Geometry and Algebra

Relating axioms for plane geometry to the field axioms

by

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Summary for laypersons

Before the Renaissance, geometry and arithmetic were considered separate branches of mathematics. In the Renaissance these two were united, this union has lead to major advances in mathematics and its applications. Numbers can be used in geometry by means of a coordinate system. In this thesis the relation between geometrical axioms and axioms for fields (generalised number systems that make up a coordinate system) are investigated. It is shown how to construct a coordinate system based solely on geometrical concepts and that this coordinate system is isomorphic (i.e. it has the same form, all geometric terms and relations can be expressed in the coordinate system) to the original plane. Additionally, it is shown that there are geometric axioms that have a direct counterpart in the field axioms.

Summary

In 1908, the mathematician Felix Klein published a book *Elementary Mathematics from an Advanced Standpoint: Geometry*. This title aptly characterizes the focus of this thesis. This thesis introduces the axioms for Euclidean and projective plane geometry. Afterwards an arithmetic of lengths, based solely on these axioms, is constructed. By establishing a connection between the geometric axioms and the field axioms, it is demonstrated that these lengths form a field. It is shown that the original geometric plane is isomorphic to the Cartesian plane of lengths. Additionally, the thesis highlights the direct relation between two geometric propositions, Pappos' theorem and Desargues' theorem, and commutativity and associativity of the induced field of lengths.

Contents

Introduction

Mathematics, as a formal scientific discipline, originated in Greece around 600 BC. Mathematical problems had been studied and solved by multiple civilisations for centuries at that time, originating from practical day-to-day situations like measuring land, calculating taxes or creating calendars. The Greeks studied such problems systematically and took mathematics further than anyone else had done. In the (relatively) short period of about 400 years, they laid the foundations of mathematics as a scientific discipline. They identified the basic branches of mathematics and established important results in all of them. Moreover, they developed a method of proving mathematical propositions that remains at the heart of mathematics: the axiomatic method, that starts with axioms and demonstrates its results on the exclusive basis of deductions from these axioms.

One of several climaxes of Greek mathematics is Euclid's *Elements*, written around 300 BC. Its influence on the development and study of mathematics is unrivalled, even into the 20th century the *Elements* was the standard textbook to teach mathematics from[1] and its ability to demonstrate profound results through a ruthless adherence to logical reasoning served as inspiration for humankind's greatest scientists.

If a contemporary reader opens the *Elements*, they might be surprised to see that there are no numbers anywhere in Euclid's books on geometry. Euclid, and the Greeks more broadly, did not mix geometry and numbers in theoretical mathematics. Geometry dealt with magnitudes in space, which are continuous and (infinitely) divisible, whereas numbers dealt with multitudes, which are discrete and indivisible. Their concept of 'number' included only counting numbers.

They were of course aware that these fields are related. They identified Pythagorean triplets like $3^2 + 4^2 = 5^2$ and they also used numbers for counting within geometry, for example by considering a line segment that is three times (or any other multitude) that of another. Also in practical applications, numbers were used to measure lengths, areas and time. But within geometry as a theoretical discipline, there were no numerical measurements. In geometry, things were measured (i.e., quantitatively related to each other) by means of lines and circles.

In modern geometry, the numerical and geometrical perspectives are deeply intertwined. The perspective of $y = x^2$ as a parabola is just as familiar, maybe even more familiar, than the Greek perspective of a parabola as a certain intersection of a cone and a plane.

Demonstrating geometrical propositions directly by reasoning about points, lines and shapes is done by means of the axiomatic approach. In this thesis, I will relate this to the practice of solving geometrical problems by means of numbers and algebraic calculation (coordinate geometry). After giving a general introduction to geometry in chapter 1, I will spend three chapters on Euclidean geometry. I will start with the axiomatic approach in chapter 2, I will construct an arithmetic of lengths on the basis of such axioms in chapter 3 and I will relate this arithmetic to coordinate geometry in chapter 4. Afterwards I will go through these steps again for projective geometry in chapters 5 to 7.

The fundamental structure of this thesis and many of the propositions demonstrated for Euclidean geometry are based on Hartshorne's *Geometry: Euclid and Beyond*. Hartshorne uses Hilbert's axioms for Euclidean geometry and sometimes I will make use of the primary source: Hilbert's *Foundations of Geometry*. For projective geometry I made use of Stillwell's *The Four Pillars of Geometry* and Heyting's *Axiomatic projective geometry*. For the historical aspects, I primarily used Kline's *Mathematical Thought from Ancient to Modern Times*.

[1](Wilson, 2009, p.278)

1

What is geometry?

"And *geometry* can boast that with so few principles obtained from other fields, it can do so much."

— Isaac Newton

1.1. Position, shape and geometry

Before discussing geometric axioms, let us consider what geometry is. I do not intend to give a comprehensive overview or a full answer, nor do I indent to present all the different historical views. The goal is to present some ideas that will aid our investigation and form the context for the later chapters. I will start with the view of geometry before 19^{th} century developments and use that as a framework to discuss some important changes in the 19^{th} and 20^{th} century that formed the modern field.

Let us take an Aristotelian approach and start with what we see when we open our eyes. We observe entities interacting with their surroundings: some move, some grow and some dissolve. Entities have two attributes relevant to geometry: position and shape. If I take a book from the shelf and put it on a table, then it changes position but not shape; if I mold a piece of clay on my desk, then it changes shape but not position.

Position Aristotle identified that position is a relational concept. (He used 'place' instead of 'position', I take these to be the same thing.) He thought that an entity's position consists, not of something within the entity, but of something outside the entity: its relation to the surrounding entities.[1] For example, the position of my book is on the shelf in between these other two books.

For Aristotle, there is no such a *thing* as space. 'Space' is just the totality of all positions. We can use space to refer to such a totality, but we should recognize that it is not a 'thing' (the view of space as a container) but a relation among things. Fundamental concepts in the study of space are those of distance and direction, which relate two positions to each other.

Shape What about shape? Shape too is an aspect of entities. It is not the entity as a whole — a book has a publication date and an owner, but these are not part of its shape — but only one central aspect that determines many of its physical properties. Important concepts related to shapes are: solid, surface, line, angle, size, etc…

Aristotle thought that a point is 'a unit having position'.[2] It cannot be analysed further by breaking it down into different shapes. Line segments, surfaces or figures like triangles also have a position but can also be broken down into other shapes.

The traditional realist view is that geometry studies the properties of geometrical shapes in position. Two different views were offered on the nature of these shapes. Plato thought that they existed as actual

[1](Morison, 2002)
[2](Mendell, 2019, sec.7)

objects in another realm and that the shapes we observe around us are merely imperfect reflections of those objects. Aristotle thought that they were aspects of actual entities and that they have no separate existence outside these entities. A physical stroke on paper can be considered *qua* line by considering its extension in one direction at a given position and considering its width and the molecular departures from straightness as immaterial. This does not imply that there exist strokes without any width in reality, or more broadly that geometrical characteristics exist independent from their objects in reality. It is a selective focus on some characteristics as against others. For Aristotle, geometrical shapes exist, but they are actual shapes considered from an abstract perspective.

There were fierce disagreements between Platonists and Aristotelians on the reality of geometry, but there was agreement that, whatever the ultimate nature of geometry was, geometric axioms were fundamental and indubitable truths that described it and geometry concerned itself with studying geometrical objects.

A major problem with this view becomes evident when we consider where geometrical shapes come from. Geometrical shapes are not found in nature, they are found in human civilization. Triangles, cylinders and rectangles are the shapes that make up buildings, phones and kitchenware, not forests, rivers or rocks.

Why do we create them? Two reasons: to measure other objects, the straight line is the shape of a ruler and the square is the shape by which we measure areas; or when designing man-made objects. When creating objects with geometrical shapes, the role of geometry is that of demonstrating that a given shape has a desirable quantitative property. If we want a heating pipe to lose as little heat as possible, a circular shape makes sense, as a circle maximizes the area for a given perimeter (a geometrical proposition). If we want to fit as many containers as possible in a ship then it makes sense to make them rectangular and not circular.

Geometry, properly speaking, is not about describing shapes, but about measuring them. It bases itself on facts about space, but geometry does not give an account of the shapes around us; it creates shapes, with demonstrable characteristics, to measure the shapes around us. Other sciences then use these concepts to measure, among other things, the distances between cities, the size of the earth and the motion of the planets.

Geometry is not about the *physical* process of measurement, like how to use a ruler, draw a circle or use a telescope, it is about the *cognitive* process of measurement: the use of some physical measurements to infer others that are not measured directly. Newton wrote the following about how these two relate:

> "To describe straight lines and to describe circles are problems, but not problems in *geometry*. *Geometry* postulates the solution of these problems from *mechanics* and teaches the use of the problems thus solved. And *geometry* can boast that with so few principles obtained from other fields, it can do so much."[3]

This view of geometry that I just described bases itself completely on geometry before 19^{th} century developments. I chose to start here because I think that it is the proper context for understanding the transformative developments of geometry in the 19^{th} century.

1.2. Non-Euclidean geometry

One of Euclid's axioms stood out from the rest and had been scrutinized thoroughly from the start: the parallel postulate.

> "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."[4]

Its form seems more like that of a theorem, not an axiom. For centuries mathematicians had tried to derive — and some had claimed to derive — the parallel postulate from Euclid's other axioms. This turned out to be in vain, in the 19^{th} century mathematicians developed a model of non-Euclidean geometry. In such a model all Euclid's axioms hold, except the parallel postulate. Though properties

[3] (Newton, 1999, p.382)
[4] (Euclid, 2008, p.7)

of figures in this model were counterintuitive — for example, triangles having an angle sum larger or smaller than two right angles — they were not contradictory. This development caused people to challenge the fundamentality of Euclidean geometry.

Gauss recognized that the question of whether space is Euclidean or non-Euclidean is a *physical* question, to be determined by experiment, not something that can be established by mathematical reasoning.[5] If you thought that geometry describes objects in space and/or that Euclidean geometry is in some sense the intrinsic nature of space, à la Kant, then the development of non-Euclidean geometry is a major blow to your worldview. One of the indubitable truths on which you have based all your reasoning turns out to be not so fundamental and its denial gives rise to interesting mathematics.

But if you think that geometry is about the measurement of shapes, then the axioms are properly speaking not truths, they are a starting point for logical inference. Euclid's axioms already express this attitude. His Postulate 1 reads: "to draw a straight-line from any point to any point." For geometric theory, this means that, given these axioms — given the ability to draw straight lines between points — this is what I can measure. On such a view the development of non-Euclidean geometry is merely the development of a new toolkit, not a challenge of the fundamental nature of space. This toolkit turns out to be essential for measuring shapes and distances at an astronomical scale.

When we talk about a space being Euclidean, what we actually mean is that measurement by means of Euclid's propositions is possible. As a contrast, consider a photo. If one want to measure shapes on the basis of a photo, then Euclidean geometry is not appropriate, one needs projective geometry (we will discuss this extensively in chapter 5). The usage of Euclidean geometry does base itself on actual facts about space around us, but that does not make that space inherently Euclidean. If you want to make your life unnecessarily difficult (but not impossible), you can also study distances and shapes at a small scale using non-Euclidean geometry.

The development of coordinate systems also loosened the relationship between geometry and physical space. Traditionally, space meant position-space, points were units with position. It was recognised that coordinate systems can be used for many things that have nothing to do with position, like a chart of the stock market. We can however also consider the underlying things of the chart as a space. Consider the possible orientations of an object with three axes of rotation. As we did with the totality of all positions, we can consider the totality of all such orientations and consider them to form a space. Geometrically, this would be the space of a 3-torus (the three-dimensional equivalent of the 1-dimensional circle and 2-dimensional torus). The methods of geometry are just as applicable to this orientation-space as they were to position-space. One can form a space by considering objects with respect to any multiple dimensions of (quantiative) variation. This gives rise to ten-dimensional Euclidean spaces[6] that might be used to measure the state of a robot, which has other relevant characteristics to its functioning beyond position. Position-space is a (very important) geometrical space, but conversely, geometrical space is not necessarily position-space.

The philosopher of science Tim Maudlin makes a fundamental distinction that is worth mentioning here. He distinguishes between a geometric space and a metaphorical space. He argues that things like Euclidean space and the non-Euclidean spaces of Riemanninan geometry are geometrical spaces, but that things like the set of integers or $\mathbb{R}^3$ are not literal spaces, but spaces in a metaphorical sense.

> "The points of Euclidean space, being all intrinsically identical, form a space only because of structure that is *not* a function of their intrinsic features. Euclidean space is therefore an instance of what I mean by a *geometrical space*. [...] In contrast, the set of integers, or of real numbers, do not form a geometrical space: the elements are not all intrinsically alike and the "geometrical" notion of, for example, proximity is determined by the different intrinsic natures. The sense in which the number 2 is "closer" to the number 1 than it is to the number 100, or sense in which the number 2 "lies between" 1 and 100, has to do with the arithmetic nature of these objects, not with any extrinsic structure that unites them."[7]

I take the difference between metaphorical and actual space to be between the use of numbers (or things based on numbers, like a probability or function space) or not. Therefore I do think that the

[5](Kline, 1972, p.872)

[6]Euclidean space is a more general concept than position-space, it is any space where the Euclidean axioms apply. Moreover, at astronomical distances position-space is non-Euclidean. However, to ease the use of language, I take position-space to mean a Euclidean space unless indicated otherwise.

[7](Maudlin, 2014, p.7)

different orientations of an object or the ten-dimensional robot space are actual geometrical spaces. If we consider the state of a robot as a set of numbers however, then we are considering the space of states in a metaphorical sense.[8]

1.3. Axiomatic geometry

Let us now turn to axiomatic geometry, which considers geometrical propositions by reducing them (or trying to reduce them) to a set of axioms that form the basis for all inferences. This is contrasted with doing geometry by means of numbers, where one demonstrates a claim by reducing it to an arithmetical computation and where the validity of the claim rests on the validity of the axioms for numbers and the validity of interpreting a proposition in numerical terms, this latter approach will be studied in-depth in chapter 4.

The axioms put forward by Euclid in the *Elements* formed the foundation for geometry for millennia, until the developments of the 19^{th} century. At that time mathematicians sought to revise them in light of the new developments. Why did Euclid need a revision?

First of all, it was recognized that Euclid's axioms were incomplete. An important figure in bringing this fact to the full attention of mathematicians was Moritz Pasch (1843-1930), who had proposed concepts and axioms that Euclid had not considered, most importantly those about the order of points on a line.[9]

But the addition of missing axioms was a relatively minor revision. Their addition did not invalidate the geometric propositions that had been used throughout the centuries and the validity and necessity of their content would likely be recognized even by the Greeks if it had been pointed out to them.

The most radical change happened in ideas about the nature of axiomatic geometry itself. The explosion of different kinds of geometries (most importantly non-Euclidean geometry, but also projective geometry) lead Hilbert (1862-1943) to make a fundamental dichotomy between content and form, between the ultimate meaning of geometrical terms and the methods of reasoning used in geometry. For Hilbert, mathematics should not deal with what 'triangle,' 'line' or 'point' mean, the content; but only with the form, the way some sentences follow from others, ultimately based on a set of undefined terms whose meaning is not part of mathematics at all. Historian Kline writes the following:

> "The axioms merely express the rules by which formulas follow from one another. All signs and symbols of operation are freed from their significance with respect to content. Thus all meaning is eliminated from the mathematical symbols. In his 1926 paper Hilbert says that the objects of mathematical thought are the symbols themselves. The symbols are the essence; they no longer stand for idealized physical objects. The formulas may imply intuitively meaningful statements, but these implications are not part of mathematics."[10]

For Hilbert and the formalists, geometry did not merely establish its results by means of formal axiomatic systems, *it is about such systems*. Any external meaning you cared to give to the symbols should not be part of mathematics proper.

