarrow \varphi \vdash \psi}{\Phi' \Rightarrow \varphi' \vdash \psi'}\ (r).
$$

We say that (r) is *valid* in $\mathbb{BDRG}$ if $\Phi' \Rightarrow \varphi' \vdash \psi'$ is valid in all BDRGs validating $\Phi \Rightarrow \varphi \vdash \psi$.

6.3.2. Definition. The Ackermann lemma based calculus ALC based on BDFNL consists of the following rules:

(1) Splitting rules:

$$
\frac{\gamma \vdash \varphi \wedge \psi, \Phi \Rightarrow \chi \vdash \delta}{\gamma \vdash \varphi, \gamma \vdash \psi, \Phi \Rightarrow \chi \vdash \delta}\ (\wedge S) \qquad \frac{\varphi \vee \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\varphi \vdash \gamma, \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\vee S)
$$

[7] In [157, Section 2.5], some contraction rules are shown to guarantee certain axioms. For example, (I) follows from the weakening rule $X \cdot Y \Rightarrow X$, and (Tr) follows from Restall's contraction rule (CSyll) $X; Y \Rightarrow (X; Y); Y$.

(2) Residuation rules:

$$\frac{\psi \vdash \varphi \to \gamma, \Phi \Rightarrow \chi \vdash \delta}{\varphi \bullet \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\mathrm{RL1}) \qquad \frac{\varphi \vdash \gamma \leftarrow \psi, \Phi \Rightarrow \chi \vdash \delta}{\varphi \bullet \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\mathrm{RL2})$$

$$\frac{\Phi \Rightarrow \psi \vdash \varphi \to \gamma}{\Phi \Rightarrow \varphi \bullet \psi \vdash \gamma}\ (\mathrm{RR1}) \qquad \frac{\Phi \Rightarrow \varphi \vdash \gamma \leftarrow \psi}{\Phi \Rightarrow \varphi \bullet \psi \vdash \gamma}\ (\mathrm{RR2})$$

(3) Approximation rules:

$$\frac{\Phi \Rightarrow \varphi \vdash \psi}{p \vdash \varphi, \Phi \Rightarrow p \vdash \psi}\ (\mathrm{Ap1}) \qquad \frac{\Phi \Rightarrow \varphi \vdash \psi}{\psi \vdash p, \Phi \Rightarrow \varphi \vdash p}\ (\mathrm{Ap2})$$

$$\frac{\varphi \to \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \varphi, p \to \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\to\mathrm{Ap1}) \qquad \frac{\varphi \to \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\psi \vdash p, \varphi \to p \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\to\mathrm{Ap2})$$

$$\frac{\gamma \vdash \varphi \to \psi, \Phi \Rightarrow \chi \vdash \delta}{\varphi \vdash p, \gamma \vdash p \to \psi, \Phi \Rightarrow \chi \vdash \delta}\ (\to\mathrm{Ap3}) \qquad \frac{\gamma \vdash \varphi \to \psi, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \psi, \gamma \vdash \varphi \to p, \Phi \Rightarrow \chi \vdash \delta}\ (\to\mathrm{Ap4})$$

$$\frac{\varphi \vdash \psi \bullet \gamma, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \psi, \varphi \vdash p \bullet \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\bullet\mathrm{Ap1}) \qquad \frac{\varphi \vdash \psi \bullet \gamma, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \gamma, \varphi \vdash \psi \bullet p, \Phi \Rightarrow \chi \vdash \delta}\ (\bullet\mathrm{Ap2})$$

$$\frac{\varphi \bullet \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\varphi \vdash p, p \bullet \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\bullet\mathrm{Ap3}) \qquad \frac{\varphi \bullet \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\psi \vdash p, \varphi \bullet p \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\bullet\mathrm{Ap4})$$

$$\frac{\varphi \wedge \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\varphi \vdash p, p \wedge \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\wedge\mathrm{Ap5}) \qquad \frac{\varphi \wedge \psi \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}{\psi \vdash p, \varphi \wedge p \vdash \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\wedge\mathrm{Ap6})$$

$$\frac{\varphi \vdash \psi \vee \gamma, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \psi, \varphi \vdash p \vee \gamma, \Phi \Rightarrow \chi \vdash \delta}\ (\vee\mathrm{Ap1}) \qquad \frac{\varphi \vdash \psi \vee \gamma, \Phi \Rightarrow \chi \vdash \delta}{p \vdash \gamma, \varphi \vdash \psi \vee p, \Phi \Rightarrow \chi \vdash \delta}\ (\vee\mathrm{Ap2})$$

where p is a fresh variable, i.e., a variable which does not occur in previous derivation.

(4) Ackermann rules:

$$\frac{\varphi_1 \vdash p, \ldots, \varphi_n \vdash p, \Phi, \Phi' \Rightarrow \chi \vdash \delta}{\Phi[\bigvee_{i=1}^{n} \varphi_i/p], \Phi' \Rightarrow (\chi \vdash \delta)^*}\ (\mathrm{RAck})$$

where (i) p does not occur in Φ' or φ_i for $1 \leq i \leq n$; (ii) $\Phi = \{\psi_j \vdash \gamma_j \mid \psi_j(+p), \gamma_j(-p), 1 \leq j \leq m\}$ and $\Phi[\bigvee_{i=1}^{n} \varphi_i/p] = \{\psi_j[\bigvee_{i=1}^{n} \varphi_i/p] \vdash \gamma_j[\bigvee_{i=1}^{n} \varphi_i/p] \mid \psi_j \vdash \gamma_j \in \Phi\}$; and (iii) either p does not occur in $\chi \vdash \delta$ and $(\chi \vdash \delta)^* = \chi \vdash \delta$, or $\chi \vdash \delta$ is positive in p and $(\chi \vdash \delta)^* = \chi[\bigvee_{i=1}^{n} \varphi_i/p] \vdash \delta[\bigvee_{i=1}^{n} \varphi_i/p]$.

$$\frac{p \vdash \varphi_1, \ldots, p \vdash \varphi_n, \Phi, \Phi' \Rightarrow \chi \vdash \delta}{\Phi[\bigwedge_{i=1}^{n} \varphi_i/p], \Phi' \Rightarrow (\chi \vdash \delta)^*}\ (\mathrm{LAck})$$

where (i) p does not occur in Φ' or φ_i for $1 \leq i \leq n$; (ii) $\Phi = \{\psi_j \vdash \gamma_j \mid \psi_j(-p), \gamma_j(+p), 1 \leq j \leq m\}$ and $\Phi[\bigwedge_{i=1}^{n} \varphi_i/p] = \{\psi_j[\bigwedge_{i=1}^{n} \varphi_i/p] \vdash \gamma_j[\bigwedge_{i=1}^{n} \varphi_i/p] \mid \psi_j \vdash \gamma_j \in \Phi\}$; and (iii) either p does not occur in $\chi \vdash \delta$ and $(\chi \vdash \delta)^* = x \vdash \delta$, or $\chi \vdash \delta$ is negative in p and $(\chi \vdash \delta)^* = \chi[\bigwedge_{i=1}^{n} \varphi_i/p] \vdash \delta[\bigwedge_{i=1}^{n} \varphi_i/p]$.

The double line in above rules means that the above and the below supersequents can be derived from each other. A supersequent rule (r) is said to be *derivable* in ALC if there is a derivation of the conclusion from the premiss of (r) using only rules in ALC.

6.3.3. Theorem (Correctness). *All rules in* ALC *are valid in* $\mathbb{BDRG}$.

Proof:
The proof is routine. For details, see e.g. [55].

□

Given a set of $\mathcal{L}_{\mathrm{SI}}$-sequents Φ, let $\mathsf{Alg}(\Phi)$ and $\mathsf{Alg}^{+}(\Phi)$ be the class of all BDIs and the class of all BDRGs validating all sequents in Φ respectively. Similarly, given a set of $\mathcal{L}_{\bullet}$-sequents Ψ, let $\mathsf{Alg}^{+}(\Psi)$ be the class of all BDRGs validating all sequents in Ψ. Obviously, an $\mathcal{L}_{\mathrm{SI}}$-sequent $\varphi \vdash \psi$ corresponds to an $\mathcal{L}_{\bullet}$-sequent $\varphi' \vdash \psi'$ over BDRGs if and only if $\mathsf{Alg}^{+}(\varphi \vdash \psi) = \mathsf{Alg}^{+}(\varphi' \vdash \psi')$.

6.3.4. Proposition. *Given an* $\mathcal{L}_{\mathrm{SI}}$*-sequent* $\varphi \vdash \psi$ *and an* $\mathcal{L}_{\bullet}$*-sequent* $\chi \vdash \delta$*, if the rule*

$$\frac{\Rightarrow \varphi \vdash \psi}{\Rightarrow \chi \vdash \delta}(r)$$

is derivable in ALC*, then* $\varphi \vdash \psi$ *algebraically corresponds to* $\chi \vdash \delta$ *over BDRGs.*

Proof:
Assume that the rule (r) is derivable in ALC. By the correctness of ALC, the premiss $\varphi \vdash \psi$ and the conclusion $\chi \vdash \delta$ defines the same BDRGs, i.e., $\mathsf{Alg}^{+}(\varphi \vdash \psi) = \mathsf{Alg}^{+}(\varphi' \vdash \psi')$.

□

By Proposition 6.3.4, one obtains a proof-theoretic tool for algebraic correspondence over BDRGs between the languages $\mathcal{L}_{\mathrm{SI}}$ and $\mathcal{L}_{\bullet}$.

6.3.5. Example. Some $\mathcal{L}_{\mathrm{SI}}$-sequents (inequalities) in Table 6.1 and their algebraic correspondents in $\mathcal{L}_{\bullet}$ are listed in Table 6.4.

(Tr) One proof is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow (p \to q) \wedge (q \to r) \vdash (p \to r)}{s \vdash (p \to q) \wedge (q \to r) \Rightarrow s \vdash p \to r}(\text{AAp1})}{s \vdash p \to q, s \vdash q \to r \Rightarrow s \vdash p \to r}(\wedge\text{S})}{p \bullet s \vdash q, q \bullet s \vdash r \Rightarrow p \bullet s \vdash r}(\text{RL1, RR1})}{p \bullet s \vdash q \Rightarrow p \bullet s \vdash q \bullet s}(\text{RAck})}{\Rightarrow p \bullet s \vdash (p \bullet s) \bullet s}(\text{RAck})$$

Other pairs of corresponding sequents can be proved similarly. See Appendix 6.6.