The formal axiomatic approach swept across mathematics in the 20^{th} century. All branches were put on such a foundation under the impression that this would guarantee their validity. The ambition to base all of mathematics on exclusively formal reasoning, independent of observation or intuition, was shattered in the 1930s when Gödel proved that any formal system is limited: in any formal system that is powerful enough to do arithmetic, one can construct a sentence (i.e. string of symbols) that cannot be proved within that system.

I started off this chapter by distinguishing the view of geometry as a descriptive science and geometry as a science of measurement. The formalists were dealing with this same issue, they also recognized that geometry is not a description of nature. But the formalists took this too far when they argued that this means that geometry is therefore without meaning and concerns itself only with internal structure. It is true that geometry is not essentially about what the terms mean. Geometry is about how to infer measurements from others, what follows from what. Such inference can be done in different contexts for different types of spaces, but that does not mean that geometry is not about any of those spaces.

[8] I am not sure if Maudlin shares this way of looking at the difference between actual and metaphorical space.

[9] (Stillwell, 2005, p.44)

[10] (Kline, 1972, p.1204)

Consider the following analogy:[11] suppose the only kind of triangles that you knew were equilateral triangles. If you then discover the existence of isosceles and scalene triangles, then that does not make the term 'triangle' refer to *neither* equilateral, isosceles or scalene triangles. It means a triangle can be *either* equilateral, isosceles or scalene. Similarly, the geometric terms should have *some* meaning and they refer to things in *some* space, but this may be in *any* number of different spaces. The meaning of the terms differs depending on what particular application you consider. The wonderful fact is that we can use the *same* geometrical theory to reason about *different* spaces, so our formal theory need not specify a particular application of the theory.

What is the cash value of all this? Poincaré objected to Hilbert's system that "we do not know from whence [the axioms] come."[12] Indeed, the idea that geometry is about what follows from arbitrarily chosen primitive terms and axioms without any meaning, makes one wonder why the axioms are what they are. But if one recognises that the terms must have some meaning, but may have any number of different meanings depending on the space we consider, then we can take two different perspectives on what we are doing: the 'purpose perspective' and the 'formal perspective'.

The purpose perspective. From this perspective we have in mind the space to which we want to apply a formal system of geometrical axioms. In this chapter we work towards a formalisation of Euclidean position-space, therefore we can use this goal to motivate our choices and explain what facts about physical space around us these axioms reflect and why they are necessary.

The formal perspective. From this perspective, we consider the generality of our formal system and its applicability beyond any specific space one might have in mind. We make sure to only use the things that we have explicitly postulated through axioms or proved on their bases. This ensures that we do not bring assumptions into the formal system are theory that are not explicitly mentioned and thereby unknowingly limit its applicability.

I will take Hilbert's axiomatisation of Euclidean geometry as the starting point. But instead of ignoring intuition as the formalists would like us to, I will make extensive use of physical observations to motivate our terms and axioms (the purpose perspective) without any feeling of guilt for using such 'intuitions', while sharing the goal of making all assumptions explicit using axioms (the formal perspective).

1.4. Terminology

Let us end this chapter by giving a summary of some of the important terms I used.

Geometrical and metaphorical space

Geometrical space is a constellation of objects in multiple dimensions, these can be shapes in physical space but also orientations in an orientation-space (an orientation is not an object, but an object can be considered with respect to its orientation).

Geometrical spaces were contrasted with metaphorical spaces, which do not consist of actual objects but of numbers. Metaphorical spaces are very related to spaces, as we can use numbers to measure objects in space. Most importantly, the coordinate spaces that we will discuss in chapter 4 are an extremely powerful tool for measurement in geometrical spaces. They are themselves metaphorical spaces, but they have given rise to considering new geometric spaces.

Formal geometric theory and models

A formal geometric theory is a hierarchical organisation of geometric propositions in which each proposition is based only on axioms or previously demonstrated results. Such a theory is characterised by a set of primitive terms and axioms. Formal geometric theories are used for inferring measurements of things within a space. The specific space is left unspecified in the formal theory, hence the terms and relationships are not defined.

In the context of formal geometric theories, there exists another important thing: a model. Models are specific spaces in which all axioms of a given formal theory are satisfied. Most models are metaphorical spaces, $\mathbb{R}^2$ is for example a model for the Euclidean plane.

[11] Adapted from (Binswanger, 2014, p.118)
[12] (Reid, 1996, p.63)

Important other concepts in the study of axiomatic systems are *independence*: are the axioms independent of each other or can some be derived from others? *Consistency*: do the axioms contradict each other? And *categoricalness*: is there a unique model that satisfies a set of axioms (up to isomorphism)? These properties are studied by means of models. For example, by giving a model for a set of axioms, one proves that they are consistent.

2

Axiomatic Euclidean geometry

To present a formal theory of Euclidean geometry, we use five primitive terms: point and line, together with three relations: that of incidence, a relation between points and lines; of betweenness, a relation between three points; and of congruence, a binary relation that exists in two forms: between line segments and between angles. (Line segments and angles will be defined later in terms of the primitive terms and relations). Symbolically they are defined using a pair S, L where S is the set of points, which is our universe, and to which we might refer as 'the plane' and L is a set of lines.

Today is customary to define lines in L as subsets of S, this is what Hartshorne does. One can raise objections to *defining* lines as sets of points, as opposed to merely stating that lines contain points.[1] This is not strictly necessary for formal geometry, as all axioms will establish how points and lines relate. Hilbert did not do this, he simply postulates the points and lines as two different things.[2] This same issue will come up when trying to define line segments and angles in terms of the primitive terms. Here again, Hilbert did not define line segments as the set of points between them but as pairs of two points (the extremities) and did not define angle as the union of points on two rays but simply as the pair of two rays. However, Hilbert did define circles as sets of points. I will follow Hartshorne and define all such curves as sets of points because it offers better consistency and simpler expressions within the formal system.

When we want to refer to the collection of points, lines, relations and axioms as a whole we refer to it as a (plane) geometry Π. Formally we can write this as:

Definition 2.1. A Plane Geometry Π is described by:

- A set of points S, a set of lines L (subsets of S), n relations between and within elements of S and L symbolised by $R_1, ..., R_n$. The total can be symbolised as

$$\langle S, L, R_1, ..., R_n \rangle$$

- A set of axioms about points, lines and their relations.

2.1. Axioms of Incidence

The axioms of incidence form the first group of Hilbert's axioms. They characterise the relationship between points and lines through the concept of incidence. They are the most fundamental, the incidence relationship is sufficient to form a geometry, but without incidences, there is no geometrical structure. Hilbert also gave axioms that relate points and lines to planes. These axioms are necessary for three-dimensional geometry. We are doing plane geometry and do not discuss geometry beyond a single plane, so I have omitted these.

Axiom *(I.1)*. *For any two distinct points, there exists a unique line incident with them.*

Axiom *(I.2)*. *Every line is incident with at least two points.*

[1] See for example (Mendell, 2019, sec. 9) for some interesting comments by Aristotle on this issue.

[2] See (Hilbert, 1971) for Hilbert's definitions.

Axiom *(I.3)*. *There exist three non-collinear*[3] *points.*

When I say a point lies on a line or that a line contains a point, I mean the same thing as that the line and the point are incident with each other. In an application to Euclidean planes, 'lines' are infinitely extended straight lines. In space around us, we can determine a direction between two points; straight lines are lines of constant direction, so they are determined by two points. We see these facts reflected in *(I.1)* (while there is a concept of direction in position-space, geometry is also applicable to spaces without such a concept, therefore we do not use it). If singular points were lines, then the concept line loses its meaning and there would be no way to distinguish points from lines, which is reflected in *(I.2)*. *(I.3)* is a non-degeneracy axiom, if all points were on the same line there is no plane geometry or geometrical properties to study.

A sphere, considered as a geometrical space, is a space where one of these axioms is violated: going in any constant direction from the north pole will get you to the south pole, so *(I.1)* does not hold.

As an indication of the value of this abstract approach let us consider a geometric space that is wildly different from the Euclidean plane. The geometry stems from Kirkman's schoolgirl problem:

Example 2.1 (Kirkman)**.** 'Fifteen young ladies in a school walk out in rows of three for seven days in succession: it is required to arrange them daily so that no two shall walk twice in the same row.'[4]

One can consider this as a geometrical problem as follows: consider the ladies as points, groups of them as lines, and each day as a set of lines. Different constellations of girls form a geometric space. The axioms are determined by the conditions specified in the problem. We need *(I.1)* and *(I.3)* and

(K.1) All lines contain three points.

(K.2) Each line is contained in a unique set of five lines.

(K.1) is a further restriction of *(I.2)* and *(K.2)* is the specific axiom that makes us consider only spaces that form solutions to the original problem. There are multiple solutions to this problem[5] and hence multiple spaces that satisfy these axioms. Consider the model of this geometry displayed in fig. 2.1, where lines of the same colour form a set.

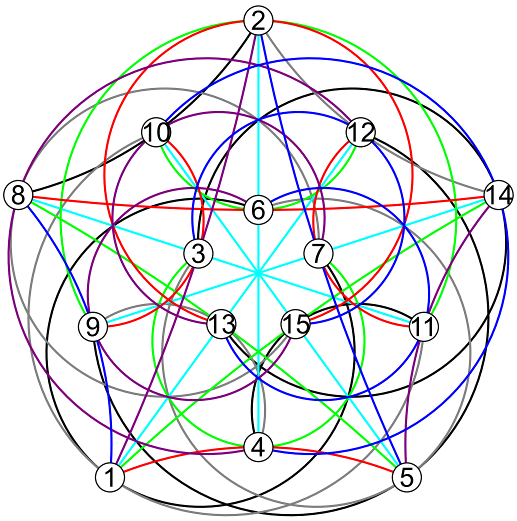

Figure 2.1: A model for a geometrical solution to the Kirkman problem; note that the figure is not symmetric under rotations. Source: (Weisstein, n.d.)

One can visually check that all axioms are satisfied, hence that these pencils form a solution for the original problem. The points and lines in this plane are vastly different from those in a Euclidean plane. a Euclidean plane is a continuum that is infinitely extended in all directions, while the Kirkman model contains a finite number of points (to reiterate the terminology: points and lines in figure form a

[3]not on the same line

[4]Adapted from (Falcone & Pavone, 2011)

[5]See (Falcone & Pavone, 2011) for different solutions

model for the *geometrical plane* of ladies in rows on different days); a point in a Euclidean plane has a position, in the model a point has a position too, but in a much more derivative sense (of a lady in a specific row of three on a specific day).

The Kirkman geometry demonstrates the wide applicability of geometric axiomatic systems. For this problem we started with something that might not have a solution: we need to check for the consistency of our axioms, that they are not in contradiction with each other. By providing a concrete model, we have proved the consistency of the axioms. The fact that more than one model exists demonstrates that the axioms are non-categorical.

2.2. Axioms of Order

The axioms of order are about the order of points on a line, which is characterised by means of the concept 'betweenness'. Euclid did not consider any such axioms of order at all. The axioms will give our lines the topological structure (i.e. sub-metrical structure) of lines that is familiar to us from Euclidean geometry: infinite in extent (beyond any point there is a next one), infinite in divisibility (between any points there is another one) and the existence of a linear order.

Definition 2.2. A *linear order* on a set S is a relation, symbolized by '$<$', such that for all $a, b, c \in S$:

1. Not $a < a$ (irreflexive)
2. If $a \neq b$, either $a < b$ or $b < a$ (anti-symmetric and total)
3. If $a < b$ and $b < c$, then $a < c$ (transitive)

Linear orders can be defined for arbitrary sets, we are interested in establishing them for the points on a single line. Points on a line do not have an intrinsic orientation, for two points on a line, there is no way in which A is intrinsically greater than B, a linear order however, does have a specified orientation. This is a reason we use the concept of betweenness as the axiomatic concept. We will construct a linear order afterwards.

We encourage the reader to consider whether infinite in extent, infinite divisibility and the existence of a linear order actually characterise the topology of a Euclidean line (i.e. a line in Euclidean space) but save critical questions about divisibility and continuity for section 2.5, when we discuss the axioms of continuity. This will further narrow our formal geometry and exclude application to spaces with lines that are finite or not ordered at all (like in the Kirkman geometry) and spaces with lines that have a cyclic order (like lines on a circle or a cylinder).

Axiom *(O.1)*. *If three points A, B, C lie on a line so that B lies between A and C ($A * B * C$), then B also lies between C and A ($C * B * A$).*

Axiom *(O.2)*. *For two points A, B on a line there exists a point C on that line such that B is between A and C.*

Axiom *(O.3)*. *For three distinct points on a line one, and only one, is between the other two.*

(O.2) might seem to imply infinite extensibility, but a betweenness defined for points on a circle also satisfies this. *(O.3)* excludes circles, so together they establish infinite extensibility. For infinite divisbility and a linear order, we will have to do more work. Most importantly, we need a fourth axiom of order. To state this axiom, we first introduce some definitions of concepts familiar to us from Euclidean geometry.

We will symbolise the line through the points A, B as $\overleftrightarrow{AB}$. This line is infinite in extent, we will now introduce finite line segments. Such segments preceded infinite lines in Euclid's original axioms. Some geometrical spaces lack line segments however (because they lack betweenness) and because we seek a general framework, we choose to go the other way around. Nevertheless, the change in attitude between Euclid and Hilbert is noteworthy: while Euclid's axioms are formulated in such a way as to not mention any kind of infinity at all, Hilbert starts with infinite lines and defines line segments in terms of them.

Definition 2.3. A *line segment* AB is the set of points containing A, B such that, a point is in AB if and only if it is a point in $\overleftrightarrow{AB}$ between A and B or it is A or B itself.

There is only one such line segment for a given A and B by *(I.1)* and AB is the same as BA by *(O.1)*.

Definition 2.4. A *separation of a plane* S by a point ℓ defines a pair of subsets S_1, S_2 of S, two half-planes, such that:

- Any A and B are in the same set if, and only if, AB does not intersect ℓ.
- Any A and B are in different sets if, and only if, AB intersects ℓ.

Axiom *(O.4)*. *Any line separates the plane.*

Hilbert did not use *(O.4)* as an axiom, we will first introduce his axiom and show that it is equivalent to *(O.4)* (i.e. they imply each other). The axiom that Hilbert used is due to Pasch, who was a central figure in bringing the axioms of order into formal geometry. Pasch's axiom will help us in demonstrating the topological properties of our lines.

Definition 2.5. For three distinct points A, B, C, a *triangle* $\triangle ABC$ is the union of three line segments AB, BC and AC. A, B, C are called its *vertices* and AB, BC, AC are called its *sides*.

Axiom *(Pasch)*. *Let A, B, C be the vertices of a triangle and ℓ a line not containing A, B, C. If ℓ contains a point D on the segment AB, then it must also contain a point on either BC or AC, but not both.*

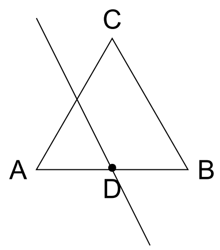

Figure 2.2: Pasch's axiom

Pasch's axiom in ordinary language states that if a line enters a triangle, then it must also exit it.

Proposition 2.1. *(O.4) and (Pasch) imply each other in the context of (I.1) to (I.3) and (O.1) to (O.3).*

Proof. Let us prove that *(O.4)* implies *(Pasch)*, the other implication can be found in (Hartshorne, 2000, p.74).

Assume the conditions of *(Pasch)* hold for points A, B, C, D and a line ℓ. Let the pair $\langle S_1, S_2 \rangle$ be a separation of the plane by ℓ. AB intersects ℓ in D, therefore A and B are in different sets, without loss of generality let $A \in S_1$ (and thus $B \in S_2$). C is not on ℓ so it belongs either to S_1 or S_2. If C belongs to S_1 then AC does not intersect ℓ and BC intersects ℓ; if C belongs to S_2 then AC does intersect ℓ and BC does not intersect ℓ.

Therefore if ℓ contains a point D on AB, then it must also contain a point on either BC or AC, but not both: *(O.4)* implies *(Pasch)*. □

The choice of *(O.4)* gives rise to the same geometrical theory as Hilbert's choice of *(Pasch)*. Our particular choice stems from the fact that the existence of such a separation is much more obvious for Euclidean space than is Pasch's axiom. The formal statement is also shorter. Greater clarity at no cost of generality is a win. In the proof of the other topological properties of the line, we will use both.

Proposition 2.2. *For any points A, C on a line ℓ, there exists a point D on ℓ between them ($A * D * C$).*

Proof (Adapted from Hilbert). Choose E not on ℓ *(I.3)*, take F on $\overleftrightarrow{AE}$ such that $A * E * F$ *(O.2)*. Similarly, take G on $\overleftrightarrow{FC}$ such that $F * C * G$, then G is not between F and C *(O.3)*. $\overleftrightarrow{GE}$ intersects the $\triangle ACF$ in E, thus by *(Pasch)* $\overleftrightarrow{GE}$ must also intersect AC in some point D. Therefore there exists a point between A and C. □

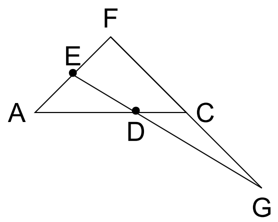

Figure 2.3: Proposition 2.2

The following lemma will enable us to establish the linear order of lines.[6]

Lemma 2.3. *Let A, B, C, D be four distinct points on a line ℓ.*

- *If $A * B * C$ and $A * C * D$ then $A * B * C * D$*[7]
- *If $B * C * D$ and $A * B * D$ then $A * B * C * D$*
- *If $A * B * C$ and $B * C * D$ then $A * B * C * D$*

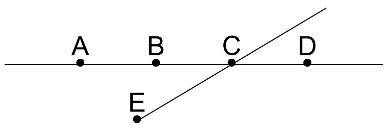

Figure 2.4: Lemma 2.3

Proof. We will prove only the first case as all three are very similar. Suppose for arbitrary and distinct A, B, C, D on ℓ such that $A * B * C$ and $A * C * D$. Choose E not on ℓ *(I.3)*. Divide the plane by the line $\overleftrightarrow{EC}$ *(O.4)*. A, D are on opposite sides as C is in between them, so $\overleftrightarrow{EC}$ intersects AD. A, B are on the same side as C is not in between A, B thus AB does not intersect $\overleftrightarrow{EC}$. B, D are therefore on opposite sides of $\overleftrightarrow{EC}$ as the two sides are disjunct. We conclude $B * C * D$, which means $A * B * C * D$. □

Theorem 2.4. *For any line ℓ, (O.1) to (O.4) induce a well-defined linear order.*

Proof. Choose two arbitrary points $A, B \in \ell$ to define an orientation by setting $A < B$. Then for $C \in \ell$:

- If $A * B * C$, define $A < C$ and $B < C$
- If $A * C * B$, define $A < C$ and $C < B$
- If $C * A * B$, define $C < A$ and $C < B$

We now define the relation for two arbitrary points $D, E \in \ell$ that are distinct from A, B as $D < E$ if one of the following holds:

- $D * E * A * B$
- $D * A * E * B$
- $D * A * B * E$
- $A * D * E * B$
- $A * D * B * E$
- $A * B * D * E$

Irreflexivity follows from the fact that betweenness is only defined for distinct points and totality/anti-symmetry follows from *(O.3)*. Transitivity follows from Lemma 2.3, we omit the details. □

We do not claim the uniqueness of this order, because it is not unique. One can define new linear orders on the basis of the one used in the proof by taking two points and interchanging their position in the order. There are however only two that actually reflect the order of points in the line, the order such that $A * B * C \implies A < B < C$ or $C < B < A$. We call these two orders the canonical linear orders of

[6]These are adapted from (Beifang, 2013)

[7]A betweenness relation between four points is defined by all the resulting betweenness relations of three points resulting from the removal of one point.

a line. The two different orders two are related by the choice of orientation. One can define the other orientation $\tilde{<}$ by $A\tilde{<}B \iff A > B$.

From the axioms of order and incidence, we have derived that all lines we consider have the topology of the Euclidean line: infinitely extensible and divisible and they admit a linear order. In the next section, we will establish the metric properties for lines in our geometry. Let us first define some more concepts that are familiar from Euclidean geometry and that we will use in the later axioms. Analogous to our separation of the plane, we define:

Definition 2.6. A *separation of a line* ℓ by a point A defines a pair of subsets ℓ_1, ℓ_2 of ℓ, the two sides, such that:

- B, C are on the same side of A if and only if A is not between B and C
- B, C are on the opposite side of A if and only if A is between B and C

The side of a line separated by a point together with that point is also called a *ray* $\overrightarrow{AB}$ from A. Note that you need at least one other point to determine a line, and thus a ray. Different points B can be chosen to designate the same ray from A however.

Definition 2.7. An *angle* is a pair of two rays, $\overrightarrow{OA}$ and $\overrightarrow{OB}$, emanating from the same point O where A and B do not lie on the same line, we write $\angle AOB$ or $\angle\alpha$, if the specific points are unspecified.

Following both Euclid and Hilbert, we do not include the degenerate angles (angles between the same ray or between rays that form a line together) in our definition. In contrast with Euclid, but following Hilbert, we only define angles for two straight lines that meet and not for two curves in general. Let us conclude this paragraph by giving another model related to the axioms discussed so far that is different from Euclidean space.

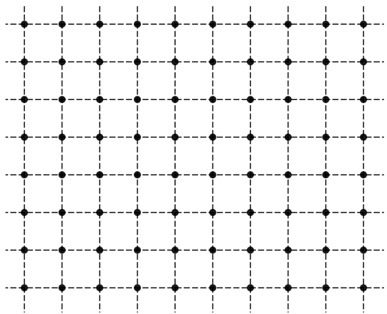

Figure 2.5: A lattice

Example 2.2 (Lattice)**.** Consider a lattice fig. 2.5. We introduce a model for a lattice using integers. Consider $\mathbb{Z}^2$ whose elements define our points, and define all lines by the solutions to equations of the form $px + qy + r = 0$ for $p, q, r \in \mathbb{Z}$.

Lines in our lattice are not infinitely divisible, so we expect some of the axioms to be violated. Interestingly $\mathbb{Z}^2$ satisfies all axioms except *(O.4)*! (Because it does not satisfy the equivalent axiom *(Pasch)*.) We will give a counterexample, the reader can check for themselves that all other axioms are satisfied. Consider the triangle with vertices $(0,0), (0,1), (2,0)$ and the line $x + 2y - 1 = 0$. This line cuts the triangle in $(1,0)$, but nowhere else! It 'floats' through the other side. *(Pasch)* has an important role in establishing infinite divisibility of lines, we will return to that topic when we discuss the axioms of continuity because *(O.4)* alone is not enough to prevent other types of curves from 'floating' through a line.

2.3. Axioms of Congruence

The axioms of congruence characterise the relation of equality in magnitude, which means the same thing as congruence. We denote this relationship by the symbol $\cong$, both for lines and for angles as they share the same conceptual genus ('congruent to'). Bear in mind that congruence of lines is different from congruence of angles and one cannot compare a line to an angle.

Definition 2.8 (Adding line segments)**.** The *addition* of a line segment CD to a line segment AB ($AB + CD$) defines the line segment AE, where E is a point on the ray $\overrightarrow{AB}$ with $A * B * E$ so that $BE \cong CD$. We call AE the *sum* of AB and CD.

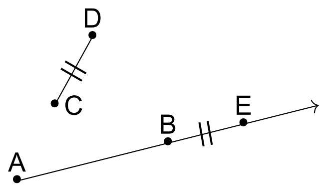

Figure 2.6: Addition of two line segments

Note that this definition is *not* a definition of adding lengths, but of adding line segments. $AB + CD$ and $CD + AB$ define line segments incident with different points, though they are congruent. To consider length independent of any particular line segment requires another level of abstraction, which will be considered in chapter 3. This is also why the axioms are stated in terms of facts about line segments, not in terms of lengths. That such a point E as specified in the definition exists follows from *(C.1)*, to which we turn now.

Axiom *(C.1)*. *Given a line segment AB and a ray $\overrightarrow{CD}$, there exists a unique point E on $\overrightarrow{CD}$ such that $AB \cong CE$.*

Axiom *(C.2)*. *Line segments congruent to the same segment are congruent to each other.*

Axiom *(C.3)*. *If congruent line segments are added to congruent line segments, their sums are congruent.*

Axiom *(C.4)*. *Given an angle $\angle BAC$ and a ray $\overrightarrow{DF}$, there exists a ray $\overrightarrow{DE}$ on a given side of $\overrightarrow{DF}$ such that $\angle BAC \cong \angle EDF$.*

Axiom *(C.5)*. *Angles congruent to the same angle are congruent to each other.*

Both *(C.1)* and *(C.4)* are propositions that Euclid proves. Instead of *(C.1)*, Euclid assumes that we can draw circles (Book I Postulate 3). The lines of the midpoint of a circle to a point on the circle are by definition congruent, so in fact, his axiom requires C to be the same point as A. Through construction, he then proves that one can construct a congruent line segment on any other line. The same goes for *(C.4)*, which corresponds to the *Elements* Book I Proposition 23. *(C.2)* and *(C.5)* are two separate cases of Euclid's Common Notion 1. One could condense it into Euclid's single axiom 'things congruent to the same thing are congruent to each other', understanding that 'thing' means either 'line' or 'angle.'

Though these differences in choices between Euclid and Hilbert are interesting, we choose not to discuss them because there are other axioms (most notably *(Pasch)*, *(SAS)* and *(P)*) that are more important, and discussing the different merits and proving the equivalences for *all* axioms would make us go too far astray of the goal of this thesis. Therefore we will simply take Hilbert's version for the sake of completeness. Hilbert does not assume uniqueness for *(C.1)* but does for *(C.4)* and then proves uniqueness for *(C.1)* on the basis of an axiom that is to follow. We will just postulate uniqueness for both.

(C.3) is an adaptation of Euclid, who postulated it in general as a common notion. We restrict it to line segments because we only defined addition for them and it is the only thing we need. With a definition of congruence, we also define an inequality of line segments.

Definition 2.9. AB is *less than* CD ($AB < CD$) if there is a point E in between CD such that $AB \cong CE$. We also say CD *is greater than* AB ($CD > AB$).

The five axioms of congruence introduced so far are not sufficient for Euclidean geometry, we need another axiom. Let us consider (a modern formulation of) Book I Proposition 4 of the *Elements*.

Axiom *(SAS)*. *Given triangles $\triangle ABC$ and $\triangle DEF$, if $AB \cong DE$ and $AC \cong DF$ and $\angle BAC \cong \angle EDF$ then the triangles are congruent, i.e. $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.*

The proposition is often referred to as side-angle-side (SAS) as it states that the congruences of a triangle are characterised by a given side-angle-side combination. Euclid claimed to prove this proposition by means of superposition: the method of placing the congruent side-angle-side combination on top of each other and arguing why the other side and angles must be congruent as well. Throughout history, mathematicians and philosophers have raised objections to this method, because the process of superposition involves motion and seems much more physical than the other axioms. Nevertheless one cannot escape the fact that this proposition is independent of all other axioms. Hilbert took *(SAS)* as an axiom, we are going to do something different.

The fundamental fact that gives rise to the need for this axiom is the homogeneity of space: shapes are independent of any particular position. *(SAS)* establishes this for triangles. Given a SAS-combination, there is only one way in which a third line segment can be drawn to connect the other two *(I.1)*. *(SAS)* then establishes that the other parts of the two triangles must also be congruent, hence that the two shapes are actually the same. This can only be the case if the plane is homogeneous. However, one cannot just take a proposition like 'the plane is homogeneous' as an axiom in our formal system, we have to precisely define what homogeneity is in this context. We will do this by postulating the existence of certain symmetries as axioms in our space.

The idea of using a group of symmetries to characterise a geometry stems from Felix Klein's (1849-1925) *Erlangen program*. His essential idea is that one can characterise a geometric space by the existence of certain symmetries in the geometry: transformations under which figures remain invariant. A symmetry is always a symmetry of certain properties. Until now we have discussed incidences, betweenness and congruences, which will be our invariants. This kind of symmetry is called Euclidean isometry (or 'rigid motion', but I personally do not like this term, because it suggests that we are actually talking about moving things whereas this isometry can also just be a more abstract kind of relating). Mathematically, we can define it as follows.

Definition 2.10. Let Π be a geometry consisting of the primitive terms of point, line, betweenness and congruences, we define an *Euclidean isometry* to be a map $\phi : \Pi \to \Pi$ such that:

1. ϕ is a 1-to-1 mapping of points onto itself
2. ϕ sends lines to lines
3. ϕ preserves betweenness of collinear points
4. For any two points $AB \cong \phi(A)\phi(B)$
5. For any angle $\angle(BAC) \cong \angle(\phi(B)\phi(A)\phi(C))$

A Euclidean isometry preserves all properties we have discussed so far. Note that beyond the isometry of mapping each point to itself (the identity), there is no guarantee that such isometries exist in a given geometry. Let us demonstrate that a geometry might lack symmetries by an example.

Example 2.3. Consider the following the metric space $(\mathbb{R}^2, d(\cdot,\cdot))$ with

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^4 + (y_2 - y_1)^2}$$

This is a model based on a Euclidean plane such that all distances grow quadratically faster with respect to a given origin and direction (the $x-$axis). (Why would we ever measure distances differently for Euclidean space? I cannot possibly give a full answer to such a broad question here, but think about the following interesting application: in designing a world for Virtual Reality (VR), it is desirable to have a space that has the topology of physical space around us, but that the distances are altered in such a way that walking around in VR world will keep you within the limited domain of a VR setup). We will not consider all attributes, just the congruence of line segments. Congruences between line segments can be measured via the distance function: two line segments are congruent if the distance between their endpoints is the same. A translation does not change the distance between two $x-$coordinates. A rotation does change them, so there exist no rotational symmetries. The metric we defined is still relatively regular, but as the possible application to VR indicates, such metrics can be highly irregular.

Characteristic for Euclidean space is that moving things around, rotating them and flipping them over does not change the congruences of lines and angles. Therefore let us introduce the existence of Euclidean isometries (EEI) axiom.

Axiom *(EEI)*.

Translation: *For any two points, A, A' there is an isometry that maps A to A'.*

Rotation: *For any three points O, A, A' there is an isometry that maps the ray $\overrightarrow{OA}$ to the ray $\overrightarrow{OA'}$.*

Reflection: *For any line there is an isometry such that any point on that line gets mapped to itself, but the separation of the line changes sides.*

As an example of a symmetry that does not exist in the geometry defined so far, take the operation of scaling the whole space with respect to a center point. Scaling preserves (1)-(3) and (5) of Definition 2.10, but it does not preserve congruences.

Proposition 2.5 (Hartshorne 17.1)**.** *(EEI) and (SAS) imply each other in a plane satisfying (I.1) to (I.3), (O.1) to (O.4) and (C.1) to (C.5).*

Proof. *(EEI)* $\implies$ *(SAS)*: Suppose that for two triangles $\triangle ABC$ and $\triangle A'B'C'$, $AB \cong A'B'$ and $A'C' \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$. First note that the composition of isometries is an isometry *(C.2)* and *(C.5)*. Define an isometry by first mapping A' to A (translation), then mapping the rays $\overrightarrow{AB'}$ to $\overrightarrow{AB}$ leaving $A = A'$ in place (rotation) and lastly reflecting in AB if C and C' are on different sides of AB (else leaving them in place). This isometry sends A' to A, B' to the ray $\overrightarrow{AB}$ and C' to the same side of that line as C. $A' = A$, thus $AB \cong AB'$ and as B and B' are on the same ray from A, $B' = B$ *(C.1)*. $\angle BAC \cong \angle BAC'$, thus $\overrightarrow{AC}$ and $\overrightarrow{AC'}$ are the same ray *(C.4)*. Now again by congruence of AC and AC', $C = C'$ *(C.1)* which completes one direction of the proof.