Table 6.4: Some Algebraic Correspondents

$\mathcal{L}_{\mathrm{SI}}$-sequent		$\mathcal{L}_{\bullet}$-sequent	
(I)	$q \vdash p \to p$	(wl)	$p \bullet q \vdash p$
(Tr)	$(p \to q) \wedge (q \to r) \vdash p \to r$	(tr)	$p \bullet s \vdash (p \bullet s) \bullet s$
(MP)	$p \wedge (p \to q) \vdash q$	(ct)	$p \vdash p \bullet p$
(W)	$p \vdash q \to p$	(wr)	$q \bullet p \vdash p$
(RT)	$p \to q \vdash r \to (p \to q)$	(rt)	$p \bullet (r \bullet s) \vdash p \bullet s$
(B)	$p \to q \vdash (r \to p) \to (r \to q)$	(b)	$r \bullet (s \bullet t) \vdash (r \bullet t) \bullet s$
(B′)	$p \to q \vdash (q \to r) \to (p \to r)$	(b')	$p \bullet (t \bullet s) \vdash (p \bullet s) \bullet t$
(C)	$p \to (q \to r) \vdash q \to (p \to r)$	(c)	$p \bullet (q \bullet s) \vdash q \bullet (p \bullet s)$
(Fr)	$p \to (q \to r) \vdash (p \to q) \to (p \to r)$	(fr)	$p \bullet (u \bullet s) \vdash (p \bullet u) \bullet (p \bullet s)$
(W′)	$p \to (p \to q) \vdash p \to q$	(w')	$p \bullet r \vdash p \bullet (p \bullet r)$

6.3.6. REMARK. Some inductive $\mathcal{L}_{\mathrm{SI}}$-sequents have algebraic correspondents in $\mathcal{L}_{\bullet}$ using ALC. But it is not clear whether all inductive sequents in $\mathcal{L}_{\mathrm{SI}}$ have algebraic correspondents in $\mathcal{L}_{\bullet}$. Consider the sequents (Sym), (Euc) and (D). Our conjecture is that these sequents never correspond to any $\mathcal{L}_{\bullet}$-sequents. Conversely, we conjecture that not all $\mathcal{L}_{\bullet}$-sequents have their algebraic correspondents in $\mathcal{L}_{\mathrm{SI}}$. Consider the inverse of (tr) in Table 6.4. We start from $(p \bullet s) \bullet s \vdash p \bullet s$ and apply ALC. The first step is to use approximation rule, and we get

$$p \bullet s \vdash q \Rightarrow (p \bullet s) \bullet s \vdash q$$

Using residuation rules, we get

$$s \vdash p \to q \Rightarrow s \vdash (p \bullet s) \to q$$

The next step is to consider using the left Ackermann rule because the term $p \bullet s$ on the right hand side takes a negative position. Then we have

$$t \leq p \bullet s, s \vdash p \to q \Rightarrow s \vdash t \to q$$

Then there is no way to continue ALC. It is rather likely that the sequent $(p \bullet s) \bullet s \vdash p \bullet s$ has no algebraic correspondent in $\mathcal{L}_{\mathrm{SI}}$. The general question on the expressive power of ALC will be explored in future work.

Let Φ be a set of $\mathcal{L}_{\mathrm{SI}}$-sequents and Ψ a set of $\mathcal{L}_{\bullet}$-sequents. We use the notation $\Phi \equiv_{\mathsf{ALC}} \Psi$ to denote that Ψ consists of $\mathcal{L}_{\bullet}$-sequents obtained from sequents in Φ using ALC. Let $\mathbf{S}_{\mathbb{BDI}}(\Phi)$ be the algebraic sequent system obtained from $\mathsf{S}_{\mathbb{BDI}}$ by adding all instances of sequents in Φ as axioms. Similarly, let BDFNL(Ψ) be the algebraic sequent system obtained from BDFNL by adding all instances of sequents in Ψ as axioms. Clearly $\mathbf{S}_{\mathbb{BDI}}(\Phi)$ is sound and complete with respect to Alg(Φ), and BDFNL(Ψ) is sound and complete with respect to $\mathsf{Alg}^{+}(\Psi)$.

6.3.7. LEMMA. *Let Φ be a set of inductive $\mathcal{L}_{SI}$-sequents and Ψ a set of $\mathcal{L}_{\bullet}$-sequents. Assume $\Phi \equiv_{\mathsf{ALC}} \Psi$. For every algebra $\mathbb{A}$ in $\mathsf{Alg}^{+}(\Psi)$, its $(\wedge, \vee, \bot, \top, \rightarrow)$-reduct is an algebra in $\mathsf{Alg}(\Phi)$.*

Proof:
Let $\mathbb{A} \in \mathsf{Alg}^{+}(\Psi)$. Then $\mathbb{A} \models \Psi$. By $\Phi \equiv_{\mathsf{ALC}} \Psi$, one gets $\mathbb{A} \models \Phi$. Hence the $(\wedge, \vee, \bot, \top, \rightarrow)$-reduct of $\mathbb{A}$ is an algebra in $\mathsf{Alg}(\Phi)$.

□

6.3.8. LEMMA. *Let Φ be a set of inductive $\mathcal{L}_{SI}$-sequents and Ψ a set of $\mathcal{L}_{\bullet}$-sequents. Assume $\Phi \equiv_{\mathsf{ALC}} \Psi$. For every algebra $\mathbb{A} = (A, \rightarrow)$ in $\mathsf{Alg}(\Phi)$, its canonical extension $\mathbb{A}^{\delta} = (A^{\delta}, \rightarrow^{\pi}, \bullet, \leftarrow)$ is in $\mathsf{Alg}^{+}(\Psi)$.*

Proof:
Obviously, $\mathbb{A}^{\delta}$ is a BDRG. Moreover, $(A^{\delta}, \rightarrow^{\pi}) \in \mathsf{Alg}(\Phi)$ because every inductive sequent in Φ is canonical. By $\Phi \equiv_{\mathsf{ALC}} \Psi$, one gets $\mathbb{A}^{\delta} \models \Psi$.

□

By Lemma 6.3.7 and Lemma 6.3.8, one gets the following theorem immediately:

6.3.9. THEOREM. *Let Φ be a set of inductive sequents in $\mathcal{L}$ and Ψ a set of $\mathcal{L}_{\bullet}$-sequents. Assume $\Phi \equiv_{\mathsf{ALC}} \Psi$. The algebraic sequent $\mathsf{BDFNL}(\Psi)$ is a conservative extension of $\mathsf{S}_{\mathbb{BDI}}(\Phi)$.*

6.3.10. EXAMPLE. Notice that (I) $q \vdash p \rightarrow p$ corresponds to (wl) $p \bullet q \vdash p$, and (Tr) $(p \rightarrow q) \wedge (q \rightarrow r) \vdash p \rightarrow r$ corresponds to (tr) $p \bullet s \vdash (p \bullet s) \bullet s$. Both (I) and (Tr) are inductive sequents. The algebras defined by (wl) and (tr) are BDRGs satisfying the conditions: (wl) $a \bullet b \leq a$ and (tr) $a \bullet b \leq (a \bullet b) \bullet b$. We call such algebras *residuated weak Heyting algebras*, and the class of such algebras is denoted by $\mathbb{RWH}$. By Theorem 6.3.9, the algebraic sequent system $\mathbf{S}_{\mathbb{RWH}}$ is a conservative extension of $\mathbf{S}_{\mathbb{WH}}$. For sequents in Example 6.3.5, one can get similar conservativity results.

6.4 Gentzen-style sequent calculi

In this section, we will first introduce a Gentzen-style cut-free sequent calculus $\mathsf{G}_{\mathsf{BDFNL}}$ for BDFNL[8], which will be presented by introducing structure operators separately for connectives $\wedge$ and $\bullet$. By the conservativity of BDFNL over $\mathsf{S}_{\mathsf{BDI}}$, and the subformula property of $\mathsf{G}_{\mathsf{BDFNL}}$, one gets a cut-free sequent calculus for $\mathsf{S}_{\mathsf{BDI}}$. Let $\mathbf{S}_{\mathbb{BDI}}(\Phi)$ be an extension of $\mathbf{S}_{\mathbb{BDI}}$ with inductive sequents in Φ as axioms which have algebraic

[8] The sequent system for BDFNL defined in e.g. [25, 26] does not admit cut elimination. When the distributivity is added as an axiom, the sequent $\varphi \wedge (\psi \vee (\chi \vee \delta)) \vdash (\varphi \wedge \psi) \vee ((\varphi \vee \chi) \vee (\varphi \wedge \delta))$ cannot be proved without cut.

correspondents in $\mathcal{L}_\bullet$. One can transform these axioms into analytic rules, and if these rules which are added to $\mathsf{G}_{\mathsf{BDFNL}}$ does not effect the subformula property, one gets a cut-free sequent system for $\mathbf{S}_{\mathbb{BDI}}(\Phi)$ by omitting additional rules for the two additional operators $\bullet$ and $\leftarrow$. Our presentation follows [131].

6.4.1 The sequent calculus $\mathsf{G}_{\mathsf{BDFNL}}$

6.4.1. Definition. Let $\odot$ and $\oslash$ be structural operators for the product $\bullet$ and $\wedge$ respectively. The set of all *structures* is defined inductively as follows:

$$\Gamma ::= \varphi \mid (\Gamma \odot \Gamma) \mid (\Gamma \oslash \Gamma),$$

where $\varphi \in \mathcal{L}_{\mathrm{LC}}$. We use Γ, Δ, Σ etc. with indexes to denote structures. Each structure Γ is associated with a term $\tau(\Gamma) \in \mathcal{L}_{\mathrm{LC}}$ defined inductively by

- $\tau(\varphi) = \varphi$, for every $\varphi \in \mathcal{L}_{\mathrm{LC}}$;
- $\tau(\Gamma \odot \Delta) = \tau(\Gamma) \bullet \tau(\Delta)$;
- $\tau(\Gamma \oslash \Delta) = \tau(\Gamma) \wedge \tau(\Delta)$.

A *consecution* (sequent) is $\Gamma \vdash \varphi$ where Γ is a structure and φ is an $\mathcal{L}_{\mathrm{LC}}$-formula.

Given a BDRG $\mathbb{A}$ and an assingnment μ in A, for any structure Γ, define $\mu(\Gamma) = \mu(\tau(\Gamma))$. We say that $\Gamma \vdash \varphi$ is *valid* in $\mathbb{A}$ if $\mu(\Gamma) \leq \mu(\varphi)$ for every assignment in $\mathbb{A}$. We use the notation $\mathbb{BDRG} \models \Gamma \vdash \varphi$ to denote that $\Gamma \vdash \varphi$ is valid in every BDRG. Obviously $\mathbb{BDRG} \models \Gamma \vdash \varphi$ iff $\mathbb{BDRG} \models \tau(\Gamma) \vdash \varphi$.

A *context* is a structure $\Gamma[-]$ with a single hole $-$ for a structure. Formally, contexts are defined inductively by the following rule:

$$\Gamma[-] ::= [-] \mid \Gamma[-] \odot \Delta \mid \Delta \odot \Gamma[-] \mid \Gamma[-] \oslash \Delta \mid \Delta \oslash \Gamma[-],$$

where Δ is a structure. For any context $\Gamma[-]$ and structure Δ, let $\Gamma[\Delta]$ be the structure obtained from $\Gamma[-]$ by substituting Δ for the hole $-$. For a context $\Gamma[-]$, let $\tau(\Gamma[-])$ be the formula which contains a hole. In particular, let $\tau([-]) = -$.