(EEI) $\impliedby$ *(SAS)*: The direction of this proof can be found in (Hartshorne, 2000, Proposition 17.4). □

Taking *(EEI)* as an axiom has a certain merit: it makes much more explicit what the underlying assumptions about space are. It is also closer to the kind of assumption that we make implicitly when thinking about space around us and it will make it easier to relate the axiomatic and coordinate-based approach to each other. Nevertheless, it is true that this axiom has a completely different form from the other axioms and that *(SAS)* is a simpler statement if the goal is to stay as close as possible to the formalism.

2.4. Axiom of Parallels

Now we turn to the most famous of Euclid's axioms: the parallel postulate. In the introduction we discussed some of the mathematical history surrounding it, here we will elaborate on some of the conceptual and physical underpinnings of the parallel postulate.

This axiom is very central to the nature of Euclidean space and what differentiates it from other geometrical spaces, most of the other geometrical spaces studied in mathematics and used in physics are non-Euclidean. In position-space, what ability does this axiom presuppose? It is the ability to consider direction and position independently, the same direction that exists between A and B's can exist between C and other D's. In position-space parallel lines are lines with the same direction[8], at different positions. In other spaces where the parallel postulate does not hold, directions from different starting points are not comparable. Multiple or no parallel lines might exist from a point outside the line, so there is no way to determine the same direction at a different position (or the concept direction loses its meaning because there are multiple that are 'the same'). By means of the concept of parallelism and the parallel postulate, we can form an abstraction of direction that is independent of a specific position. This observation will become even clearer in chapter 5 when we discuss projective spaces and where 'direction' can be characterised by a point on the horizon. Let us start with a formal definition of parallel lines.

Definition 2.11. Two lines are *parallel* if they do not intersect.

We also consider a line to be parallel to itself, as a degenerate case. In an arbitrary geometry, there might exist parallel lines or none at all, outside the degenerate case. The modern form of the parallel postulate is due to Playfair:

Axiom *(P)*. *Let ℓ be a line and A a point not on ℓ, then there is a unique line through A parallel to ℓ.*

[8]Here we consider the opposite directions A to B and B to A as equivalent.

It is not strictly necessary to assume the existence of such a line, that can be proven,[9] but postulating uniqueness is absolutely necessary. Let us finish this section by introducing another geometry where parallel lines play an important role.

Example 2.4. An *affine plane* is a plane satisfying *(I.1)* to *(I.3)* and *(P)*. We will consider an example of a real-world space that is affine. Consider the motion of a particle moving in a single dimension over time.[10] In the two-dimensional space (one of position one of time) 'points' have significance, they are the state of the particle at a given point in time, 'lines' have significance as well, they are the path of a particle over time at constant velocity. However, there is no significance to congruences of line segments nor of angles, because the two different dimensions cannot be compared directly. Only when one chooses a unit to measure and relate the displacement and time dimensions do these properties become meaningful. (There is the small exception that parallel lines can be compared in terms of congruences.) It might not come as a big surprise, given this example, that affine spaces will arise very naturally when considering coordinate geometry.

2.5. Axioms of Continuity

The most famous example of something Euclid assumed but that cannot be derived from his axioms is the fact that if circles overlap, then they have points of intersection. He uses this fact immediately in Proposition 1, but it is not derivable from his axioms. In section 2.2 we claimed that we had derived all topological structure of the Euclidean line. Here we swept the differences between infinite divisibility and continuity under the rug. Let us now consider this subject.

(O.4) or *(Pasch)* gives an infinite divisibility induced by triangles: we can intersect our line by constructing a triangle with a point outside our line. The existence of such intersections does not necessarily imply the existence of intersections between two circles. Coordinate geometry is a very appropriate setting to explain what gives rise to this difference, so we defer such explanations to chapter 4, here we will just state the axiom.

Definition 2.12. A *circle* with centre O and radius OA is the set of points such that a point B is in the set if and only if $O * B * A$, $B = O$ or $B = A$.

A point B is *inside* the circle if for A on the circle $OB < OA$, it is *outside* the circle if $OB > OA$. Two circles c_1, c_2 *overlap* if there exist points A, B on c_1 so that A is inside and B is outside c_2. That c_1, c_2 overlap is the same as that c_2, c_1 overlap.

Axiom *(CCI)*. *If two circles overlap, then they intersect.*

With all axioms considered so far, one can prove all of Euclid's propositions from Book I-IV of the *Elements* that do not use area, which we have not defined or axiomatised. For the other books of Euclid, one needs additional axioms that are necessary to develop a theory of proportion (starting in Euclid's Book V). Let us state the most important of these axioms: Archimedes' axiom. It states that any length can be used to measure any other length.

Axiom *(A)*. *If AB and CD are any segments, then there exists a number n such that n segments CD placed next to each other will be greater than AB.*

With these axioms, one is equipped to prove all of Euclid's propositions about plane geometry. That does not exhaust plane geometry however…Modern developments include the real numbers and the formalisation of continuity, to integrate with this theory we need another axiom. Hilbert chose the following axiom:

Axiom *(Line Completeness)*. *It is impossible to extend the set of points of a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follow from (I.1) to (I.3), (O.1) to (O.4), (C.1) to (C.5), (EEI), (CCI) and (A).*

In essence, this axiom states that the line is 'complete', we cannot add extra points to a line to construct a new line that would preserve all original properties of the line. This axiom implies Archimedes'

[9] See for example *Elements* Book I Proposition 27.

[10] This example is inspired by the introduction to affine planes in (Bamberg & Sternberg, 1991).

axiom and *(CCI)*. We state it separately, because taking it as an axiom masks distinctions that are worth studying. Hartshorne did not take *(Line Completeness)* as an axiom and instead took an alternative axiom, Dedekind's axiom:

Axiom *(Dedekind)*. *Suppose the points on a line are divided into a partition S, T such that no point of S is between points of T, and no point of T is between points of S. Then there exists a unique point P such that for any $A \in S$ and any $B \in T$, either $A = P$ or $B = P$ or $A * P * B$.*

Dedekind's axiom says that any 'cut' of the line defines a unique point. We will not try to demonstrate that these two are equivalent as it goes too far astray from our theme. We will see later that either one of these two axioms is necessary to get a categorical model of Euclidean geometry.

This concludes the axioms. We have now built up Euclidean geometry synthetically using Hilbert's axioms. We finish with two definitions of planes that interest us.

Definition 2.13. A *Hilbert plane* Π is a geometry that satisfies the axioms *(I.1)* to *(I.3)*, *(O.1)* to *(O.4)*, *(C.1)* to *(C.5)* and*(EEI)*.

A Hilbert plane does not necessarily satisfy *(P)* and *(CCI)*. We also define the Euclidean plane, which is the plane for which Euclid's propositions about plane geometry hold.

Definition 2.14. An *Euclidean plane* Π is a Hilbert plane satisfying *(P)* and *(CCI)*.

3

Geometrical arithmetic

3.1. Synthesis versus Analysis

After a long period of stagnation and decline during the Dark Ages, mathematics revived again in Europe when the Greek works were rediscovered. The new renaissance of geometry reached a first culmination in the work of Descartes.[1] In this chapter we will reproduce an essential ingredient of his achievement but in the framework of Hilbert's axioms.

In Euclid's *Elements*, all propositions are proved by giving a construction that bases itself purely on established (or postulated) knowledge. We reason from the known to the unknown. But, we could object to Euclid's construction in the same way Poincarè objected to Hilbert's axioms: 'we do not know from whence they come.' How did the Greeks come up with their constructions?

It was known that next to synthetic constructions, Greek geometry also consisted of something they called *analysis*. This is what the Greek mathematician Pappos (c. 290 - c. 350 AD) had to say about it:

> "[A]nalysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method 'analysis,' as if to say *anapalin lysis* (reduction backward). In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call 'synthesis.'"[2]

Analysis[3]is a method of *finding* solutions by reasoning backwards from a supposed solution, synthesis is a deductive *demonstration* of solutions on the exclusive basis of axioms, without reference to unknown facts or assumptions. Synthesis is Euclid's method in the *Elements*, but analysis is precisely this missing element that explains where the constructions come from. Pappos also makes the important observation that analysis precedes synthesis. One first needs to find a solution, which in general is not trivial at all and requires a methodological approach, before one can demonstrate the validity of such a solution.

Unfortunately, most of the Greek works presented synthetic constructions without any reference to the analysis that preceded it and most of that analysis has been lost. We can only awe in imagination at the kind of methods they must have had to find solutions to new mathematical problems. Some works did survive, see for example Euclid's *Data*.[4]

As a philosopher, Descartes was very concerned with method. In mathematics too, his goal was to create general methods to solve *all* mathematical problems. In his work *La Géométrie* (1637), he

[1] For an extensive study of this development, see (Bos, 2012).

[2] (Bos, 2012, p.95)

[3] The modern usage of the term 'analysis' in mathematics is different, I will therefore limit my usage of this term.

[4] (Euclid, 1993)

developed a new method to solve geometrical problems: analytic geometry. In Descartes' analytic geometry, one assigns symbols to different line segments — known and unknown —– and one determines the length of the unknown line segments based on the relationships *in length* of the different line segments and the arithmetical operations (addition, multiplication, subtraction, division and root extraction) defined for lengths. It is a form of *analysis* that works using algebra.

Algebra originated in the Islamic world as a method of analysis for *numerical* problems, not geometrical problems. Descartes' achievement is a full application and integration of this method in geometry. Algebra works by means of the arithmetical operations, however, so before discussing such a thing as algebra in geometry, let us introduce arithmetic in geometry.

3.2. Arithmetic of lengths

Hilbert's axioms include the congruence relationship, which for line segments means equality in length. But congruence itself is a relationship between line segments, not between lengths. To compare lengths directly without any reference to positions of line segments, one needs to consider lengths abstracted from any specific line segment. We need to introduce a second level of abstraction.

Euclid presents the theory of proportions in Book V, which provides a general framework to compare magnitudes, including length. It could be used for this. But to develop a full theory of magnitudes with the same level of rigour and understanding we have today, one basically needs to develop the real numbers. Moreover, it can be a very interesting pedagogical lesson into the nature of numbers and their applications to see what we can do based on *only* the geometry we have defined so far and to develop an arithmetic that is specifically made for lengths only. We are going to take inspiration from (constructions of) line segments themselves and develop an arithmetic of lengths directly, without any reference to numbers.

Length is an attribute of line segments. To consider the length of the line segment, we have to abstract away from its other attributes. In modern mathematics, this can be done by means of an equivalence class. One could define the length of a line segment to be the equivalence class of congruent line segments. But such a definition raises some real philosophical objections as it blurs the distinction between classification and description, and thus between attributes and groups.[5] We take a different approach of simply defining an attribute[6]length that line segments have and define how different lengths relate to each other.

Definition 3.1. A *length* a is an attribute of a line segment AB such that for any other line segment CD with length b, $a = b \iff AB \cong CD$.

We define inequalities of lengths in the same way, based on inequalities for line segments. We defined the addition of two line segments as the line segment constructed by putting one next to the other, we can use this to induce a definition for the addition of lengths.

Definition 3.2. Given two lengths a, b, we define their *sum* $(a + b)$ to be the following length c. For collinear points A, B, C such that segments AB and BC have lengths a and b respectively the line segment AC has length c.

That the sum of lengths a, b produces a unique length c follows from *(C.3)*. A length is an abstraction, so it cannot exist independently of any line segment. Therefore, in our definition of the addition of lengths, we used specific line segments AB, BC. Our definition does not depend on these specific line segments, any other congruent line segment could be substituted for them and by *(C.2)* this is equivalent.

Addition is the operation of combining quantities of the same kind along their domain of commensurability (adding lengths to lengths, time to time, area to area, units to units), inducing a definition of addition for lengths can easily be done based on the established geometrical properties of line segments alone. Multiplication, however, introduces more difficulties. Multiplication is the operation of interrelating two quantities of a different kind and results in a quantity different from the other two: multiplying length and width produces area, multiplying speed and time produces distance, multiplying weight by acceleration produces force.

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[5]See (Binswanger, 2014, p.175-178) for the importance of this issue).

[6]This can be taken as a primary concept for knowledge in general, attributes are things that entities are made of.

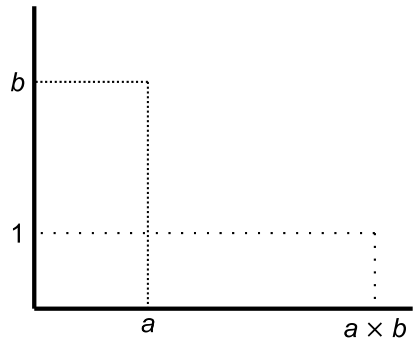

Figure 3.1: Multiplication of lengths through equality in area

The simplest case is that of using length[7]and width to produce area, as this does not involve the choice of any fixed unit. If one wants to measure the product by the same unit by which one measures one of the factors, one needs the selection of a unit, however. Consider the case of counting the number of items n in a group and counting the total number of groups m. The total number of items is then $n \times m$, *if the groups are uniform.* That is, if n is not only the number of items in the counted group but n is also the number of items in the other groups. This fact enables one to go from 'n items in the counted group' to 'n items per group' and multiply the number of items per group by the number of groups to obtain a number of items. We will see that if we are going to go from a length to another length, we will have to fix a unit along the way.

Multiplying two lengths has no natural definition, what that would require is a way to relate two line segments uniquely to a third line segment. The closest thing to a product of two lengths arises from considering one of the lengths as a width and producing an area by multiplying them. One could then relate this area back to a unique length by choosing a unit width (see fig. 3.1).

This is arguably the simplest way to think about it, but it is not what we are going to do, mainly for pedagogical reasons. It is an interesting lesson both about the nature of multiplication and its meaning and about the nature of formal systems to do it in a way that avoids the usage of axioms and propositions for areas.

Let us first state the way Hartshorne defined multiplication for line segments, which is the simplest way within our formal system.[8]

Definition 3.3 (Triangle-multiplication)**.** Given two lengths a, b and a unit length 1. We define their product $a \cdot b$ as follows. First, make a right triangle $\triangle ABC$ with AB length 1 and BC length a with the right angle at B. Let α be the angle $\angle BAC$. Now make a new right triangle $\triangle DEF$ with DE length b and having angle α at D. Then we define $a \cdot b$ to be the length of EF on this new triangle.

This way of defining the multiplication of line segments is also very similar to the way Hilbert and Descartes did it.[9] There is however something unsatisfying with this definition because it is not clear *why* it works. *Why* is this a valid definition for multiplication? Let us, therefore, give an interpretation of this definition, after we have demonstrated two results about this multiplication.

Lemma 3.1. *For lengths a, b, c, d with respect to a unit length 1 the following equivalences hold: $a < b < c \iff ad < bd < cd$ and $a = e \iff ad = ed$.*[10]

Proof. On a ray $\overrightarrow{OI}$ consider the points A, B, C with OA length a, OB length b and OC length c. By definition, $A{*}B{*}C$. If we now consider the point I' with OI' length 1 and $\angle OAI'$ a right angle, then we have the three triangles $\triangle OI'A, \triangle OI'B, \triangle OI'C$. Moreover, it is clear that $A{*}B{*}C \iff \angle OI'A < \angle OI'B < \angle OI'C$. If D lies on the ray $\overrightarrow{OI'}$ such that OD has length d, then the three points with the multiplied lengths lie on $\overrightarrow{OI}$. Now the angles of these triangles stay the same, hence the order of the points as well. The

[7]'Length' is used in two ways: in the wide sense as a concept for 1-dimensional extension and in a narrow sense in contrast with width and height, the other two 1-dimensional extensions. Normally we use it in the wide sense, but here it is used in the narrow sense.

[8]Hartshorne treats lengths as equivalence classes of lines, which we do not take over.

[9](Hilbert, 1971, p.52) and (Bos, 2012, ch.21)

[10]Note that these are not identities for number in general. They are for our lengths, which correspond to positive numbers.