6.4.2. Definition. The sequent calculus $\mathsf{G}_{\mathsf{BDFNL}}$ consists of the following axioms and rules:

- Axioms:

$$(\mathrm{Id})\ \varphi \vdash \varphi, \quad (\top)\ \Gamma \vdash \top, \quad (\bot)\ \Gamma[\bot] \vdash \varphi,$$

- Logical rules:

$$\frac{\Delta \vdash \varphi \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\Delta \odot (\varphi \to \psi)] \vdash \gamma}(\to\vdash), \quad \frac{\varphi \odot \Gamma \vdash \psi}{\Gamma \vdash \varphi \to \psi}(\vdash\to),$$

$$\frac{\Gamma[\varphi] \vdash \gamma \quad \Delta \vdash \psi}{\Gamma[(\varphi \leftarrow \psi) \odot \Delta] \vdash \gamma}(\leftarrow \vdash), \quad \frac{\Gamma \odot \psi \vdash \varphi}{\Gamma \vdash \varphi \leftarrow \psi}(\vdash \leftarrow),$$

$$\frac{\Gamma[\varphi \odot \psi] \vdash \gamma}{\Gamma[\varphi \bullet \psi] \vdash \gamma}(\bullet \vdash), \quad \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma \odot \Delta \vdash \varphi \bullet \psi}(\vdash \bullet),$$

$$\frac{\Gamma[\varphi \owedge \psi] \vdash \gamma}{\Gamma[\varphi \wedge \psi] \vdash \gamma}(\wedge \vdash), \quad \frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma \owedge \Delta \vdash \varphi \wedge \psi}(\vdash \wedge),$$

$$\frac{\Gamma[\varphi] \vdash \gamma \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\varphi \vee \psi] \vdash \gamma}(\vee \vdash), \quad \frac{\Gamma \vdash \varphi_i}{\Gamma \vdash \varphi_1 \vee \varphi_2}(\vdash \vee)(i = 1, 2),$$

- Structural rules:

$$\frac{\Gamma[\Delta \owedge \Delta] \vdash \varphi}{\Gamma[\Delta] \vdash \varphi}(\owedge\mathrm{C}), \quad \frac{\Gamma[\Delta] \vdash \varphi}{\Gamma[\Sigma \owedge \Delta] \vdash \varphi}(\owedge\mathrm{W}),$$

$$\frac{\Gamma[\Delta \owedge \Lambda] \vdash \varphi}{\Gamma[\Lambda \owedge \Delta] \vdash \varphi}(\owedge\mathrm{E}), \quad \frac{\Gamma[(\Delta_1 \owedge \Delta_2) \owedge \Delta_3] \vdash \varphi}{\Gamma[\Delta_1 \owedge (\Delta_2 \owedge \Delta_3)] \vdash \varphi}(\owedge\mathrm{As}).$$

A derivation in $\mathsf{G}_{\mathsf{BDFNL}}$ is an instance of an axiom or a tree of applications of logical or structural rules. The height of a derivation if the greatest number of successive applications of rules in it, where an axiom has height 0. A formula with the connective in a logical rule is called the *principal* formula of that rule. A sequent $\Gamma \vdash \varphi$ is derivable in $\mathsf{G}_{\mathsf{BDFNL}}$ if there is a derivation ending with $\Gamma \vdash \varphi$ in $\mathsf{G}_{\mathsf{BDFNL}}$. A rule of sequents is derivable in $\mathsf{G}_{\mathsf{BDFNL}}$ if the conclusion is derivable whenever the premisses are derivable in $\mathsf{G}_{\mathsf{BDFNL}}$.

6.4.3. Fact. The following structural rules are derivable in $\mathsf{G}_{\mathsf{BDFNL}}$:

$$(\owedge\mathrm{W}')\ \frac{\Gamma[\Delta] \vdash \varphi}{\Gamma[\Delta \owedge \Sigma] \vdash \varphi}, \quad (\owedge\mathrm{As}')\ \frac{\Gamma[\Delta_1 \owedge (\Delta_2 \owedge \Delta_3)] \vdash \varphi}{\Gamma[(\Delta_1 \owedge \Delta_2) \owedge \Delta_3] \vdash \varphi}.$$

We will now prove the admissibility of cut rule in $\mathsf{G}_{\mathsf{BDFNL}}$. The standard cut rule for a 'deep inference' system using contexts with a hole is the following:

$$\frac{\Delta \vdash \varphi \quad \Gamma[\varphi] \vdash \psi}{\Gamma[\Delta] \vdash \psi}\ (\mathrm{cut})$$

Consider the cut in which the right premiss is obtained by ($\owedge$C) and the left premiss is an axiom ($\top$):

$$\frac{\Delta \vdash \top \quad \dfrac{\Gamma[\top \owedge \top] \vdash \psi}{\Gamma[\top] \vdash \psi}\ (\owedge\mathrm{C})}{\Gamma[\Delta] \vdash \psi}\ (\mathrm{cut})$$

To eliminate the cut here, one need to cut simultaneously the two occurrences of $\top$ in the premiss of ($\owedge$C). Then we will consider Gentzen's multi-cut or mix rule of which the cut rule is a special case. We use multiple-hole contexts of the form $\Gamma[-]\ldots[-]$ to formulate the mix rule.

6.4.4. Theorem. *The mix rule*

$$\frac{\Delta \vdash \varphi \quad \Gamma[\varphi] \ldots [\varphi] \vdash \psi}{\Gamma[\Delta] \ldots [\Delta] \vdash \psi}\text{(mix)}$$

is admissible in $\mathsf{G}_{\mathsf{BDFNL}}$.

Proof:
We prove (mix) by simultaneous induction on (i) the complexity of the mixed formula φ; (ii) the height of the derivation of $\Delta \vdash \varphi$; (iii) the height of the derivation of $\Gamma[\varphi] \vdash \psi$. Assume that $\Delta \vdash \varphi$ is obtained by R_1, and $\Gamma[\varphi] \vdash \psi$ by R_2. We have four cases:

(I) At least one of R_1 and R_2 is an axiom. We have two cases:

Case 1. Both R_1 and R_2 are axioms. We have the following subcases:

(1.1) $R_1 = (\bot)$ or $R_2 = (\top)$. The conclusion of (mix) is an instance of $(\bot)$ or $(\top)$.

(1.2) $R_1 =$ (Id). Then $\Delta = \varphi$. The conclusion of (mix) is obtained by R_2.

(1.3) $R_1 = (\top)$, $R_2 =$ (Id). Then $\varphi = \top = \psi$. The conclusion of (mix) is obtained by $(\top)$.

(1.4) $R_1 = (\top)$, $R_2 = (\bot)$. Then $\varphi = \top$, and $\bot$ occurs in $\Gamma[\top] \ldots [\top]$. The conclusion of (mix) is obtained by $(\bot)$.

Case 2. Exactly one of R_1 and R_2 is an axiom. We have the following subcases:

(2.1) $R_1 =$ (Id). Then the conclusion is the same as the right premiss of (mix).

(2.2) $R_1 = (\bot)$. Then the conclusion of (mix) is an axiom.

(2.3) $R_1 = (\top)$. Then $\varphi = \top$. We have subcases according to R_2. If R_2 is a right rule of a logical connective. We first apply (mix) to $\Delta \vdash \top$ and the premiss(es) of R_2, and then apply the rule R_2. If R_2 is a left rule of a logical connective, the proof is similar to Case 6. If R_2 is a structural rule, the proof is similar to Case 4.

(2.4) $R_2 =$ (Id). The conclusion of (mix) is the same as the left premiss of (mix).

(2.5) $R_2 = (\top)$. The conclusion of (mix) is an axiom.

(2.6) $R_2 = (\bot)$. If $\varphi \neq \bot$, then the conclusion of (mix) is an instance of $(\bot)$. Suppose $\varphi = \bot$. We have subcases according to R_1. Clearly R_1 cannot be a right rule of a logical connective. If R_1 is a left rule of a logical cognitive, the proof is similar to Case 5. If R_1 is a structural rule, the proof is similar to Case 3.

(II) At least one of R_1 and R_2 is a structural rule. We have two cases:

Case 3. R_1 is a structural rule. By induction (ii), the (mix) can be push up to the premiss of R_1 and then apply R_1. For example, let $R_1 = (\owedge\mathrm{C})$. The derivation

$$\dfrac{\dfrac{\Delta'[\Sigma \owedge \Sigma] \vdash \varphi}{\Delta'[\Sigma] \vdash \varphi}(\owedge\mathrm{C}) \quad \Gamma[\varphi] \ldots [\varphi] \vdash \psi}{\Gamma[\Delta'[\Sigma]] \ldots \Gamma[\Delta'[\Sigma]] \vdash \psi}\text{(mix)}$$

is transformed into

$$\dfrac{\dfrac{\Delta'[\Sigma \owedge \Sigma] \vdash \varphi \quad \Gamma[\varphi] \ldots [\varphi] \vdash \psi}{\Gamma[\Delta'[\Sigma \owedge \Sigma]] \ldots [\Delta'[\Sigma \owedge \Sigma]] \vdash \psi}\text{(mix)}}{\Gamma[\Delta'[\Sigma]] \ldots \Gamma[\Delta'[\Sigma]] \vdash \psi}(\owedge\mathrm{C}^*)$$

where ($\oslash$C*) stands for the application of ($\oslash$C) multiple times.

Case 4. R_2 is a structural rule. Suppose that φ is obtained by ($\oslash$W) in R_2. The derivation

$$\dfrac{\Delta \vdash \varphi \qquad \dfrac{\Gamma[\varphi]\ldots[\Delta']\ldots[\varphi] \vdash \psi}{\Gamma[\varphi]\ldots[\Sigma[\varphi] \oslash \Delta']\ldots[\varphi] \vdash \psi}\,(\oslash\mathrm{W})}{\Gamma[\Delta]\ldots[\Sigma[\Delta] \oslash \Delta']\ldots[\Delta] \vdash \psi}\,(\mathrm{mix})$$

is transformed into

$$\dfrac{\dfrac{\Delta \vdash \varphi \quad \Gamma[\varphi]\ldots[\Delta']\ldots[\varphi] \vdash \psi}{\Gamma[\Delta]\ldots[\Delta']\ldots[\Delta] \vdash \psi}\,(\mathrm{mix})}{\Gamma[\Delta]\ldots[\Sigma[\Delta] \oslash \Delta']\ldots[\Delta] \vdash \psi}\,(\oslash\mathrm{W})$$

For the remaining cases of R_2, by induction (ii), the (mix) can be push up to the premiss of R_2 and then apply R_2.

(III) At least one of R_1 and R_2 is a logical rule, but the mixed formula is not principal. We have two cases:

Case 5. The mixed formula φ is not principal in the left premiss. Then we have subcases according to R_1. Clearly R_1 cannot be a right rule of a logical connective. Assume $R_1 = (\to\vdash)$. The derivation ends with

$$\dfrac{\dfrac{\Delta' \vdash \chi \quad \Delta[\delta] \vdash \varphi}{\Delta[\Delta' \odot (\chi \to \delta)] \vdash \varphi}\,(\to\vdash) \qquad \Gamma[\varphi]\ldots[\varphi] \vdash \psi}{\Gamma[\Delta[\Delta' \odot (\chi \to \delta)]]\ldots[\Delta[\Delta' \odot (\chi \to \delta)]] \vdash \psi}\,(\mathrm{mix})$$

Firstly we push up (mix) as below:

$$\dfrac{\Delta[\delta] \vdash \varphi \quad \Gamma[\varphi]\ldots[\varphi] \vdash \psi}{\Gamma[\Delta[\delta]]\ldots[\Delta[\delta]] \vdash \psi}\,(\mathrm{mix})$$

Then we apply $(\to\vdash)$ to $\Delta' \vdash \chi$ and $\Gamma[\Delta[\delta]]\ldots[\Delta[\delta]] \vdash \psi$ multiple times, and we get the conclusion.

Assume $R_1 = (\vee\vdash)$. The derivation ends with

$$\dfrac{\dfrac{\Delta[\chi] \vdash \varphi \quad \Delta[\delta] \vdash \varphi}{\Delta[\chi \vee \delta] \vdash \varphi}\,(\vee\vdash) \qquad \Gamma[\varphi]\ldots[\varphi] \vdash \psi}{\Gamma[\Delta[\chi \vee \delta]]\ldots[\Delta[\chi \vee \delta]] \vdash \psi}\,(\mathrm{mix})$$

The rule (mix) is push up to sequents with less height of derivation in multiple steps. For the first occurrence of φ in $\Gamma[\varphi]\ldots[\varphi] \vdash \psi$, mix it with $\Delta[\chi] \vdash \varphi$ and $\Delta[\delta] \vdash \varphi$ respectively, and by $(\vee\vdash)$ one gets $\Gamma[\Delta[\chi \vee \delta]][\varphi]\ldots[\varphi] \vdash \psi$. Repeat this process multiple times and we achieve the conclusion $\Gamma[\Delta[\chi \vee \delta]]\ldots[\Delta[\chi \vee \delta]] \vdash \psi$.

The remaining cases $R_1 = (\leftarrow\vdash)$, $(\bullet\vdash)$ or $(\wedge\vdash)$ are similar.

Case 6. The mixed formula φ is principal only in the left premiss. Then we have subcases according to R_2. Assume $R_2 = (\to\vdash)$. If φ does not occur in Σ, then the derivation

$$\dfrac{\Delta \vdash \varphi \qquad \dfrac{\Sigma \vdash \chi \quad \Gamma'[\delta][\varphi]\ldots[\varphi] \vdash \psi}{\Gamma'[\Sigma \odot (\chi \to \delta)][\varphi]\ldots[\varphi] \vdash \psi}\,(\to\vdash)}{\Gamma'[\Sigma \odot (\chi \to \delta)][\Delta]\ldots[\Delta] \vdash \psi}\,(\mathrm{mix})$$

is transformed into

$$\dfrac{\Sigma \vdash \chi \qquad \dfrac{\Delta \vdash \varphi \quad \Gamma'[\delta][\varphi]\ldots[\varphi] \vdash \psi}{\Gamma'[\delta][\Delta]\ldots[\Delta] \vdash \psi}\,(\text{mix})}{\Gamma'[\Sigma \odot (\chi \to \delta)][\Delta]\ldots[\Delta] \vdash \psi}\,(\to\vdash)$$

Suppose that $\Sigma = \Sigma'[\varphi]$. The derivation

$$\dfrac{\Delta \vdash \varphi \qquad \dfrac{\Sigma'[\varphi] \vdash \chi \quad \Gamma'[\delta][\varphi]\ldots[\varphi] \vdash \psi}{\Gamma'[\Sigma'[\varphi] \odot (\chi \to \delta)][\varphi]\ldots[\varphi] \vdash \psi}\,(\to\vdash)}{\Gamma'[\Sigma'[\Delta] \odot (\chi \to \delta)][\Delta]\ldots[\Delta] \vdash \psi}\,(\text{mix})$$

is transformed into

$$\dfrac{\dfrac{\Delta \vdash \varphi \quad \Sigma'[\varphi] \vdash \chi}{\Sigma'[\Delta] \vdash \chi}\,(\text{mix}) \qquad \dfrac{\Delta \vdash \varphi \quad \Gamma'[\delta][\varphi]\ldots[\varphi] \vdash \psi}{\Gamma'[\delta][\Delta]\ldots[\Delta] \vdash \psi}\,(\text{mix})}{\Gamma'[\Sigma'[\Delta] \odot (\chi \to \delta)][\Delta]\ldots[\Delta] \vdash \psi}\,(\to\vdash)$$

The remaining cases $R_2 = (\leftarrow\vdash)$, $(\bullet\vdash)$, $(\wedge\vdash)$, or $(\vee\vdash)$ are similar.

(IV) Both R_1 and R_2 are logical rules, and the mixed formula is principal. Then we prove it by induction on the complexity of φ. Assume that $\varphi = \varphi_1 \bullet \varphi_2$. The derivation

$$\dfrac{\dfrac{\Delta_1 \vdash \varphi_1 \quad \Delta_2 \vdash \varphi_2}{\Delta_1 \odot \Delta_2 \vdash \varphi}\,(\vdash\bullet) \qquad \dfrac{\Gamma[\varphi]\ldots[\varphi_1 \odot \varphi_2]\ldots[\varphi] \vdash \psi}{\Gamma[\varphi]\ldots[\varphi_1 \bullet \varphi_2]\ldots[\varphi] \vdash \psi}\,(\bullet\vdash)}{\Gamma[\Delta_1 \odot \Delta_2] \vdash \psi}\,(\text{mix})$$

is transformed into

$$\dfrac{\Delta_1 \vdash \varphi_1 \qquad \dfrac{\Delta_2 \vdash \varphi_2 \qquad \dfrac{\Delta_1 \odot \Delta_2 \vdash \varphi \quad \Gamma[\varphi]\ldots[\varphi_1 \odot \varphi_2]\ldots[\varphi] \vdash \psi}{\Gamma[\Delta_1 \odot \Delta_2]\ldots[\varphi_1 \odot \varphi_2]\ldots[\Delta_1 \odot \Delta_2] \vdash \psi}\,(\text{mix})}{\Gamma[\Delta_1 \odot \Delta_2]\ldots[\varphi_1 \odot \Delta_2]\ldots[\Delta_1 \odot \Delta_2] \vdash \psi}\,(\text{mix})}{\Gamma[\Delta_1 \odot \Delta_2]\ldots[\Delta_1 \odot \Delta_2]\ldots[\Delta_1 \odot \Delta_2] \vdash \psi}\,(\text{mix})$$

Note that the (mix) rule is push up to sequents with lesser height in the derivation. The remaining cases $\varphi = \varphi_1 \to \varphi_2$, $\varphi_1 \leftarrow \varphi_2$, $\varphi_1 \wedge \varphi_2$, or $\varphi_1 \vee \varphi_2$ are quite similar.

□

In all rules of $\mathsf{G}_{\mathsf{BDFNL}}$, no formula disappears in from the premiss(es) to the conclusion. Hence we get the subformula property of $\mathsf{G}_{\mathsf{BDFNL}}$ immediately:

6.4.5. Theorem. *If a consecution $\Gamma \vdash \varphi$ has a derivation in $\mathsf{G}_{\mathsf{BDFNL}}$, then all formulas in the derivation are subformulas of Γ, φ.*

Now we will prove the completeness of $\mathsf{G}_{\mathsf{BDFNL}}$ with respect to $\mathbb{BDRG}$. Firstly, we have the following lemma on the invertibility of some rules in $\mathsf{G}_{\mathsf{BDFNL}}$:

6.4.6. Lemma. *The following rules are admissible in* $\mathsf{G_{BDFNL}}$*:*

$$\frac{\Gamma \vdash \varphi \to \psi}{\varphi \odot \Gamma \vdash \psi}(\vdash\to\uparrow), \quad \frac{\Gamma[\varphi \bullet \psi]\ldots[\varphi \bullet \psi] \vdash \gamma}{\Gamma[\varphi \odot \psi]\ldots[\varphi \odot \psi] \vdash \gamma}(\bullet \vdash \uparrow),$$

$$\frac{\Gamma \vdash \varphi \leftarrow \psi}{\Gamma \odot \psi \vdash \varphi}(\vdash\leftarrow\uparrow), \quad \frac{\Gamma[\varphi \wedge \psi]\ldots[\varphi \wedge \psi] \vdash \gamma}{\Gamma[\varphi \oslash \psi]\ldots[\varphi \oslash \psi] \vdash \gamma}(\oslash \vdash \uparrow).$$

Proof:
The proof is done by induction on the height of the derivation of the premiss. Here we prove only the admissibility of $(\vdash\to\uparrow)$ and $(\bullet \vdash \uparrow)$. The remaining rules are shown similarly. Assume that the premiss is obtained by R.

For $(\vdash\to\uparrow)$, if R is an axiom, one can get $\varphi \odot \Gamma \vdash$ easily. If R is a left rule of a connective, or a rule for $\oslash$, we push up $(\vdash\to\uparrow)$ to the premiss(es) of R and then apply the rule R. If R is a right rule, it can only be $(\vdash\to)$ and then one gets $\varphi \odot \Gamma \vdash \psi$.

For $(\bullet \vdash \uparrow)$, assume that $\Gamma \vdash \varphi \to \psi$ is obtained by R. We have the following cases:

Case 1. R is an axiom. When R is $(\bot)$ or $(\top)$, the conclusion is also $(\bot)$ or $(\top)$. Assume $R = (\mathrm{Id})$. The conclusion $\varphi \odot \psi \vdash \varphi \bullet \psi$ can be derived by $(\vdash \bullet)$ obviously.

Case 2. R is a logical rule. If R is a rule of $\to$, $\leftarrow$, $\wedge$, or R is $(\vdash \bullet)$, one can push up $(\vdash\to\uparrow)$ to the premiss of R and then apply the rule R. If $R = (\bullet \vdash)$, one can push up $(\bullet \vdash \uparrow)$ to the premiss of R and obtain the conclusion directly.

Case 3. R is a structural rule. Apply $(\bullet \vdash \uparrow)$ to the premiss of R and then apply R.

□

6.4.7. Lemma. *If* $\varphi \vdash \psi$ *is derivable in* BDFNL*, then* $\varphi \vdash \psi$ *is derivable in* $\mathsf{G_{BDFNL}}$.

Proof:
By induction on the derivation of $\varphi \vdash \psi$ in BDFNL.

Case 1. $\varphi \vdash \psi$ is an axiom. The cases of (Id), $(\top)$ and $(\bot)$ are clear. For (D), one derivation is

$$\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \quad \psi \vdash \psi}{\varphi \oslash \psi \Rightarrow \varphi \wedge \psi}(\vdash \wedge)}{\varphi \oslash \psi \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)}(\vdash \vee) \quad \dfrac{\dfrac{\varphi \vdash \varphi \quad \gamma \vdash \gamma}{\varphi \oslash \gamma \Rightarrow \varphi \wedge \gamma}(\vdash \wedge)}{\varphi \oslash \gamma \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)}(\vdash \vee)}{\dfrac{\varphi \oslash (\psi \vee \gamma) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)}{\varphi \wedge (\psi \vee \gamma) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)}(\wedge \vdash)}(\vee \vdash)$$

Case 2. $\varphi \vdash \psi$ is obtained by a rule. Obviously, rules for $\wedge$ and $\vee$ are derivable in $\mathsf{G_{BDFNL}}$. The rule (cut) is a special case of (mix) in $\mathsf{G_{BDFNL}}$. For residuation rules, (Res1) is shown by the rule $(\bullet \vdash \uparrow)$ in Lemma 6.4.6 and $(\bullet \vdash)$. (Res2) is obtained by the rule $(\vdash\to\uparrow)$ in Lemma 6.4.6 and $(\bullet \vdash)$. The remaining residuation rules are shown similarly.

□

6.4.8. Lemma. *If a consecution* $\Gamma \vdash \varphi$ *is derivable in* $\mathsf{G_{BDFNL}}$*, then* $\tau(\Gamma) \vdash \varphi$ *is derivable in* BDFNL.

Proof:
By induction on the height of the derivation of $\Gamma \vdash \varphi$ in $\mathsf{G_{BDFNL}}$.