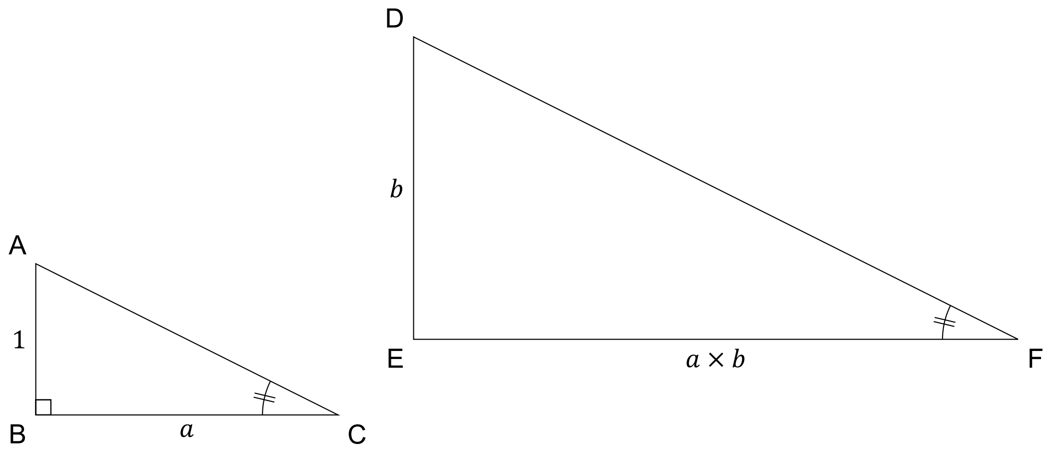

Figure 3.2: Triangle-multiplication

relevant implications are all both ways so $a < b < c \iff ad < bd < cd$. The second statement is by definition true. $\square$

Let us now consider an operation that directly relates line segments to line segments: stretching. We are going to relate any line segment by its length to a unique stretching operation and use this stretch to stretch another line segment. The length of the resulting line segment is the product of the two lengths. We will start by defining a stretch for a ray, for which we use the more formal term of dilation.

Definition 3.4. A *linear dilation* of a ray $\overrightarrow{OI}$ is a map $\phi : \overrightarrow{OI} \to \overrightarrow{OI}$ such that for fixed $I, A \in \overrightarrow{OI}$ with OI of length 1 and OA of length a (which is by definition also the magnitude of the dilation):

1. ϕ is a bijection
2. $O = \phi(O), A = \phi(I)$

   For any $B, C, D, E \in \overrightarrow{OI}$,
3. $B * C * D \iff \phi(B) * \phi(C) * \phi(D)$ (preservation of order)
4. $BC \cong DE \iff \phi(B)\phi(C) \cong \phi(D)\phi(E)$ (preservation of line congruences)

Notice how the magnitude of a linear dilation is directly related to the length a. In ordinary language, linear dilation is the ray you get by holding O in place and pulling I to A while keeping the order and relative lengths the same. One could define multiplication by means of this stretch:

Definition 3.5 (Dilation-multiplication)**.** Given two lengths a, b and a unit length 1. We define their product $a \cdot b$ to be the following length c. Consider the line segment OI with length 1. On the ray $\overrightarrow{OI}$ consider the line segment OA with length a. Define the linear dilation ϕ on the ray $\overrightarrow{OI}$ determined by b with respect to 1 and consider the point $C := \phi(A)$. Then we define the product $a \cdot b$ to be the length of OC.

This definition has an important advantage from the viewpoint of understanding multiplication. Like defining it by using area, it makes clear what it means to multiply two line segments. Any such multiplication involves relating one line segment to a map (the dilation), only that map relates a line segment back to another line segment.

A problem arises if one wants to put this into a formal system, however. While it is easy to understand what such a dilation map means and that these are actually the requirements that induce a unique stretch for each length (something we will demonstrate), proving the existence of such a function from the axioms alone is a lot of work and will involve using the triangle-definition of multiplication. From the formalist standpoint, it might be better to see dilation-multiplication as giving an application of the definition of multiplication. Let us prove that such a dilation exists and is unique based on triangle-multiplication (that line segment exists by direct construction). Let us start with another definition.

Definition 3.6. A circular dilation of a plane Π with respect to a point O and with magnitude a is a map $\phi : \Pi \to \Pi$ such that any ray from O gets mapped to itself, and the restriction of ϕ to a ray is a linear dilation of magnitude a.

Again, we still have not proved that such a function exists but it is easily seen how circular and linear dilations are uniquely related to each other. We will use circular dilations to prove the existence of linear dilations.

Theorem 3.2 (Existence of circular dilation)**.** *Given a length b and a unit length 1. Consider a center O for two circles: c_1 with radius 1 and c_b with radius b. Define a function ϕ with $\phi(O) = O$ and for any $A \neq O$ with OA length a, define $\phi(A)$ as the point on the ray $\overline{OA}$ with length $a \cdot b$ (defined by triangle-multiplication). Then ϕ is a circular dilation.*

Proof. By definition, any point A gets mapped to another point on the ray $\overline{OA}$. We have to show that the restriction of ϕ to such a single ray is a linear dilation. Let us pick a point A and consider the restriction of ϕ to the ray $\overrightarrow{OA}$. We check all conditions of the definition:

1. That ϕ is a bijection is obvious.
2. By definition $\phi(O) = O$ and if I is the point on $\overrightarrow{OA}$ such that OI has length 1, then $\phi(I) = A$ also by definition.

   Let $B, C, D, E \in \overrightarrow{OA}$
3. $B*C*D \iff \phi(B)*\phi(C)*\phi(D)$. This follows from the related identity for Hartshorne's multiplication proved in Lemma 3.1.
4. $BC \cong DE \iff \phi(B)\phi(C) \cong \phi(D)\phi(E)$. This also follows from Lemma 3.1.

□

Lemma 3.3. *A circular dilation maps any line to a parallel line*

Proof. Given is a circular dilation ϕ with respect to a point O of magnitude a. If $a = 1$ then every point gets mapped to itself and the result is trivial. So, suppose $a \neq 1$, we do a proof by contradiction. Suppose a line ℓ gets mapped to a non-parallel line ℓ', then they intersect in a point A. But then on the ray $\overrightarrow{OA}$, $\phi(A) = A$ which implies every point on the ray gets mapped to itself, so the magnitude of the linear dilation is 1, but the magnitude of the linear dilation equals that of the circular dilation which is $\neq 1$, a contradiction. We conclude that any line gets mapped to a parallel line. □

Proposition 3.4 (*Elements* Book I, Prop. 29)**.** *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.*[11]

This last proposition is taken directly from Euclid, hence its language somewhat differs from ours but its meaning should be familiar to anyone who has done high-school geometry (they are sometimes called 'F-angles').

Theorem 3.5. *A given ray $\overrightarrow{OI}$, a magnitude a and unit 1 uniquely determine a linear dilation.*

Proof. Suppose there were two functions ϕ, ψ that are both linear dilations of the same ray $\overrightarrow{OI}$ with the same magnitude b. Let OB on $\overrightarrow{OI}$ have length b. Take an arbitrary point $A \in \overrightarrow{OI}$ such that OA has length a. We prove that $\phi(A) = \psi(A)$ which implies $\phi \equiv \psi$.

Any linear dilation determines a unique circular dilation, so we consider the same maps ϕ, ψ extended uniquely to a ray perpendicular to $\overrightarrow{OI}$.

On this ray, there exists a point I' such that OI' has length 1. By definition ϕ and ψ send I to B. So they must both send I' to a point B' such that $OB \cong OB'$.

A parallel line gets mapped to a parallel line, so the line $\overleftrightarrow{AI'}$ gets mapped to a parallel line CB' for some C. And by the parallel postulate, C is the same point for ϕ and ψ. As A was an arbitrary point, this is the case for any point on $\overrightarrow{OI}$, ϕ and ψ are identical. □

Dilation-multiplication is equivalent with triangle-multiplication. This follows directly because we have shown that there exists a dilation that agrees with triangle-multiplication and it is unique, one can also just look at how OC is defined and see that OC has length $a \cdot b$.

[11](Euclid, 2008, p.32)

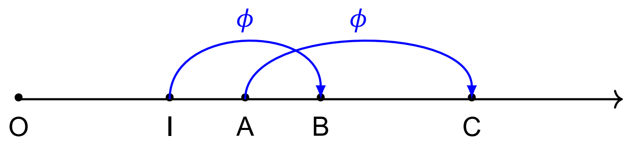

Figure 3.3: Dilation-multiplication

Without ever considering a general theory of magnitude and using numbers, we have defined addition and multiplication for lengths, giving rise to arithmetic for lengths. With arithmetic of lengths, one has the language to use algebra to solve geometric problems. This is done by thinking of the problem as solved and all lines as drawn, assigning the letters $a, b, c, \ldots$ to the known lengths and $z, y, x, \ldots$ to unknown ones (yes, this convention stems from Descartes!), then one uses the geometric relations in length specified by the problem to define equations. The famous example of an algebraic relation that can be induced from geometric relations is the Pythagorean theorem: $a^2 + b^2 = c^2$. Using such equations one can then express the unknown length in terms of known ones. Note that in our arithmetic a^2 does not mean a square, but another line segment with length a^2. Deriving equations that correspond to geometric relations is a whole field in itself and hence we will not go into detail of how to solve geometric problems in this way. Examples of how to do this can be found in secondary literature or Descartes' original work.[12]

Lengths alone are not enough to do geometry. The attribute is an aspect of shape and is independent of position (at what points the line segment with the given length in question exists), so with lengths alone, it is not possible to determine the (relative) position of different line segments and therefore no geometry is possible. By specifying a standard position and relating all other positions to it, we can create a system in which we can determine both lengths and positions, that is what we do with a coordinate system.

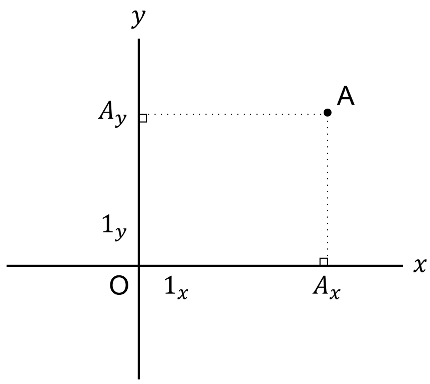

Figure 3.4: A simple coordinate system

We consider a standard position, the origin, and take two perpendicular lines on it. For both lines, we choose a unit length by choosing two unit points 1_X and 1_Y respectively.

Each point A in the geometrical space can be identified by mapping the point on the axes using perpendicular lines to A_X and A_Y respectively and considering the pair of lengths of the line segments OA_X and OA_Y.

There is one problem, however, a problem that can be seen both from the geometric and algebraic perspective. The coordinate system works well if all line segments we consider exist in the upper right quadrant, but currently, each coordinate has four referents. It is not possible to distinguish this point from three points in other quadrants.

A related problem arises from the algebra. We have not yet formally defined subtraction, but it is not hard to do so on the basis of addition. Subtraction is not problematic if one subtracts smaller from bigger lengths, if one however subtracts a bigger from a smaller length then the expression becomes meaningless.

[12]See for example chapter 22 in (Bos, 2012) or (Descartes, 1925).

This problem is inescapable in Euclidean geometry, because in Euclidean geometry lines are infinitely extendable in both directions. If our lines were mere circles, like a clock, there would be no need for negative lengths as we could simply select an orientation point and restart our coordinates if we reach this point again. If we have selected an origin for Euclidean geometry however, this does not work. Nor is it practical (or elegant) to select a point as the origin such that all relevant points in an application lie in one quadrant. Let us therefore define oriented lengths.

3.3. Oriented lengths

The problem we ran into with creating a coordinate system based on lengths is that on a line there are two line segments with the same length with respect to a point. We want to differentiate these two. The solution is of course the idea of having positive and negative lengths. We could simply introduce a '−' symbol for lengths on one side and use the familiar rules for negative and positive numbers.

From a modern viewpoint, no one would be surprised if we were to do this, but I think that by doing this we are stepping over some of the historical opposition that existed against negative numbers. What does it mean to have a length less than zero, and why is it justified to introduce something like this? Let us, therefore, stay true to our mission of defining numerical concepts in purely geometrical terms and introduce negative lengths on the basis of actual geometrical facts of our space.

With the concept of length, we only considered the congruence of line segments and treated line segments equivalent with respect to all other properties. To differentiate the two segments that lead to equal lengths, we want to also consider the orientation of the line segment so that one is 'opposite' that of another.

Let us start by clarifying two properties of line segments: position and direction. We will not give formal definitions of them, but just consider two things that characterise them in relation to other line segments.

On a segment multiple positions can be differentiated, all the points on the line are at different positions. We can consider the position of the segment as a whole by picking a specific point on it, e.g. one of its extremities or the middle. Translating a line segment changes its position. If we pick a point on it as its position and rotate the segment around that point, then the position does not change.

Direction is a relationship between points, in the context of a line segment between the extremities of a line segment. We can measure differences in direction by the angle formed between two lines (or segments). In section 2.4 we discussed how in the context of *(P)*, direction can be considered independent of position. This applies to line segments as well. In the context of *(P)*, the direction of a line segment does not change under translations, but it does change under rotations. This is exactly the opposite of what happens with position.

(When we used direction for infinite lines we treated two possible directions, the two rays into opposite directions as equivalent. This is a consequence of considering direction for infinite lines and will come up again in the context of projective geometry in chapter 5 where the extremities of a line converge to the same point. For line segments, however, it is exactly these two directions that we want to differentiate)

Let us stress here and now that we do not want to consider direction in general in our arithmetic, we only want to differentiate two directions of line segments. Considering direction in general can be done and that would lead to the complex numbers, but defining such multiplication is very involved and is not necessary for our goal of building a coordinate system. What we want to consider is only two possible directions: the two that are each other's opposite. This will mean that our arithmetic of directed line segments will have some restrictions, let us indicate why this is necessary.

When we defined addition for lengths, we defined their sum by putting the line segments next to each other on the same line. One way to view this is that we used an isometry that placed one in the extension of the other on the same line. Within this isometry, there exist two possible components: a translation and a rotation.

The translation is no problem, as we do not consider the position of line segments in the arithmetic. A rotation however, changes the direction of a line. If we are defining addition of lengths with respect to an orientation, then we cannot join two line segments with different directions to create a new line segment, as that would require changing the direction of one. Oriented length arithmetic will therefore be restricted to line segments that can be put next to each other through translation alone. This means line segments that lie on parallel lines.

Here we see the value of identifying that *(P)* enables a comparison of directions independent of positions. If *(P)* did not hold, then it would not make sense to develop this arithmetic on parallel lines, as it means that some line segments might have to be rotated to add them and others not. The restriction to parallel lines will not be a real restriction in the end, because in a coordinate system we break up a line segment into components that are parallel to the axes.

In summary, oriented arithmetic exists within a given class of parallel lines. We therefore choose a class of parallel lines and start our journey here. Note that we will at no point use any particular property of this class, only the fact that we remain in the same class of parallel lines. So our definitions are applicable to any class of parallel lines. Let us first define what a class of parallel lines is.

Definition 3.7. A *class of parallel lines* with respect to a line ℓ is the set of lines P_ℓ such that any line in it is parallel to ℓ.

The reader can check that parallelism is an equivalence relation, thus that any line within a class of parallel lines can be used to define that class. One can easily extend a linear order of a line to a relation on the class of parallel lines by considering points equivalent if they lie on the same perpendicular line.

Definition 3.8. A *linear quasi-order on a class of parallel lines* P_ℓ is a relation $<_P$ with the following properties. If $<_P$ is restricted to any particular line, then it is a linear order of that line. If two points A, B lie on different lines, let C be the point on the same line as A in this parallel class that is uniquely determined by a line that contains B and is perpendicular to any line in the class,[13] then $A <_P B \iff A <_P C$.

It is not possible to compare two points that are cut by the same perpendicular line, so it is not a real linear order. Our definition defines a relation between two arbitrary points that lie on lines in the same parallel class, the definition also directly demonstrates the existence of such an order. We will use the linear order of a parallel class to define an orientation between two points, i.e. the two possible directions.