Case 1. $\Gamma \vdash \varphi$ is an axiom. The cases of (Id) and ($\top$) are obvious. We prove $\tau(\Gamma[\bot]) \vdash \varphi$ by induction on the construction of Γ. We have the following cases:

(1.1) $\Gamma = \psi$. Then $\Gamma[\bot] = \bot = \psi$. By ($\bot$), we have $\bot \vdash \varphi$.

(1.2) $\Gamma = \Gamma' \odot \Delta$. Assume $\Gamma' = \Gamma'[\bot]$. By induction hypothesis, we have $\tau(\Gamma'[\bot]) \vdash \varphi \leftarrow \tau(\Delta)$. Then by (Res4) in BDFNL, one gets $\tau(\Gamma'[\bot]) \bullet \tau(\Delta) \vdash \varphi$. Assume $\Delta = \Delta[\bot]$. By induction hypothesis, $\tau(\Delta[\bot]) \vdash \tau(\Gamma) \to \varphi$. By (Res2), one gets $\tau(\Gamma) \bullet \tau(\Delta[\bot]) \vdash \varphi$.

(1.3) $\Gamma = \Gamma' \oslash \Delta$. Then $\tau(\Gamma[\bot]) = \tau(\Gamma'[\bot]) \wedge \tau(\Delta)$ or $\tau(\Gamma[\bot]) = \tau(\Gamma') \wedge \tau(\Delta[\bot])$. By induction hypothesis, one can easily obtain $\tau(\Gamma) \vdash \varphi$.

Case 2. $\Gamma \vdash \varphi$ is obtained by ($\to \vdash$) or ($\leftarrow \vdash$). We prove the case of ($\to \vdash$) and the other one is similar. By inductive hypothesis, we have $\tau(\Delta) \vdash \chi$ and $\tau(\Sigma[\xi]) \vdash \varphi$. Our goal is to prove $\tau(\Sigma[\Delta \odot (\chi \to \xi)]) \vdash \varphi$. Firstly, in BDFNL, from $\tau(\Delta) \vdash \chi$, one gets $\chi \to \xi \vdash \tau(\Delta) \to \xi$. Then by (Res2), one gets $\tau(\Delta) \bullet (\chi \to \xi) \vdash \xi$.

Claim. For any context $\Sigma[-]$, we have $\tau(\Sigma[\tau(\Delta) \bullet (\chi \to \xi)]) \vdash \tau(\Sigma[\xi])$.

Proof of Claim. By induction on the construction of $\Sigma[-]$. The case $\Sigma[-] = [-]$ is obvious. Assume $\Sigma[-] = \Sigma'[-] \odot \Delta'$. Then $\tau(\Sigma[-]) = \tau(\Sigma'[-]) \bullet \tau(\Delta')$. By induction hypothesis, one gets $\tau(\Sigma'[\tau(\Delta) \bullet (\chi \to \xi)]) \vdash \tau(\Sigma'[\xi])$. Then one gets $\tau(\Sigma'[\tau(\Delta) \bullet (\chi \to \xi)]) \bullet \tau(\Delta') \vdash \tau(\Sigma'[\xi]) \bullet \tau(\Delta')$. The remaining cases are similar. This completes the proof of the claim.

Now by applying (cut) to $\tau(\Sigma[\tau(\Delta) \bullet (\chi \to \xi)]) \vdash \tau(\Sigma[\xi])$ and $\tau(\Sigma[\xi]) \vdash \varphi$, one gets $\tau(\Sigma[\Delta \odot (\chi \to \xi)] \vdash \varphi)$.

Case 3. $\Gamma \vdash \varphi$ is obtained by ($\vdash \to$) or ($\vdash \leftarrow$). We prove the case of ($\vdash \to$) and the other one is similar. Let $\varphi = \chi \to \xi$. From the premiss $\chi \odot \Gamma \vdash \xi$ of ($\vdash \to$), by inductive hypothesis, one gets $\chi \bullet \tau(\Gamma) \vdash \xi$. By (Res1), one gets $\tau(\Gamma) \vdash \chi \to \xi$.

Case 4. $\Gamma \vdash \varphi$ is obtained by ($\bullet \vdash$). By induction hypothesis, one gets $\tau(\Gamma[\chi \odot \xi]) \vdash \varphi$ is derivable in BDFNL. Clearly, it is rather easy to check by induction on the construction of Γ that $\tau(\Gamma[\chi \odot \xi]) = \tau(\Gamma[\chi \bullet \xi])$. Therefore $\tau(\Gamma[\chi \bullet \xi]) \vdash \varphi$ is derivable in BDFNL.

Case 5. $\Gamma \vdash \varphi$ is obtained by ($\vdash \bullet$). Let $\varphi = \chi \bullet \xi$. By induction hypothesis, one gets $\tau(\Gamma) \vdash \chi$ and $\tau(\Delta) \vdash \xi$. By the monotonicity rules of $\bullet$, one gets $\tau(\Gamma) \bullet \tau(\Delta) \vdash \chi \bullet \xi$.

Case 6. $\Gamma \vdash \varphi$ is obtained by ($\wedge \vdash$) or ($\vdash \wedge$). The proof is similar to Case 4 or Case 5.

Case 7. $\Gamma \vdash \varphi$ is obtained by ($\vee \vdash$). By induction hypothesis, one gets $\tau(\Gamma[\chi]) \vdash \varphi$ and $\tau(\Gamma[\xi]) \vdash \varphi$. We prove $\tau(\Gamma[\chi \vee \xi]) \vdash \varphi$ by induction on the construction of Γ.

(7.1) $\Gamma[-] = [-]$. Then we have $\chi \vdash \varphi$ and $\xi \vdash \varphi$. By ($\vee$L) in BDFNL, one gets $\chi \vee \xi \vdash \varphi$.

(7.2) $\Gamma[-] = \Gamma_1[-] \odot \Gamma_2$ or $\Gamma_1 \odot \Gamma_2[-]$. The two cases are quite similar, and we specify only the first case. Clearly we have $\tau(\Gamma_1[\chi]) \bullet \tau(\Gamma_2) \vdash \varphi$ and $\tau(\Gamma_1[\xi]) \bullet \tau(\Gamma_2) \vdash \varphi$. By residuation rules, one gets $\tau(\Gamma_1[\chi]) \vdash \varphi \leftarrow \tau(\Gamma_2)$ and $\tau(\Gamma_1[\xi]) \vdash \varphi \leftarrow \tau(\Gamma_2)$. By induction hypothesis on Γ_1, one gets $\tau(\Gamma_1[\chi \vee \xi]) \vdash \varphi \leftarrow \tau(\Gamma_2)$. By residuation, one gets $\tau(\Gamma_1[\chi \vee \xi]) \bullet \tau(\Gamma_2) \vdash \varphi$.

(7.3) $\Gamma[-] = \Gamma_1[-] \oslash \Gamma_2$ or $\Gamma_1 \oslash \Gamma_2[-]$. The proof is quite similar to (7.2).

Case 8. $\Gamma \vdash \varphi$ is obtained by $(\vdash \vee)$. The proof is quite similar to Case 5.

Case 9. $\Gamma \vdash \varphi$ is obtained by ($\oslash$W). By induction hypothesis, one gets $\tau(\Gamma[\Delta]) \vdash \varphi$. Clearly one gets $\tau(\Gamma[\tau(\Delta)] \vdash \varphi$. We prove $\tau(\Gamma[\tau(\Sigma) \wedge \tau(\Delta)]) \vdash \varphi$ by induction on Γ.

(9.1) $\Gamma[-] = [-]$. Then we have $\tau(\Delta) \vdash \varphi$. In BDFNL we have $\tau(\Sigma) \wedge \tau(\Delta) \vdash \varphi$.

(9.2) $\Gamma[-] = \Gamma_1[-] \odot \Gamma_2$ or $\Gamma_1 \odot \Gamma_2[-]$. The two cases are quite similar, and we specify only the first case. Clearly $\tau(\Gamma_1[\tau(\Delta)]) \bullet \tau(\Gamma_2) \vdash \varphi$. By residuation, one gets $\tau(\Gamma_1[\tau(\Delta)]) \vdash \varphi \leftarrow \tau(\Gamma_2)$. By induction hypothesis on Γ_1, one gets $\tau(\Gamma_1[\tau(\Sigma) \wedge \tau(\Delta)]) \vdash \varphi \leftarrow \tau(\Gamma_2)$. By residuation, one gets $\tau(\Gamma_1[\tau(\Sigma) \wedge \tau(\Delta)]) \bullet \tau(\Gamma_2) \vdash \varphi$.

(9.3) $\Gamma[-] = \Gamma_1[-] \oslash \Gamma_2$ or $\Gamma_1 \oslash \Gamma_2[\chi]$. The proof is quite similar to (9.2).

Case 10. $\Gamma \vdash \varphi$ is obtained by ($\oslash$C), ($\oslash$E) or ($\oslash$As). The proof is done by lattice rules in BDFNL. The proof is quite similar to Case 9.

□

6.4.9. Lemma. *If* $\tau(\Gamma) \vdash \varphi$ *is derivable in* $\mathsf{G}_{\mathsf{BDFNL}}$*, then* $\Gamma \vdash \varphi$ *is derivable in* $\mathsf{G}_{\mathsf{BDFNL}}$.

Proof:
By induction on the construction of Γ. The case that Γ is a formula is obvious. Assume $\Gamma = \Gamma_1 \oslash \Gamma_2$. Assume $\tau(\Gamma_1) \wedge \tau(\Gamma_2) \vdash \varphi$. By induction on the construction of a structure Σ one can easily show $\Sigma \vdash \tau(\Sigma)$. Then we have $\Gamma_1 \vdash \tau(\Gamma_1)$ and $\Gamma_2 \vdash \tau(\Gamma_2)$. By $(\vdash \wedge)$, one gets $\Gamma_1 \oslash \Gamma_2 \vdash \tau(\Gamma_1) \wedge \tau(\Gamma_2)$. By (mix), one gets $\Gamma_1 \oslash \Gamma_2 \vdash \varphi$. The case $\Gamma = \Gamma_1 \odot \Gamma_2$ is similar.

□

6.4.10. Theorem. *A consecution* $\Gamma \vdash \varphi$ *is derivable in* $\mathsf{G}_{\mathsf{BDFNL}}$ *if and only if* $\mathbb{BDRG} \models \Gamma \vdash \varphi$.

Proof:
For the 'if' part, assume $\mathbb{BDRG} \models \Gamma \vdash \varphi$. Then $\mathbb{BDRG} \models \tau(\Gamma) \vdash \varphi$. By the completeness of BDFNL, $\tau(\Gamma) \vdash \varphi$ is derivable in BDFNL. By Lemma 6.4.7, $\tau(\Gamma) \vdash \varphi$ is derivable in $\mathsf{G}_{\mathsf{BDFNL}}$. By Lemma 6.4.9, $\Gamma \vdash \varphi$ is derivable in $\mathsf{G}_{\mathsf{BDFNL}}$. For the 'only if' part, assume that $\Gamma \vdash \varphi$ is derivable in $\mathsf{G}_{\mathsf{BDFNL}}$. By Lemma 6.4.8, $\tau(\Gamma) \vdash \varphi$ is derivable in BDFNL. By the completeness of BDFNL, $\mathbb{BDRG} \models \tau(\Gamma) \vdash \varphi$. Therefore $\mathbb{BDRG} \models \Gamma \vdash \varphi$.

□

6.4.2 Extensions

We will now consider some extensions of $\mathsf{S}_{\mathbb{BDI}}$ and their conservative extensions over BDFNL. Given an $\mathcal{L}_\bullet$-sequent $(\sigma)\ \chi \vdash \delta$ the propositional variables occurred in which are among $p_1, \ldots, p_n$, the structural rule corresponding to (σ) is defined as the following rule $(\odot\sigma)$:

$$\frac{\delta[\Delta_1/p_1, \ldots, \Delta_n/p_n] \vdash \varphi}{\chi[\Delta_1/p_1, \ldots, \Delta_n/p_n] \vdash \varphi}(\odot\sigma)$$

where $\delta[\Delta_1/p_1, \ldots, \Delta_n/p_n]$ and $\chi[\Delta_1/p_1, \ldots, \Delta_n/p_n]$ are obtained from δ and χ by substituting Δ_i for p_i uniformly, and substituting $\odot$ for $\bullet$.

6.4.11. Example. For weak Heyting algebras, we have the following structural rules for (tr) and (wl):

$$\frac{\Gamma[(\Lambda \odot \Delta) \odot \Delta] \vdash \varphi}{\Gamma[\Lambda \odot \Delta] \vdash \varphi}(\odot tr), \qquad \frac{\Gamma[\Delta] \vdash \varphi}{\Gamma[\Delta \odot \Sigma] \vdash \varphi}(\odot wl).$$

Let $\mathsf{G}_{\mathsf{RWH}}$ be the Gentzen-style sequent system obtained from $\mathsf{G}_{\mathsf{RBDI}}$ by adding $(\odot tr)$ and $(\odot wl)$. We can get similar sequent rules for sequents in Example 6.3.5 and Genzten-style sequent systems.

For any set of $\mathcal{L}_\bullet$-sequents Ψ, let $\odot\Psi = \{\odot\sigma \mid \sigma \in \Psi\}$ and $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$ be the Gentzen-style sequent system obtained from $\mathsf{G}_{\mathsf{BDFNL}}$ by adding all rules in $(\odot\Psi)$.

6.4.12. Theorem. *For any set of $\mathcal{L}_\bullet$-sequents Ψ, if for every sequent $\chi \vdash \delta \in \Psi$, each propositional variable in χ occurs only once, then* (mix) *is admissible in* $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$.

Proof:
Based on the proof of Theorem 6.4.4, one needs to consider only the case that the right premise of (mix) is obtained by $(\odot\sigma)$. We first apply (mix) to the left premiss of (mix) and the premiss of $(\odot\sigma)$. Then by $(\odot\sigma)$, we get the conclusion of (mix).

□

6.4.13. Remark. The condition that a propositional variable occurs at most once in χ in Theorem 6.4.12 is significant. All sequents in Example 6.3.5 satisfy this condition. When a propositional variable occurs more than once in χ, the proof strategy in Theorem 6.4.12 may not work. For example, consider the the following inverse rule of $(\odot tr)$ which is obtained from $(p \bullet q) \bullet q \vdash p \bullet q$:

$$\frac{\Gamma[\Lambda \odot \Delta[\psi]] \vdash \varphi}{\Gamma[(\Lambda \odot \Delta[\psi]) \odot \Delta[\psi]] \vdash \varphi}(\odot tr\uparrow)$$

and the derivation

$$\frac{\Sigma \vdash \psi \qquad \dfrac{\Gamma[\Lambda \odot \Delta[\psi]] \vdash \varphi}{\Gamma[(\Lambda \odot \Delta[\psi]) \odot \Delta[\psi]] \vdash \varphi}(\odot tr\uparrow)}{\Gamma[(\Lambda \odot \Delta[\Sigma]) \odot \Delta[\psi]] \vdash \varphi}(\text{mix})$$

in which only one occurrence of ψ is mixed. In such a case, we may not be able to push up (mix) to the premiss of $(\odot tr\uparrow)$.

An $\mathcal{L}_\bullet$-sequent $\chi \vdash \delta$ is said to be *good* if each propositional variable occurs at most once in χ. Then we have the following theorem about good sequents:

6.4.14. THEOREM. *For any set of good $\mathcal{L}_\bullet$-sequents Ψ, the following hold:*

(1) $\Gamma \vdash \varphi$ *is derivable in* $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$ *iff* $\mathsf{Alg}^+(\Psi) \models \Gamma \vdash \varphi$.

(2) *if every propositional variable occurred in δ also occurs in χ for each sequent $\chi \vdash \delta$ in Ψ, then $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$ has the subformula property.*

Proof:
The proof of (1) is similar to Theorem 6.4.10. It suffices to show that the algebraic sequent system $\mathsf{BDFNL}(\Psi)$ is equivalent to $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$. For (2), if every propositional variable occurred in δ also occurs in χ, then every subformula of δ is a subformula of χ. Hence the structural rule $(\odot\sigma)$ does not effect on the subformula property.

□

Let Φ be a set of inductive $\mathcal{L}_{\mathrm{SI}}$-sequents. Assume that Ψ is set of $\mathcal{L}_\bullet$-sequent such that $\Phi \equiv_{\mathsf{ALC}} \Psi$. Then the algebraic sequent system $\mathsf{BDFNL}(\Psi)$ is a conservative extension of $\mathsf{S}_{\mathbb{BDI}}(\Phi)$. If Ψ is a set of good $\mathcal{L}_\bullet$-sequent, one gets a Gentzen-style cut-free sequent calculus $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$. Furthermore, if $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$ has the subformula property, we obtain a Gentzen-style cut-free sequent calculus for $\mathsf{S}_{\mathbb{BDI}}(\Phi)$ if we omit rules for $\bullet$ and $\leftarrow$ from $\mathsf{G}_{\mathsf{BDFNL}}(\odot\Psi)$.

Table 6.5: Gentzen-style Sequent Calculi

Strict Implication Logic	Conservative Extension
G_{WH}	$\mathsf{G}_{\mathsf{RWH}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot wl) + (\odot tr)$
G_{T}	$\mathsf{G}_{\mathsf{RT}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot ct)$
G_{W}	$\mathsf{G}_{\mathsf{RW}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot wr)$
G_{RT}	$\mathsf{G}_{\mathsf{RRT}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot rt)$
G_{B}	$\mathsf{G}_{\mathsf{RB}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot b)$
$\mathsf{G}_{\mathsf{B'}}$	$\mathsf{G}_{\mathsf{RB'}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot b')$
G_{C}	$\mathsf{G}_{\mathsf{RC}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot c)$
G_{FR}	$\mathsf{G}_{\mathsf{RFR}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot fr)$
$\mathsf{G}_{\mathsf{W'}}$	$\mathsf{G}_{\mathsf{RW'}} = \mathsf{G}_{\mathsf{BDFNL}} + (\odot w')$
$\mathsf{G}_{\mathsf{BCA}}$	$\mathsf{G}_{\mathsf{RBCA}} = \mathsf{G}_{\mathsf{T}} + (\odot w)$
G_{KT}	$\mathsf{G}_{\mathsf{RKT}} = \mathsf{G}_{\mathsf{RWH}} + (\odot ct)$
G_{K4}	$\mathsf{G}_{\mathsf{RK4}} = \mathsf{G}_{\mathsf{RWH}} + (\odot rt)$
G_{S4}	$\mathsf{G}_{\mathsf{RS4}} = \mathsf{G}_{\mathsf{RKT}} + (\odot rt)$
G_{KW}	$\mathsf{G}_{\mathsf{RKW}} = \mathsf{G}_{\mathsf{WH}} + (\odot w)$

For example, the algebraic correspondents in Table 6.4 are good $\mathcal{L}_\bullet$-sequents. Then we get Gentzen-style sequent calculi in Table 6.5 for residuated BDIs defined by the corresponding $\mathcal{L}_\bullet$-sequents. These calculi admit (mix) and have the subformula property.

6.4.3 Comparison with literature

Our framework in the present chapter is to apply unified correspondence theory to proof theory of strict implication logics. The sequent calculi developed for conservative extensions are Gentzen-style. This framework is quite different from the approaches in literature. Here we compare some sequent calculi for strict implication logics in literature with these calculi listed in Table 6.5.

Two types of calculi for non-classical logics in literature are distinguished by Alenda, Olivetti and Pozzato [5]:

> "Similarly to modal logics and other extensions/alternative to classical logics two types of calculi: *external* calculi which make use of labels and relations on them to import the semantics into the syntax, and *internal* calculi which stay within the language, so that a configuration' (sequent, tableaux node ...) can be directly interpreted as a formula of the language." [5, p.15]

Obviously the sequent calculi developed in the present chapter are *internal* because every structure in an $\mathcal{L}_{\mathrm{LC}}$-sequent is directly translated into an $\mathcal{L}_{\mathrm{LC}}$-formula. Ishigaki and Kashima [120] also developed internal sequent calculi for some strict implication logics, but we have mentioned the advantages of our approach in Section 6.1.

External calculi for strict implication logics are also developed in literature. Labelled sequent calculi for intermediate logics are developed by Dyckhoff and Negri [71], and their connections with Hilbert axioms and hypersequents are investigated by Ciabattoni et al [38]. In this approach, any intermediate logic characterized by a class of relational frames that is definable by first-order *geometric axioms*[9], can be formalized in a cut-free and contraction-free labelled sequent calculus that extends the labelled sequent calculus for intuitionistic logic with geometric rules transformed from these geometric axioms. Using the same approach, Yamasaki and Sano [197] developed labelled sequent calculi for some subintuitionistic logics [62].

The development of an external calculus for a strict implication logic depends on that the logic has geometric relational semantics, i.e., it is sound and complete with respect to a class of relational frames which is definable by a set of geometric theories. Our internal calculi for strict implication logics are developed for subvarieties of BDI algebras and they do not necessarily have relational semantics. The strict implication logic $\mathbf{S}_{\mathbb{BDI}}$ is indeed an example without binary relational semantics.

The algorithm ALBA, one of the main tools in unified correspondence theory, is applied in the present chapter to the proof theory of strict implication logics. Firstly, it

[9] A geometric axiom is a first-order formula of the form $\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \supset \exists \overline{x}(M_1 \vee \ldots \vee M_n))$ where each P_i is an atomic formula, and each M_j is a conjunction of atomic formulas, and $\overline{z}$ and $\overline{x}$ are sequences of bounded variables. Each geometric axiom can be transformed into a geometric rule [71].

is used as a tool to calculate the first-order correspondents of inductive $\mathcal{L}_{\mathrm{SI}}$-sequents. If the correspondents of a set of inductive sequents are geometric axioms, they can be transformed into geometric rules, and hence some labelled sequent calculi for strict implication logics can be developed. It is unknown if ALBA can capture all geometric axioms. A general converse correspondence theory is unknown yet. Secondly, our novel application of ALBA is to calculate the algebraic correspondents of some inductive $\mathcal{L}_{\mathrm{SI}}$-sequents in the language $\mathcal{L}_\bullet$. A proof-theoretic consequence of this application is that one can obtain mix-free internal sequent calculi for the conservative extensions of some strict implication logics. However, the systematic connections between algebraic and first-order correspondents is unknown.