Definition 3.9. An *orientation* o of points A, B on lines within a parallel class P_ℓ with linear quasi-order $<_P$, is a relation between two points A, B defined as $o(A, B) \sim +1$ if $A < B$ and $o(A, B) \sim -1$ if $B < A$.

The '1' used in orientation is just a symbol for unit, it is not a length. Notice that $o(A, B) \sim -o(B, A)$. With the concept of orientation, we can define oriented line segments.

Definition 3.10. An *oriented line segment* $\overline{AB}$ on a line within a parallel class P_ℓ with linear quasi-order $<_P$ is the line segment AB with an orientation $o(A, B)$.

We use the bar as a symbol on top of the two points to indicate that an oriented line segment is not the same as a line segment. We also can use short-hand notation and write $\overline{AB} \in P_\ell$ to signify that $\overline{AB}$ lies on a line within the parallel class P_ℓ, similarly we can say $A \in P_\ell$ if A lies on a line in P_ℓ.

Definition 3.11. *An oriented length* $\pm a$ is an attribute of an oriented line segment $\overline{AB} \in P_\ell$ such that for any other oriented line segment $\overline{CD} \in P_\ell$ with oriented length $\pm c$, $\pm a = \pm c \iff AB \cong CD$ and $o(A, B) \sim o(B, C)$. We write $+a$ or $o(a) \sim +1$ if $o(A, B) \sim +1$ and $-a$ or $o(a) \sim -1$ if $o(A, B) \sim -1$.

We see that an oriented length is made up of two independent attributes, length and orientation. By definition, these determine a unique oriented length. We can consider these separately and in the context of an oriented length a, we will refer to the first as the absolute length $|a|$ and the second as the orientation $+$ or $-$. The word 'absolute' is merely used to differentiate it from oriented length, but it is the same attribute as 'length' and for absolute length, we can use addition and multiplication defined for lengths. We will define the addition and multiplication of oriented lengths through its two attributes of absolute length and orientation. Generally, we will only use the orientation symbol in the case of a negative length and in other cases omit it. The sum of two oriented lengths now is as follows:

Definition 3.12. Given two oriented lengths a, b that stem from line segments in the same parallel class, we define their *sum* $a + b$ to be the following oriented length.

- If a, b have the same orientation, then $|a + b| = |a| + |b|$ and $o(a + b) \sim o(a) \sim (b)$.

[13] That such a line exists follows from the existence of a unique perpendicular line through a given point

- If a, b have opposite orientation consider oriented line segments $\overline{OA}$ with oriented length $\pm a$ and $\overline{BA}$ with oriented length $\mp b$ that lie on the same line, then c is defined as the oriented length of oriented line segment $\overline{OB}$.

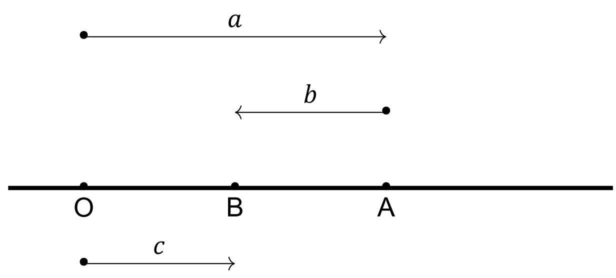

Figure 3.5: Addition of two lengths with opposite orientation

Proposition 3.6. *The sum of two oriented lengths a, b with opposite orientation and $|a| > |b|$ has the same orientation as a.*

Proof. Consider the oriented line segment $\overline{OB}$ from the definition. As $|a| > |b|$, the line segment $\overline{OB}$ is smaller than $\overline{OA}$ thus $O * B * A$. Now consider the linear order of the line. Either $O < A$, in which case also $O < B$, by theorem 2.4; or $O > A$, in which case $O > B$, by theorem 2.4 again. Therefore, $\overline{OB}$ will have the same orientation as $\overline{OA}$ □

Definition 3.13. The (oriented) length of a point $A = AA = \overline{AA}$ is defined as 0.

Note that for any (oriented) length a, $0 + a = 0 + a = a$. We can now also define subtraction.

Definition 3.14. Given two oriented lengths a, b, we define their *difference* $a - b$ to be the sum of a with the length b' that has absolute length b but opposite orientation ($a + b'$).

When multiplying two lengths, we thought of one of the lengths as a stretch. When multiplying two oriented lengths, we can think of a length with opposite orientation as switching the underlying line segment (and thus the orientation) around. This gives the effect of the following binary operation. The operation xAND ($\odot$) is a binary operation used in computer science defined as follows:

$o(a)$	$o(b)$	$o(a) \odot o(b)$
$+1$	$+1$	$+1$
$+1$	-1	-1
-1	$+1$	-1
-1	-1	$+1$

We use it to define the multiplication of two oriented lengths.

Definition 3.15. Given two oriented lengths a, b, we define their product $a \times b$ to be the following oriented length. $|a \times b| = |a| \times |b|$ and $o(a \times b) \sim o(a) \odot o(b)$

Now we have enriched our arithmetic of lengths enough to define a coordinate system, but before we do that let us go completely the other route, leave geometry behind and *only* consider numbers. We will work towards a coordinate system from this opposite direction and end the next chapter by seeing how these two perspectives come together into a coordinate system.

4

Coordinate geometry

4.1. What is a field?

We considered the fundamental concepts that give rise to a coordinate system from within geometry, what we essentially have done is develop a system that makes measurement within a geometrical space possible. Now consider the situation that we have measured some quantities and we want to study how these quantities relate to each other. This is what we do when we form a coordinate system directly out of quantities to study these quantities, not in isolation, but in relation to each other.

Therefore, we will start with something called a field, which for our purposes we can think of as a generalised number system that can be used to measure quantities. The elements of our fields will be numbers, but what numbers exactly are part of our field might differ, what will be important is that the system as a whole satisfies some axioms. (In general, a field need not consist of numbers. For example, it can also consist of functions and finite fields are ‘cyclical’, so quantity might not be the correct way to describe what they measure.)

We will not give fields and their axioms the same thorough introduction and treatment as we gave the geometrical axioms, because fields themselves are not our primary subject. But let us say at least some things that are of interest to us.

In chapter 3 we introduced the method of *analysis*, which consisted of backward reasoning by means of algebra. Introducing axioms for fields (something they did not have in the 17^{th} century), makes the distinction between analysis and synthesis different from that of geometry versus algebra, as the same synthetic axiomatic approach of geometry can be used in algebra as well. In using algebraic reasoning for geometry two issues are relevant for us.

The first relates to the combining of different numbers by means of the arithmetical operations, such operations ought to characterise a new number if we are to use them in reasoning about lengths. We spent a lot of time defining oriented lengths to make sure this is the case for the operation of subtraction. A system in which operations on numbers produce other numbers is the setting in which one can use them generally for reasoning about oriented lengths.

Next to that, we use the algebra itself to relate lengths to each other, what kind of identities can we use to reason about our lengths and relate them to each other For numbers in $\mathbb{R}$ we know all kinds of identities like $a \times b = b \times a$ and $a \times (b + c) = a \times b + a \times c$ that can be used to infer that two numbers are equal, but if we are in a more general setting can we still use all of those?

4.2. Formal definition of a Field and a Cartesian plane

In chapter 3 we worked laboriously to identify the meaning of addition and multiplication for lengths and derive them from a geometry. Here we take a completely opposite route and take the context of a set of numbers for which such operations are defined as the starting point. Here again, we see an application of the idea that the referents of a formal system need to have *some* meaning (for example the interpretation of dilation we gave to multiplication) but may have *any* meaning and in the context of algebra we are not concerned with any particular meaning of multiplication. Let us formally introduce a field:

Definition 4.1. A field is a triple $\langle F, (+, 0), (\times, 1)\rangle$ consisting of an set of elements F that contains two special elements $0, 1$ and two binary operations $+, \times$ such that:

1. $\langle F, +, 0\rangle$ forms an Abelian group:

   For $x, y, z \in F$
   (a) $x + y \in F$ (closure under addition)
   (b) $x + 0 = x$ (additive identity)
   (c) $x + (y + z) = (x + y) + z$ (associativity)
   (d) $x + y = y + x$ (commutativity)
   (e) $\exists(-x) \in F$ such that $x + (-x) = 0$ (existence of inverses)

2. $\langle F^*, \times, 1\rangle$ forms an Abelian group:

   For $x, y, z \in F^* := F \setminus \{0\}$
   (a) $x \times y \in F^*$ (closure under multiplication)
   (b) $x \times 1 = x$ (multiplicative identity)
   (c) $x \times (y \times z) = (x \times y) \times z$ (associativity)
   (d) $x \times y = y \times x$ (commutativity)
   (e) $\exists x^{-1} \in F^*$ such that $x \times x^{-1} = 0$ (existence of inverses)

   For $x, y, z \in F$

3. $x \times (y + z) = (x \times y) + (x \times z)$ (distributivity of $\times$ over $+$)

Subtraction and division are not explicitly defined by the axioms, but obviously, we can define subtraction as the addition of the additive inverse of a number and division as multiplication with the multiplicative inverse of a number. Instead of working through all axioms as we did for the geometrical axioms in chapter 2, I will just give some examples of fields and non-fields.

Example 4.1. Examples of fields are:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$
- The constructible numbers: the lengths of line segments that can be constructed in Euclidean fashion by compass and straightedge. This field can be useful in demonstrating that certain constructions are impossible.
- $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number. One can think of these as a clock that takes $p + 1$ many values.
- $\mathbb{Q}(\sqrt{2})$, that is, $\{q + p\sqrt{2} : q, p \in \mathbb{Q}\}$

Example 4.2. Examples of non-fields are:

- $\mathbb{Z}$, it forms a group under addition, but multiplicative inverses do not exist in general. It does not form a group under multiplication.
- The positive real numbers $\mathbb{R}^+$, it forms a group under multiplication, but additive inverses do not exist in general.

Lastly for contrast, let us give an example of an operation under which a field is *not* closed: square roots.

Example 4.3. If $a \in F$ then $\sqrt{a}$ need not be in F, that is, there need not be $b \in F$ such that $b \times b = a$.

As we are working towards plane geometry, it might not come as a surprise that we are going to create a two-dimensional coordinate system out of our field. Let us consider the Cartesian product F^2 of our field, which we will call the Cartesian plane Π_F over the field F. (This is a metaphorical plane)

Our plane inherently has a lot of structure that it derives from the field axioms. In the next section, we will show which of Hilbert's axioms are implied by the field axioms and what additional restrictions we need to put on our field to obtain all of Hilbert's axioms.

4.3. Geometry from field

4.3.1. Basic definitions and incidence

The elements of our Cartesian plane Π (we omit the subscript F if it is not relevant) identify points. We identify lines by means of linear equations of the form $ax + by + c = 0$ for $a, b, c \in F$ and a, b not both 0. Let us also consider a different expression for all lines. All lines can also be expressed either in the form $y = px + q$ or $x = q$, with $p, q \in F$. In the first case, we call p the slope of the line and in the second case we say that the line has a slope of ∞ (this is not an element of our field, it is merely is a symbol).

With definitions for points and lines, we can demonstrate the following result:

Proposition 4.1. *If F is any field, the Cartesian plane Π_F satisfies Hilbert's (I.1) to (I.3) and (P).*

Proof. Consider $a, b, c \in F$ such that they characterise a line (so a, b not both 0) *(I.1)* to *(I.3)* follow easily. Parallel lines are lines that do not intersect (unless they are equal), the only way this can happen is if the two lines have the same slope. But given a slope, a point uniquely determines a line. So there is always a unique parallel line through a point outside the line: the line with an equal slope. □

With these four axioms, we can identify that a Cartesian plane is an affine plane.

4.3.2. Ordered fields

Let us now turn to Hilbert's axioms of order. Like the linear ordering of a line, we will start by introducing an ordered field. This will exclude fields like $\mathbb{Z}/p\mathbb{Z}$ which behaves like a clock or other fields where there does not even exist a cyclic order.

Definition 4.2. An ordered Field is a Field F together with a linear order $<$ (see Definition 2.2) such that for any $a, b \in F$ and $0 < a, 0 < b$:

- $0 < a + b$
- $0 < a \times b$

Proposition 4.2. *If F is a field and in the Cartesian plane Π_F a betweenness relation is defined satisfying (O.1) to (O.4), then F is an ordered field. Conversely, if F is an ordered field, we can define betweenness in Π_F to satisfy (O.1) to (O.4).*

Proof. In theorem 2.4 we have shown that *(O.1)* to *(O.4)* imply the existence of a linear order on a line based on the betweenness axioms. On line $y = 0$ in Π_F, every point identifies a unique element of our field F. Using the canonical linear order of the line we induce a linear order $<$ for the field (we define its orientation by means of $(0, 0) < (0, 1)$, which means $0 < 1$ for the elements in the field). The axioms of order imply the existence of a separation of the line $y = 0$ by the point $(0, 0)$, from which we can only take the $x-$coordinate. One side of this separation consists of all elements $x \in F$ with $x > 0$. Both $a + b$ and $a \times b$ are in this set,[1] so $a + b > 0$ and $a \times b > 0$.

If F is an ordered field, we define betweenness for points $a, b, c \in F$ as $a * b * c$ if $a < b < c$ or $c < b < a$. *(O.1)* is by definition true, *(O.2)* follows from taking either the sum or the difference (i.e. sum with the inverse) of the two elements. *(O.3)* follows from the anti-symmetry and totality of a linear order. For *(O.4)*, take a line $ax + by + c = 0$, then $ax + by + c > 0$ and $ax + by + c < 0$ form such a separation (we will leave it for the reader to check this). □

4.3.3. Congruence of line segments and angles

Within an ordered field, we can use the definitions for line segments and angles that were introduced in the axiomatic construction. Any line segment can be specified by its two endpoints, because we are reasoning from numbers we would like to leverage our knowledge of magnitudes and define two line segments as congruent by means of a measure of the distance between them. For this, we will take inspiration from the Pythagorean theorem: $a^2 + b^2 = c^2$. Note that this does not necessarily have anything to do with geometrical squares in this context. Like with oriented arithmetic, we are simply relating numbers multiplied by themselves (which are themselves numbers).

Definition 4.3. A Pythagorean Field is an ordered field F for which $a, b \in F$ implies that $\sqrt{a^2 + b^2} \in F$

[1] Unfortunately I was not able to find a concise formulation of why this has to be the case. That it is true can be seen from the length arithmetic, but using that relies on results not yet demonstrated.

A Pythagorean field is a field such that in the Cartesian plane generated by it, we can measure the distance between any two points in a plane by means of a number corresponding to the length of the line between them.

Definition 4.4. For two points $A = (a_1, a_2), B = (b_1, b_2)$ in a Cartesian plane Π_F generated by a Pythagorean field F, the *distance* between A and B is defined as

$$d(A, B) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

The *square distance* is defined as $d^2(A, B) = (a_1 - b_1)^2 + (a_2 - b_2)^2$.

The square distance exists in each field, so it can be used in the case of non-Pythagorean fields. Squaring the distances will not change the existing congruences, so the definition of square distances could be substituted to give a universal definition.

Definition 4.5. Two line segments AB and CD in an Cartesian plane over an ordered field F are *congruent* if $d(A, B) = d(C, D)$

Let us define a measure for the congruence of angles as well. An angle was defined using two rays. Consider the two lines ℓ, ℓ' on which the rays lie. If their respective p, p' are such that $p \times p' = -1$, then we call the angle that they form a right angle (if one of the slopes is ∞, consider the angle is a right angle if the other line has a slope of 0). If an angle is contained in a right angle, it is called acute and if it contains a right angle it is called obtuse.

With right angles, we can use a right-angled triangle to measure angles using the traditional tangent function (opposite over adjacent). Because an opposite angle is uniquely related to an acute angle (the angle on the opposite side of one of the lines), we can use that acute angle to define a measure for obtuse angles as well.