Our framework in the present chapter may not be able to cover all such logics which have binary relational semantics. Consider strict implication logics containing (Sym) or (Euc) based on $\mathbf{S}_{\mathbb{WH}}$ in Table 6.5. Since (Sym) and (Euc) may not have correspondent in $\mathcal{L}_\bullet$, these logics may not admit Gentzen-style sequent calculi that are obtained from G_{WH} by adding structural rules about $\odot$. Another example is Visser's logic FPL (formal provability logic) [193] which is a strict implication logic that extends basic propositional logic with the Löb's axiom $(q \to p) \to p \vdash q \to p$. This axiom is not inductive. A labelled sequent calculus may be developed for FPL because it has binary relational semantics. But it is impossible to develop a Gentzen-style sequent calculus for it in our framework.

6.5 Conclusion

The present work studies the proof theory for strict impaction logic using unified correspondence theory as a proof-theoretic tool. First of all, we present general results about the semantic conservativity on DLE-logics via canonical extension. A consequence is that the strict implication logic $\mathsf{S}_{\mathsf{BDI}}$ is conservatively extended to the Lambek calculus BDFNL. The algorithm ALBA as a calculus for correspondence between DLE-logic and first-order logic and hence for canonicity, is specialized to the strict implication logic and Lambek calculus. The main contribution of the present chapter is that we obtain an Ackermann lemma based calculus ALC from the algorithm ALBA as a tool for proving algebraic correspondence between a wide range of strict implication sequents and sequents in the language $\mathcal{L}_\bullet$. This tool gives not only more conservativity results, but also analytic rules needed for introducing the Gentzen-style cut-free sequent calculi. Another contribution is that we introduce a Gentzen-style sequent calculus for BDFNL and some of its extensions with analytic rules.

The final remark is about good $\mathcal{L}_\bullet$-sequents that are used for obtaining cut-free sequent calculus. It is very likely that a hiearchy of $\mathcal{L}_\bullet$-sequents from which one obtains analytic rules can be established. Other connectives $\wedge, \vee$ and $\to$ can be in principle added into the language $\mathcal{L}_\bullet$ such that more analytic rules will be obtained. This is our work in progress. Moreover, our approach to the proof theory of strict implication may be generalized to arbitrary DLE-logics.

6.6 Appendix: algebraic correspondence

(I)

$$\begin{array}{cl}
\Rightarrow q \vdash p \to p & \\ \hline
\Rightarrow p \bullet q \vdash p & \text{RR1}
\end{array}$$

(MP)

$$\begin{array}{cl}
\Rightarrow p \wedge (p \to q) \vdash q & \\ \hline
r \vdash p \wedge (p \to q) \Rightarrow r \vdash q & \text{Ap1} \\ \hline
r \vdash p, r \vdash p \to q \Rightarrow r \vdash q & \wedge\text{S} \\ \hline
r \vdash p, p \bullet r \vdash q \Rightarrow r \vdash q & \text{RL1} \\ \hline
p \bullet p \vdash q \Rightarrow p \vdash q & \text{LAck} \\ \hline
\Rightarrow p \vdash p \bullet p & \text{RAck}
\end{array}$$

(W)

$$\begin{array}{cl}
\Rightarrow p \vdash q \to p & \\ \hline
\Rightarrow q \bullet p \vdash p & \text{RR1}
\end{array}$$

(RT)

$$\begin{array}{cl}
\Rightarrow p \to q \vdash r \to (p \to q) & \\ \hline
s \vdash p \to q \Rightarrow s \vdash r \to (p \to q) & \text{Ap1} \\ \hline
p \bullet s \vdash q \Rightarrow s \vdash r \to (p \to q) & \text{RL1} \\ \hline
p \bullet s \vdash q \Rightarrow r \bullet s \vdash p \to q & \text{RR1} \\ \hline
p \bullet s \vdash q \Rightarrow p \bullet (r \bullet s) \vdash q & \text{RR1} \\ \hline
\Rightarrow p \bullet (r \bullet s) \vdash p \bullet s & \text{RAck}
\end{array}$$

(B)

$$\begin{array}{cl}
\Rightarrow p \to q \vdash (r \to p) \to (r \to q) & \\ \hline
s \vdash p \to q \Rightarrow s \vdash (r \to p) \to (r \to q) & \text{Ap1} \\ \hline
p \bullet s \vdash q \Rightarrow s \vdash (r \to p) \to (r \to q) & \text{RL1} \\ \hline
p \bullet s \vdash q \Rightarrow (r \to p) \bullet s \vdash r \to q & \text{RR1} \\ \hline
p \bullet s \vdash q \Rightarrow r \bullet ((r \to p) \bullet s) \vdash q & \text{RR1} \\ \hline
\Rightarrow r \bullet ((r \to p) \bullet s) \vdash p \bullet s & \text{RAck} \\ \hline
\Rightarrow (r \to p) \bullet s \vdash r \to (p \bullet s) & \text{RR1} \\ \hline
\Rightarrow r \to p \vdash (r \to (p \bullet s)) \leftarrow s & \text{RR2} \\ \hline
t \vdash r \to p \Rightarrow t \vdash (r \to (p \bullet s)) \leftarrow s & \text{Ap1} \\ \hline
r \bullet t \vdash p \Rightarrow t \vdash (r \to (p \bullet s)) \leftarrow s & \text{RL1} \\ \hline
r \bullet t \vdash p \Rightarrow s \bullet t \vdash r \to (p \bullet s) & \text{RR2} \\ \hline
r \bullet t \vdash p \Rightarrow r \bullet (s \bullet t) \vdash p \bullet s & \text{RR1} \\ \hline
\Rightarrow r \bullet (s \bullet t) \vdash (r \bullet t) \bullet s & \text{RAck}
\end{array}$$

(B′)

$$
\begin{array}{cl}
\Rightarrow p \to q \vdash (q \to r) \to (p \to r) & \\
\cline{1-1}
s \vdash p \to q \Rightarrow s \vdash (q \to r) \to (p \to rt) & \text{Ap1} \\
\cline{1-1}
p \bullet s \vdash q \Rightarrow s \vdash (q \to r) \to (p \to r) & \text{RL1} \\
\cline{1-1}
p \bullet s \vdash q \Rightarrow (q \to r) \bullet s \vdash p \to r & \text{RR1} \\
\cline{1-1}
p \bullet s \vdash q \Rightarrow p \bullet ((q \to r) \bullet s) \vdash r & \text{RR1} \\
\cline{1-1}
\Rightarrow p \bullet ((p \bullet s \to r) \bullet s) \vdash r & \text{RAck} \\
\cline{1-1}
\Rightarrow (p \bullet s \to r) \bullet s \vdash p \to r & \text{RR1} \\
\cline{1-1}
\Rightarrow p \bullet s \to r \vdash (p \to r) \leftarrow s & \text{RR2} \\
\cline{1-1}
t \vdash p \bullet s \to r \Rightarrow t \vdash (p \to r) \leftarrow s & \text{Ap1} \\
\cline{1-1}
(p \bullet s) \bullet t \vdash r \Rightarrow t \vdash (p \to r) \leftarrow s & \text{RL1} \\
\cline{1-1}
(p \bullet s) \bullet t \vdash r \Rightarrow t \bullet s \vdash p \to r & \text{RR2} \\
\cline{1-1}
(p \bullet s) \bullet t \vdash r \Rightarrow p \bullet (t \bullet s) \vdash r & \text{RR1} \\
\cline{1-1}
\Rightarrow p \bullet (t \bullet s) \vdash (p \bullet s) \bullet t & \text{RAck}
\end{array}
$$

(C)

$$
\begin{array}{cl}
\Rightarrow p \to (q \to r) \vdash q \to (p \to r) & \\
\cline{1-1}
s \vdash p \to (q \to r) \Rightarrow s \vdash q \to (p \to r) & \text{Ap1} \\
\cline{1-1}
p \bullet s \vdash q \to r \Rightarrow s \vdash q \to (p \to r) & \text{RL1} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r \Rightarrow s \vdash q \to (p \to r) & \text{RL1} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r \Rightarrow q \bullet s \vdash p \to r & \text{RR1} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r \Rightarrow p \bullet (q \bullet s) \vdash r & \text{RR1} \\
\cline{1-1}
\Rightarrow p \bullet (q \bullet s) \vdash q \bullet (p \bullet s) & \text{RAck}
\end{array}
$$

(Fr)

$$
\begin{array}{cl}
\Rightarrow p \to (q \to r) \vdash (p \to q) \to (p \to r) & \\
\cline{1-1}
s \vdash p \to (q \to r), (p \to q) \to (p \to r) \vdash t \Rightarrow s \vdash t & \text{(Ap1, Ap2)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, (p \to q) \to (p \to r) \vdash t \Rightarrow s \vdash t & \text{(RL1)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, u \vdash p \to q, p \to r \vdash v, u \to v \vdash t \Rightarrow s \vdash t & (\to\text{Ap1}, \to\text{Ap2}) \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, p \bullet u \vdash q, p \to r \vdash v, u \to v \vdash t \Rightarrow s \vdash t & \text{(RL1)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, p \bullet u \vdash q, p \to r \vdash v \Rightarrow s \vdash u \to v & \text{(AAP2)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, p \bullet u \vdash q, p \to r \vdash v \Rightarrow u \bullet s \vdash v & \text{(RR1)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, p \bullet u \vdash q \Rightarrow u \bullet s \vdash p \to r & \text{(AAp2)} \\
\cline{1-1}
q \bullet (p \bullet s) \vdash r, p \bullet u \vdash q \Rightarrow p \bullet (u \bullet s) \vdash r & \text{(RR1)} \\
\cline{1-1}
p \bullet u \vdash q \Rightarrow p \bullet (u \bullet s) \vdash q \bullet (p \bullet s) & \text{(AAp2)} \\
\cline{1-1}
\Rightarrow p \bullet (u \bullet s) \vdash (p \bullet u) \bullet (p \bullet s) & \text{(RAck)}
\end{array}
$$

(W′)

$$
\begin{array}{cl}
\Rightarrow p \to (p \to q) \vdash p \to q & \\
\cline{1-1}
r \vdash p \to (p \to q) \Rightarrow r \vdash p \to q & \text{Ap1} \\
\cline{1-1}
p \bullet r \vdash p \to q \Rightarrow r \vdash p \to q & \text{RL1} \\
\cline{1-1}
p \bullet (p \bullet r) \vdash q \Rightarrow r \vdash p \to q & \text{RL1} \\
\cline{1-1}
p \bullet (p \bullet r) \vdash q \Rightarrow p \bullet r \vdash q & \text{RR1} \\
\cline{1-1}
\Rightarrow p \bullet r \vdash p \bullet (p \bullet r) & \text{RAck}
\end{array}
$$

Chapter 7

Conclusion

7.1 Summary of the thesis

The results of this dissertation pertain to a line of research dubbed ‘unified correspondence theory’, which investigates the phenomenon of correspondence and canonicity uniformly for a wide class of propositional logics using algebraic tools. The contributions of this dissertation are summarized below:

- In Chapter 3, we develop a unified correspondence treatment of the Sahlqvist theory for possibility semantics, extending the results in [196] from Sahlqvist formulas to the strictly larger class of inductive formulas, and from the full possibility frames to filter-descriptive possibility frames. Specifically, we define the possibility semantics version of the algorithm ALBA, and an adapted interpretation of the expanded modal language used in the algorithm. We prove the soundness of the algorithm with respect to both (the dual algebras of) full possibility frames and (the dual algebras of) filter-descriptive possibility frames. We make some comparisons among different semantic settings in the design of the algorithms, and fit possibility semantics into this broader picture.