Definition 4.6. In a Cartesian plane over a Pythagorean field consider an angle α, formed by two distinct rays $\overrightarrow{OA}$, $\overrightarrow{OB}$. Consider three cases:

- If α is acute, then consider a third line through B that forms a right angle with $\overrightarrow{OB}$ and intersects $\overrightarrow{OA}$ in C. We define the tangent of α as:

$$\tan \alpha = \frac{d(O, C)}{d(O, B)}$$

- If α is obtuse, consider C on $\overleftrightarrow{OA}$ such that $C * O * A$. Then $\overrightarrow{OC}$ and $\overrightarrow{OB}$ form an acute angle β. We define the tangent of α as $-\tan(\beta)$.
- If the angle is a right angle we say that the tangent is ∞.

That such a point C from the definition exists follows from the fact that (1) the angle α is an acute angle and (2) our field is Pythagorean, so all square roots exist. The definition of tangents for obtuse angles makes sense because these angles are ‘the opposite’ of the acute angle that complements it.

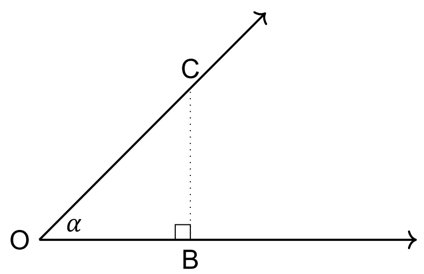

Figure 4.1: Defining the tangent of an acute angle

We will end this section by stating some propositions from Hartshorne’s book,[2] the proofs can be found there as well.

[2](Hartshorne, 2000, Chapter 3)

Proposition 4.3 (Hartshorne 16.1). *Let F be an ordered Pythagorean field and let Π_F be the associated Cartesian plane. Then Π_F satisfies (C.1) to (C.5) and (EEI).*

Proposition 4.4 (Hartshorne 16.2). *Let Π_F be the Cartesian plane over an ordered field F. The following three conditions are equivalent:*

- *Π_F satisfies the circle-circle intersection property (CCI).*
- *Π_F satisfies the line-circle intersection property (LCI).*
- *The field F is a Euclidean ordered field ($a \in F$ implies $\sqrt{a} \in F$).*

From the algebraic perspective, a difference between the intersection of circles and intersections of lines becomes apparent: lines are first-degree equations, they are linear. Circles on the other hand are second-degree equations ($x^2 + y^2 = r^2$).

Proposition 4.5 (Hartshorne 16.4). *Let F be the set of all real numbers that can be obtained from the rational numbers by a finite number of operations $+, -, \times, \div$ and $\sqrt{a}$ for $a > 0$. Then K is a Euclidean ordered field.*

Theorem 4.6 (Hartshorne 17.3). *If F is a Pythagorean ordered field, then the plane Π_F is a Hilbert plane satisfying (P). The plane Π_F will be Euclidean if and only if F is Euclidean.*

4.4. Field from geometry

In the last section, we have developed a coordinate system that is suited to reason about geometry. We have seen that by creating a Cartesian plane out of a field, we obtain something that is very suited to reason about the geometry of a plane. By narrowing the kind of fields we consider (ordered, Pythagorean, Euclidean), we obtained a field such that the associated Cartesian plane is powerful enough to reason about all of Euclidean geometry, it describes a plane that satisfies all of Hilbert's axioms. Let us now go the other way and instead of starting with a field and demonstrating its applicability to geometry, we start with a geometry and finish our construction of a coordinate system by constructing a field within the geometry. Then we will show that the obtained coordinate system is isomorphic to the original geometry: no information about the plane is lost by adopting a coordinate system.

Definition 4.7. Let Π and Π' be two Hilbert planes. An *isomorphism* between Π and Π' is a map ϕ that preserves all terms and relationships:

- ϕ is a bijection
- ℓ is a line in Π if and only if $\phi(\ell)$ is a line in Π'

  Given points $A, B, C, D \in \Pi$:
- $A * B * C$ if and only if $\phi(A) * \phi(B) * \phi(C)$ in Π'.
- $AB \cong CD$ if and only if $\phi(A)\phi(B) \cong \phi(C)\phi(D)$.
- Given two $O, O' \in \Pi$ $\angle OAB \cong \angle O'CD \iff \angle\phi(O)\phi(A)\phi(B) \cong \angle\phi(O')\phi(C)\phi(D)$.

As we can see, the definition of an isomorphism has many similarities to that of Euclidean isometry (Definition 2.10). They both use the same concepts, but the difference is that the Euclidean isometry establishes a congruence relationship between segments and angles before and after the map, this only makes sense if the isometry is a map of the plane to itself. An isomorphism on the other hand does not compare a plane with itself, it can compare two different planes with each other and there need not be any meaning to comparing elements of one to that of the other. As an example outside geometry, the possible rotations of a cube are isomorphic to the permutations of four beads. Any two rotations of a cube are intrinsically related to each other by a third rotation, but relating a rotation of a cube with an order of marbles only makes sense *after establishing an isomorphism.* This example also indicates that isomorphism is a broader term, here we only consider isomorphisms between Hilbert planes.

Let us now bring back the oriented lengths of chapter 3 and relate them to the fields we have discussed.

Theorem 4.7. *The set of oriented lengths of a geometry Π forms a field.*

Proof. We will defer the proof of associativity and commutativity to chapter 7, where these will play a special role. We prove the others.

1(a) Closure under addition. This follows from *(C.1)* and the definition of the addition of lengths.

1(b) Additive identity. Any point has length 0 by definition, so there exists an additive identity.

1(e) Additive inverses. For any oriented length, the same line segment with opposite orientation gives the additive inverse.

2(a) Closure under multiplication. This follows from *(C.1)* and the definition of multiplication of lengths.

2(b) The definition of multiplication required us to select a unit length 1, which is the multiplicative identity.

2(e) Multiplicative inverses. For a given length x define a linear dilation ϕ^{-1} that maps x to 1. Then the magnitude of the linear dilation, a^{-1}, is associated with a length such that $x \times a^{-1} = 1$

3 Distributivity. Let us demonstrate this with a picture, the result does not depend on the particular lengths in the picture but it is much clearer than writing out the symbols and the relevant equalities are straightforward.[3]

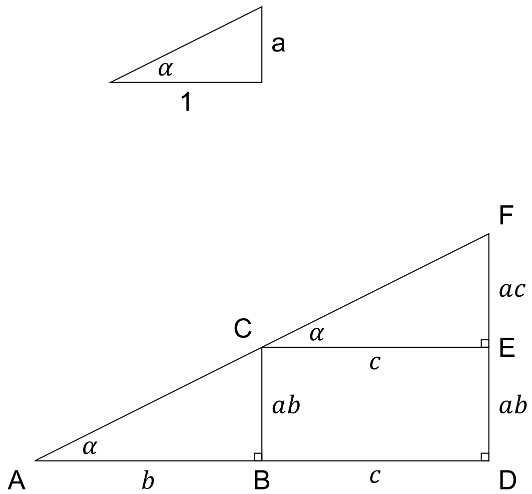

Figure 4.2: Proof of distributivity

It can be seen that $a(b+c) = ab + ac$.

□

As the lengths of a geometry form a field, the pairs of lengths form a Cartesian plane. We obtain a profound result:

Theorem 4.8 (Hartshorne)**.** *Let Π be a Hilbert Plane satisfying (P). Let F be the ordered field of oriented lengths in Π. Then F is Pythagorean, and Π is isomorphic to the Cartesian plane F^2 over the field F.*

We will state how to construct the isomorphism, but we will not demonstrate that it is an isomorphism, the reason is that with all results established so far, it is relatively straightforward to check it, but working out all the details will take up multiple pages that can also be spend on more interesting mathematics.[4]

[3] This drawing is adapted from (Hartshorne, 2000, p.172).
[4] The details are all worked out in (Hartshorne, 2000, p.187-191).

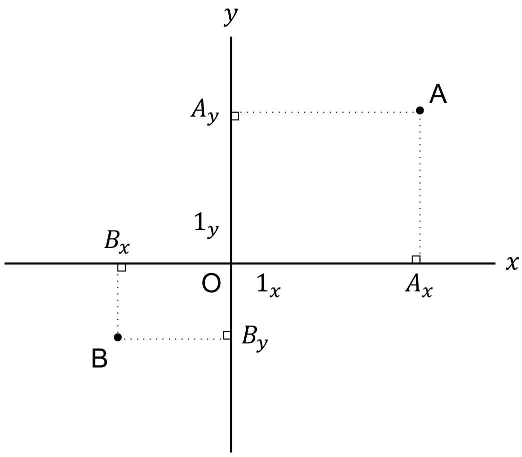

Figure 4.3: Defining coordinates for points

Proof. Define the map $\phi : \Pi \to F^2$ based on how we defined coordinates. We start by choosing two perpendicular lines as axes and identify each point by projecting it on the axis. This maps a point P to a unique pair of oriented lengths $(\pm a, \pm b)$. ϕ now defines an isomorphism. □

Theorem 4.9. *A geometry Π satisfying (I.1) to (I.3), (O.1) to (O.4), (C.1) to (C.5), (EEI), (P), (Dedekind) is isomorphic to $\mathbb{R}^2$, the geometry is categorical.*

Proof. The axiom *(Dedekind)* can be interpreted as an axiom for the elements of the field as well. For elements of our field, this axiom establishes the existence of Dedekind cuts, which implies the field is isomorphic to the real numbers (we will not demonstrate this fact). Using theorem 4.8, Π is isomorphic to F^2 which is isomorphic to $\mathbb{R}^2$. □

4.5. Conclusion of chapters 2 to 4

In the past three chapters, we have taken a deep delve into Euclidean geometry. We have demonstrated two different approaches: the axiomatic approach which starts with geometrical axioms and bases all proofs of propositions on these axioms alone, and the approach of coordinate geometry which demonstrates geometrical propositions by algebraic calculation and rests on the field axioms.

Moreover, we have constructed an arithmetic of oriented lengths based purely on geometric axioms. This result was in part Hilbert's goal. He sought to establish the consistency of the axioms for Euclidean geometry.[5] Such a demonstration was crucial for formalist mathematics, because if the formal system has no inherent meaning then there is nothing that guarantees that the axioms are consistent. By demonstrating that arithmetic can be constructed from within geometry, we have proven the relative consistency of Euclidean geometry: *if* arithmetic is consistent, *then* Euclidean geometry is too. Little did Hilbert know at this time that proving the consistency of arithmetic without reference to another formal system is a vain dream, something Gödel demonstrated later. If one however adopts the view that these axioms are not arbitrary and meaningless, but merely general methods of inference, then the axioms are ultimately based on observations and facts. This makes proving the consistency of the axioms not that interesting.

The most important result we have demonstrated is the isomorphism between a geometrical plane and the coordinate system of oriented lengths that is based on it. The coordinate system can measure every position and relationship we have defined in the plane. Given the enormous power of algebraic reasoning, it is no surprise that this integration greatly widened the study of geometry.

[5](Kline, 1972, p.1014)

5

Projective coordinate geometry

5.1. Introduction

By studying the relations between axiomatic and coordinate geometry in the context of Euclidean geometry, we have laid a foundation to study this issue in a different context: projective geometry.

Euclidean geometry is about measuring the intrinsic geometrical properties of shapes. Everything from houses, cars, and smartphones to skyscrapers is built based on such knowledge. When we observe the world around us, however, we do not just intuit the shape of objects, we always perceive them from a given perspective. This perspective alters the way objects appear to us. If you look at a railroad track that extends far in the distance, the parallel lines of the tracks seem to converge, even though you know that in fact, they do not.

In the 15^{th} century, painters became interested in accurately representing a three-dimensional scene on a two-dimensional plane. This required taking into account from which perspective a scene was viewed and the effect that this had on how objects appear to a viewer: the mathematical subject of projective geometry was born. Today, the field of computer graphics is focused on recreating a three-dimensional world on a two-dimensional screen and conversely, computer vision aims to recreate a three-dimensional model of the world from two-dimensional photos and videos. Both require the study of shapes, not with respect to their intrinsic characteristics, but with respect to their appearance from a given perspective.

Projective geometry studies shapes *in perspective*. Inference in projective geometry rests on the properties of shapes that remain invariant under different perspectives. Therefore, one might see projective geometry characterised as studying the geometrical properties that remain invariant under projective transformations (the transformations that relate two perspectives). We will restrict ourselves to studying planes in perspective and are concerned with the properties that are characteristic of shapes in perspective, what does *not* change under different perspectives?

This time we will start with the coordinate geometry and consider the axiomatic approach afterwards because it is easier to see the connection with Euclidean geometry from a coordinate-centred approach. It will make it easier to understand the projective axioms afterwards.

For all explanations in projective geometry, we will also assume that we are considering a position-space, i.e. that points are positions and lines and planes are actual lines and planes. This is simply to aid in explaining what a projective space is, it will not affect the formal theory in any way.

5.2. Construction of a projective plane from three-dimensional Euclidean space

In chapter 2 we made the distinction between Euclidean space (the totality of all positions ‘out there’), Euclidean geometry (an axiomatic theory) and $\mathbb{R}^3$ (a model). We will use this distinction again here. Instead of immediately defining coordinates, we will first motivate such definitions by showing how a projective plane arises in the context of three-dimensional Euclidean space. Afterwards, we will use $\mathbb{R}^3$ to define coordinates in this space.

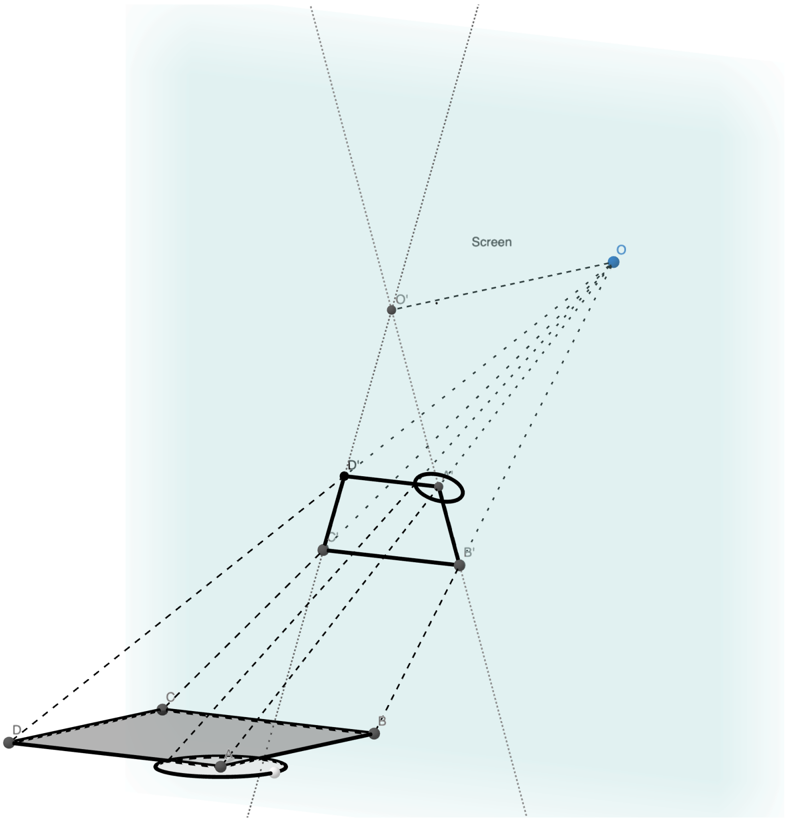

Figure 5.1: Projective transformation

Consider the following image of the projection of a square and circle on a screen, viewed from a point O.

The square is transformed into a quadrilateral: the congruence relationships between lines and between angles both change, its area is different and sides that were parallel are not parallel anymore. The circle completely changes its shape and becomes an ellipse. At the same time, two important things stay the same: straight line segments remain straight line segments and the incidences (of the square and circle) remain incidences. The screen is what gives rise to a projective plane.