- In Chapter 4, we use the algorithm ALBA to reformulate the proof in [14] and [19] that over modal compact Hausdorff spaces, the validity of Sahlqvist sequents are preserved from open assignments to arbitrary assignments. In particular, we prove an adapted version of the topological Ackermann lemma based on the Esakia-type lemmas proved in [14] and [19].

- In Chapter 5, we examine whether the alternative route ‘via translation’ could be effective for obtaining Sahlqvist-type results of comparable strength for nonclassical logics. This route consists in suitably embedding nonclassical logics into classical polyadic modal logics via some Gödel-type translations, and then obtaining Sahlqvist theory for nonclassical logics as a consequence of Sahlqvist theory of classical polyadic modal logic. We analyze the power and limits of this alternative route for logics algebraically captured by normal distributive lattice expansions, and various sub-varieties thereof. Specifically, we provide a new proof, 'via translation' of the correspondence theorem for inductive inequalities of arbitrary signatures of normal distributive lattice expansions. We also show that canonicity-via-translation can be obtained in a similarly straightforward manner, but only for normal modal expansions of *bi-intuitionistic logic*. We also provide a detailed explanation of the difficulties involved in obtaining canonicity-via-translation outside this setting.

- In Chapter 6, we specialize unified correspondence theory to strict implication logics and apply it to the proof theory of these logics. We conservatively extend a wide range of strict implication logics to Lambek Calculi over the bounded distributive full non-associative Lambek calculus (BDFNL) as a consequence of a general semantic consevativity result. By a suitably modified version of the Ackermann lemma based algorithm ALBA, we transform many strict implication sequents into analytic rules employing one of the main tools of unified correspondence theory, and develop Gentzen-style cut-free sequent calculi for BDFNL and its extensions with analytic rules which are transformed from strict implication sequents.

7.2 Methological reflections

Our methodology. The main focus of this thesis is on the methodology, which is based on algebraic and order-theoretic notions and duality-theoretic insights. Mathematical dualities link algebras with their dual topological spaces, and transform definitions and ideas from one side to another. By duality-theoretic methods, it is possible to use the insights from the algebraic side to solve problems on the topological space side. The beneficial point of using algebraic method is that it is easy to generalize results to broader settings in a modular way, and it also provides insight to proof-theoretic problems and is useful in the formulation and proof of constructive results.

Algebraic correspondence theory and duality. Correspondence theory is the reflection into logic of mathematical dualities. The phenomenon of Sahlqvist correspondence and canonicity is in its essence algebraic and order-theoretic, and the definitions of Sahqvist formulas and inequalities can be captured uniformly in every logical signature and purely depend on the order-theoretic properties of the algebraic interpretations of the logical connectives, which thanks to the generality of algebraic method. The algorithm ALBA, originally defined for the purpose of effectively computing the first-order correspondents of input formulas and inequalities, has a much wider range of applicability than its original purpose. Its applicability is independent from the existence of a relational semantics and from the choice of the relational semantics in case more than one exists for a given logic, and invests independently motivated issues such as the design of sequent calculi in structural proof theory.

Duality and correspondence in other fields. The interplay between duality and correspondence is not only useful to develop the logics themselves, but also provides a new approach to problems in social choice theory and categorization theory. Thanks to dualities, important research questions can be formalized and the theories can be improved.

- *Duality and Social Choice.* Social choice theory is the study of collective decision making. The most famous problem is Arrow's preference aggregation problem of how individual preferences can be aggregated into a group preference in a rational way. In recent years, logic has shown its usefulness in generalizing Arrow-type impossibility results and providing new insights. The ultrafilter argument makes it possible to generalize the Arrow-type impossibility results, and duality theory is the natural mathematical tools behind it. The fact that ultrafilters are the semantic counterpart of maximally consistent sets of propositions guarantees that ultrafilters have exactly the properties used in the study of aggregation problems. This fact is the keypoint for the relationship among logical languages, algebraic semantics and model-theoretic semantics, which duality theory studies.

- *Correspondence theory and categorization theory.* Categories are understood as classes of objects sharing similar properties. In [154], a multimodal language is defined and used to reason about perception, default, and belief. Concepts in categorization theory like legitimation and identity, can then be studied as consequences of the perceptions, defaults and beliefs of agents. In [154], a model is built to represent processes of perception and belief, which are essentially correspondence results. Therefore, by applying correspondence theory, we can extend and sharpen the model building techniques.

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Samenvatting

Correspondentietheorie is de studie van de relatie tussen formules uit de modale logica en formules uit de eerste-orde logica geinterpreteerd over Kripke-frames. We zeggen dat een modale formule en een eerste-orde formule met elkaar corresponderen indien zij geldig zijn op dezelfde klasse van Kripke-frames. Canoniciteitstheorie is nauw verbonden met correspondentietheorie. We noemen een formule uit de modale logica canoniek indien die geldig is op haar canoniek frame of, in andere woorden, indien haar geldigheid behouden blijft onder uitbreiding van een modale algebra naar haar canonieke extensie, of van een descriptief algemeen frame (descriptive general frame) naar het onderliggend Kripke-frame. Canoniciteit is sterk gerelateerd aan het concept van volledigheid (completeness) uit de logica. Indien een modale formule canoniek is, dan geldt dat de normale modale logica -die door deze formule geaxiomatiseerd wordt -volledig is, met betrekking tot de klasse van Kripke-frames die erdoor wordt gedefinieerd.

In de ontwikkeling van correspondentietheorie krijgen algoritmische aspecten meer en meer aandacht. De Sahlqvist-van Benthem stelling voorziet in een algoritme om een klasse van modale formules, die later Sahlqvist formules genoemd werden, in hun corresponderende eerste-orde formule te transformeren. Het algoritme genaamd SQEMA geeft een procedure om een formule uit de modale logica eerst om te zetten in een zuivere modale formule behorende tot een taalextensie van de oorspronkelijke taal, om deze laatste tenslotte in een eerste-orde expressie te transformeren. Het SQEMA-algoritme werkt voor een strikt grotere klasse van modale formules, die inductief worden genoemd.

Geünificeerde correspondentietheorie is recentelijk gebaseerd op dualiteitstheorie en op orde-algebraische inzichten. Deze benaderingswijze staat ons toe een syntactische definitie van Sahlqvist- en inductieve formules te geven die uniform toepasbaar is op iedere logische signatuur, en die bovendien uitsluitend afhangt van de orde-theoretische eigenschappen van de algebraische interpretaties der logische connectieven. Daarbovenop beschikken wij over het ALBA-algoritme, dat gebaseerd is op het lemma van Ackermann en dat een generalisatie is van het SQEMA-algoritme op basis van orde-theoretische en algebraische inzichten. Dankzij dit algoritme kunnen eerste-orde correspondenten worden berekend van input formules/ongelijkheden die gegarandeerd succesvol zijn op Sahlquist- en inductieve klassen van formules/ongelijkheden.

Deze dissertatie behoort tot het onderzoeksgebied van de correspondentietheorie. In Hoofdstuk 3 passen we de methodologie van correspondentietheorie toe op mogelijkheidssemantiek (possibility semantics) en geven we alternatieve bewijzen voor de correspondentieresultaten à la Sahlqvist zoals eerder uiteengezet in [196]. Bovendien geven we een generalisatie van die resultaten, van Salhqvistformules naar de strikt grotere klasse van inductieve formules, en van de volledige-mogelijkheidsframes (full possibility frames) tot de filterdescriptieve mogelijkheidsframes (filter-descriptive possibility frames). Hoofdstuk 4 past correspondentietheorie toe op modale compacte Hausdorff-ruimtes en geeft alternatieve bewijzen voor het type van canoniciteitspreservatie-resultaten zoals gegeven in [14]. In Hoofdstuk 5 gaan we de toepassingskracht- en beperkingen na van deze vertaalmethode bij het verkrijgen van correspondentie- en canoniciteitsresultaten. Hoofdstuk 6 gaat over een toepassing van geünificeerde correspondentietheorie op de bewijstheorie van strikte-implicatielogicas (strict implication logics). Dit toont aan hoe geünificeerde correspondentietheorie nuttig kan zijn in het construeren van analytische Gentzencalculi, bij het bijzonder wanneer het gaat om het berekenen van de corresponderende analytische bewijstheoretische regel (rule) van een gegeven sequent.

Abstract

Correspondence theory originally arises as the study of the relation between modal formulas and first-order formulas interpreted over Kripke frames. We say that a modal formula and a first-order formula correspond to each other if they are valid on the same class of Kripke frames. Canonicity theory is closely related to correspondence theory. We say that a modal formula is canonical if it is valid on its canonical frame, or equivalently,if its validity is preserved from a modal algebra to its canonical extension, or from a descriptive general frame to its underlying Kripke frame. Canonicity is closely related to completeness. If a modal formula is canonical, then the normal modal logic axiomatized by this modal formula is complete with respect to the class of Kripke frames defined by it.

In the development of correspondence theory, the algorithmic aspect receives increasing attention. The Sahlqvist-van Benthem theorem provides an algorithm to transform a class of modal formulas, which are later called Sahlqvist formulas, into their corresponding first-order formulas. The algorithm SQEMA provides a modal language-based algorithm to transform a modal formula into a pure modal formula in an expanded language, and then translate the pure modal formula into the first-order language. SQEMA succeeds on a strictly larger class of modal formulas, which are called inductive formulas.

In recent years, unified correspondence theory is developed based on duality-theoretic and order-algebraic insights. In this approach, a very general syntactic definition of Sahlqvist and inductive formulas is given, which applies uniformly to each logical signature and is given purely in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives. In addition, the Ackermann lemma based algorithm ALBA, which is a generalization of SQEMA based on order-theoretic and algebraic insights, is given, which effectively computes first-order correspondents of input formulas/inequalities, and is guaranteed to succeed on the Sahlqvist and inductive classes of formulas/inequalities.

This dissertation belong to the line of research of unified correspondence theory. Chapter 3 applies the unified correspondence methodology to possibility semantics,

and gives alternative proofs of Sahlqvist-type correspondence results to the ones in [196], and extends these results from Sahlqvist formulas to the strictly larger class of inductive formulas, and from the full possibility frames to filter-descriptive possibility frames. Chapter 4 applies the unified correspondence methodology to modal compact Hausdorff spaces, and gives alternative proofs of canonicity-type preservation results to the ones in [14]. Chapter 5 examines the power and limits of the translation method in obtaining correspondence and canonicity results. Chapter 6 is about an application of unified correspondence theory to the proof theory of strict implication logics, showing the usefulness of unified correspondence theory in the design of analytic Gentzen sequent calculi, especially when it comes to computing the corresponding analytic rules of a given sequent.

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(2) **Constructive Canonicity for Lattice-based Fixed Point Logics**, with Willem Conradie, Andrew Craig and Alessandra Palmigiano, in Logic, Language, Information, and Computation - 24th International Workshop, WoLLIC 2017, LNCS 10388, pp 92-109 (2017)

(3) **Universal Models for the Positive Fragment of Intuitionistic Logic**, with Nick Bezhanishvili, Dick de Jongh, Apostolos Tzimoulis, in Logic, Language, and Computation: 11th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2015, LNCS 10148, pp.229-250 (2017)

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