In a three-dimensional Euclidean space let us consider a specified point (the origin) from which we take different two-dimensional perspectives (the screen). We consider what happens to a plane under projections of one perspective to another with respect to the origin. One could see this as considering what happens to all shapes drawn on the ground under different perspectives, what happens when we move the screen around. The transformation between different perspectives is a projective transformation. An informal definition of a projective transformation would be a transformation obtained by mapping each point in a plane to another plane by means of a line from a specified point outside these planes (the origin). In fig. 5.1 it is the transformation that maps the square and circle on the ground to the glass screen.

If two points in three-dimensional space are on the same line from the origin, they will get mapped to the same point in the projective plane. We could consider any point on our projective plane by considering the lines from each point in the projective plane that also go through the origin, we then consider points in three-dimensional space equivalent if they are on the same line. Each point in our plane is identified by means of a unique *line* through that point and the origin, but from our perspective-view, this line is not a line but a *point*. The fact that some objects might not be visible to an observer because other things are in the line of sight will be very familiar to all. If we map planes to planes, there will be no such obstruction.

There are however also lines of sight that do not intersect the plane: the lines that are parallel to the plane. These lines (i.e. points from the projective perspective) turn out to be a central part of our projective plane! If we look at our original image, we see that the lines that were parallel before they were projected, meet exactly on this line. In a given perspective, these points are the vanishing points on the horizon where parallel lines 'meet'. This will be a feature of our projective space. To identify a perspective we need to add such vanishing points. With respect to the Euclidean plane considered without perspective, they represent a certain horizon of our Euclidean plane, 'points at infinity.' However, considered as points of our projective space, they are like any other point. To exemplify this, look at the painting in fig. 5.2. As observers, we can see lines that would be parallel in the corresponding Euclidean space independent of a perspective (the red lines). In such a space these lines would never intersect. But considered as part of a perspective on a scene (a point in the painting) there is such a point where they intersect, and in the painting, this point is just like any other. (Some difficulty arises because the painting is depicting a three-dimensional scene, so Aristotle and Plato are obstructing the vanishing point, but if we just considered the perspective on a plane, one would see a full horizon).

Figure 5.2: The School of Athens, masterfully painted in perspective. Can you spot the geometers?

The points on lines that are above the horizon are lines that intersect the plane behind us. We are considering a perspective, not just in one direction (how we would see it when we open our eyes), but in all directions. The right view is that of a 3-dimensional camera that projects everything on a single screen. This is demonstrated in fig. 5.3. It also shows that there is only one horizon, the vanishing point of one direction of a line is the same as the other direction of that line.

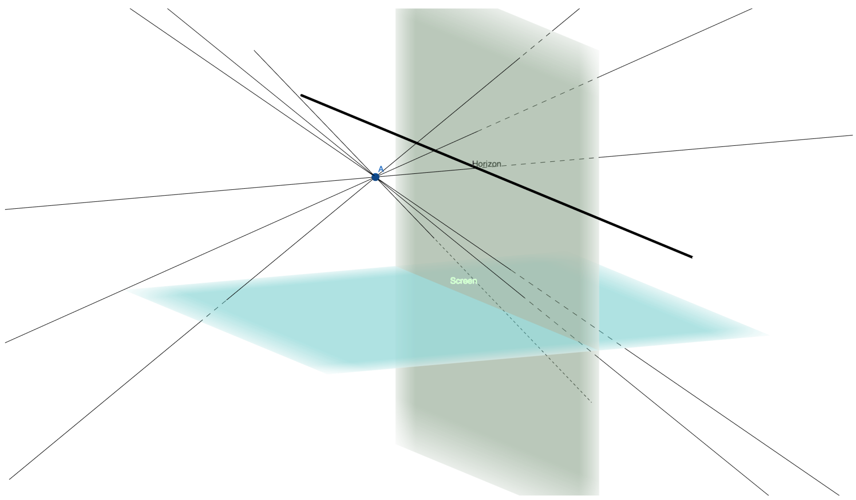

Figure 5.3: A projection on the screen from all directions

Reconstructing a Euclidean plane from a projective plane

Consider two important themes in applying projective geometry, the first is that of creating a projective geometry based on Euclidean three-dimensional space. This is the problem painters faced and this is what the field of computer graphics faces today. Another theme is that of recovering Euclidean data (lengths and angles) based on a projective space. Starting with a projection alone, what do we need to measure lengths and angles?

Suppose we are given a projective plane, this can be in the form of an image (pixels of data) or just as a mathematical description. Let us also suppose that the shapes we are interested in stem from a projection of two-dimensional shapes and not of three-dimensional shapes (if we want to identify three-dimensional data projected on a two-dimensional plane, we need multiple projections).

The most basic thing we can do is to identify the horizon and remove it. Then we have all the points that make up a Euclidean plane because we derived a projective plane from a Euclidean plane precisely by adding this line. Moreover, we can identify parallel lines, lines are parallel if they intersect the horizon at the same point.

But unfortunately, this is where it stops. Consider the projected square of fig. 5.1. If we were given only the glass screen and the horizon line, then we would be able to infer that $A'B'C'D'$ stems from a parallelogram, but this is all we can know about the shape. We would not be able to discern if it came from a square, a rectangle or simply a parallelogram. Removing the horizon line gives us an affine space, a space with parallelism but no congruences of line segments or angles. We do have the weaker sense of congruence: that between segments on parallel lines.

To do better we need more information of underlying shapes in the projective plane. To identify angles, we would have to know that $A'B'C'D'$ stems from a rectangle. In that case, we have a standard angle in the projective plane to measure other angles by. Because of the parallel postulate, these angles are independent of position, so we could have a general measure for angles. To identify line congruences, we would need to know that $A'B'C'D'$ stems from a square, then we can compare the different perpendicular directions with each other and obtain all congruence relationships.

5.3. Homogeneous coordinates

Let us now introduce coordinates to identify points and calculate in such a space. We will start by considering coordinates for three-dimensional Euclidean space and build them out to *homogeneous* coordinates for our projective space. The reason we will use homogeneous coordinates and not the familiar Cartesian coordinates $\mathbb{R}^2$ is that we need to add the horizon points. Simply adding them to our Cartesian coordinates would make them special points, whereas, from our projective perspective, they are not special at all. Let us, therefore, start with $\mathbb{R}^3$, coordinates for three-dimensional Euclidean space, and consider the point $(0, 0, 0)$ the origin from which we take different perspectives.

Any plane that does not contain $(0, 0, 0)$ will induce a perspective. In this perspective, any point on the same line that passes through the origin is considered equivalent: they get mapped to the same projective point. Lines through the origin in $\mathbb{R}^3$ can all be identified by the equation $ax + by + cz = 0$, where $a, b, c \in \mathbb{R}$ (not all equal to 0) characterise a specific line and all combinations of $x, y, z \in \mathbb{R}$ that solve this equation are points on the line.

Note that a, b, c characterise a line up to a non-zero scalar factor. For any $\lambda \neq 0$,

$$ax + by + cz = 0 = \lambda(ax + by + cz) = (\lambda a)x + (\lambda b)y + (\lambda c)$$

characterises the same line. This uniqueness up to one scalar factor stems from the one-dimensionality of our line. To identify a unique line, we consider the three numbers as a *ratio* and not as three distinct values. One might think that we could get around this by explicitly considering the ratios $(a/c, b/c, 1)$ as coordinates. The last coordinate will always be 1 in this case so that we can drop it. This gives us a special line, however, that of $c = 0$, when such division is not defined. In relating projective and Euclidean spaces this is important. If you know what the horizon line is in a projective plane, then choosing that line to be the line with $c = 0$ is part of recovering a coordinate system in which the Euclidean concepts of length and angles do have physical significance.

In Cartesian coordinates for the plane, we wrote (x, y) to identify a point. For homogeneous coordinates, we identify a point as $(x : y : z)$, where the double colons indicate that we are considering their ratio and not the particular values. $(x : y : z) = (x' : y' : z')$ if there is a scalar $\lambda \neq 0$ such that $(x, y, z) = (\lambda x', \lambda y', \lambda z')$. There is one combination of numbers that does not characterise a line in $\mathbb{R}^3$: $(0 : 0 : 0)$.

Note that because of the symmetry of the homogeneous linear equation $ax + by + cz = 0$ we can think of the coordinates to be the numbers that characterise a line or as numbers that characterise points that are all on the same line. We use the symbols x, y, z for a coordinate because this is the convention, but it might make more sense to think of a projective point as describing a line of sight in three dimensions and not solutions to a linear equation. Lines through the origin of $\mathbb{R}^3$ are also called one-dimensional vector spaces. This gives rise to the concise definition:

Definition 5.1. The *real projective plane* $\mathbb{P}^2$ is the set of one-dimensional vector spaces of $\mathbb{R}^3$:

$$\mathbb{P}^2\mathbb{R} := \{U \subset \mathbb{R}^3 | (0,0,0) \in U, \dim(U) = 1\}$$

5.3.1. Points in general position

The coordinate system of $\mathbb{R}^3$ has given rise to points of our projective plane, but we are not in a place to use these coordinates. The parallel in the Euclidean situation would be that we know that points on two lines are going to form our coordinates, but we have not yet selected the standard directions and unit lengths. In a Euclidean plane, we considered an origin and selected two unit line segments with respect to it. So in effect, we have chosen three non-collinear points, the origin and the extremities of the two unit line segments. This enabled us to identify any point in our space with unique pair of coordinates. In the language of vector spaces, this means selecting a basis for our vector space (we only considered selecting the standard basis of unit lengths in perpendicular directions).

In our projective plane, we will see that three points are not enough, we will need a fourth point. This stems from the fact that the *same* point in our projective space can be identified by *different* coordinate numbers (the ones that are scalar multiples of each other). We need an extra point that makes the coordinate system 'rigid.'

Consider selecting three non-collinear points in the Projective plane. These three points will be the equivalent of unit line segments in Euclidean space, so we can consider taking the basic coordinates for these three points, $A = (0 : 0 : 1), B = (1 : 0 : 1), C = (0 : 1 : 1)$.

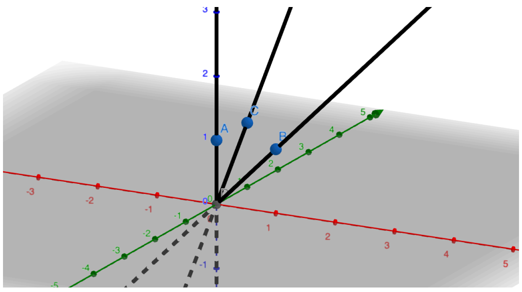

Figure 5.4: The lines through the points $(0, 0, 1), (1, 0, 1), (0, 1, 1)$

In the underlying vector space $\mathbb{R}^3$, these points are three lines. From the projective perspective, all points on these lines are equivalent. However, there are different planes in $\mathbb{R}^3$ that intersect these three lines. The three basic coordinates cannot distinguish between these planes, but if we were to take a fourth point that is non-collinear with these three basis points, then it would have *different* coordinates depending on which plane corresponds to our projective plane. Consider two different planes and consider the point that would correspond to $(1 : 1 : 1)$ in each of them (see fig. 5.5 where this point is D). These two points do not lie on the same line, so they are different points in the projective space.

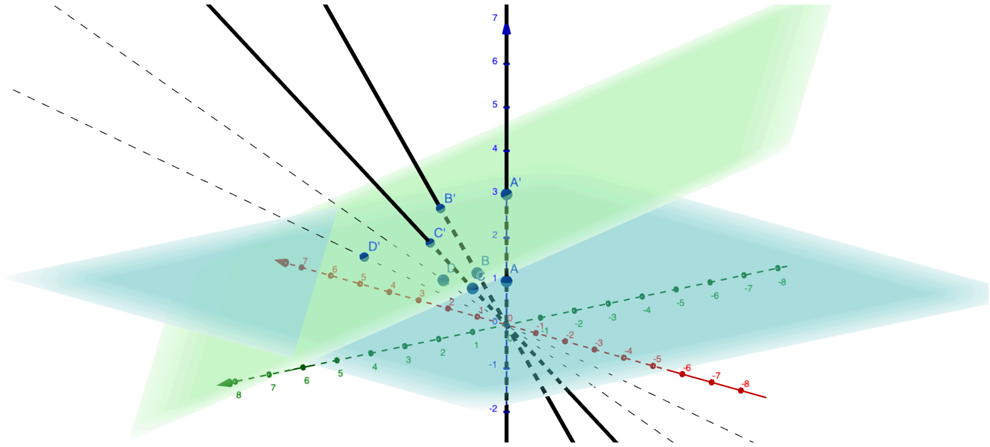

Figure 5.5: Two different planes that all cut the three lines at different points

By selecting a fourth point, we will fix the orientation of the plane as that fourth coordinate needs to be kept in place as well. We can assign the coordinates $D = (1 : 1 : 1)$ to this point. Now we still have a degree of freedom in our plane, we could scale it away from the origin in a way that does not tilt or rotate the plane. This will however not change the coordinates of any of the points in the projective plane, as these are equivalent with respect to one degree of scaling. This is reflected in fig. 5.6, where two different planes are shown but all points lie on the same line through the origin.

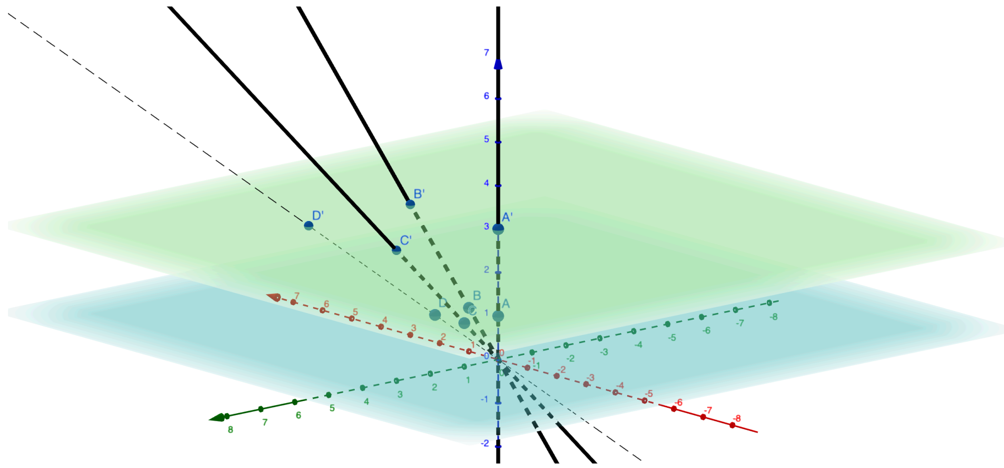

Figure 5.6: Two different planes that all cut the three lines at different points

We call four points, if no three are collinear, points in general position. They are the projective equivalent of a basis for our $\mathbb{R}^2$ coordinate space.

What happened to the horizon line when selecting such coordinates? Consider the plane in fig. 5.6 with respect to the lines through the origin. If we take the associated coordinates, then the horizon points are those of the form $(x : y : 0)$. These points correspond to lines in the plane $z = 0$, which is perpendicular to the plane we are considering.

5.3.2. Reconstructing $\mathbb{R}^2$ from $\mathbb{P}^2\mathbb{R}$

We have discussed how to identify coordinates for a projective plane based on coordinates for Euclidean space. Let us end this section by discussing the opposite problem: given a projective coordinate system, what do we need to identify coordinates for the actual positions *outside perspective*, coordinates for a Euclidean plane $\mathbb{R}^2$. We will follow the exact same steps as we did when we discussed this problem without coordinates in section 5.2.

The first step is already discussed, if we know what the horizon line is, we can choose our coordinates in such a way that this line corresponds to $z = 0$. This makes it possible to consider $(x/z, y/z)$

for $z \neq 0$ as defining points in an affine plane where each point corresponds to an actual point in the underlying Euclidean plane.

The next step is the recovery of angles, for which we have to identify two lines in the projective plane that form a right angle in the underlying Euclidean plane. If we were to know a right angle, then we could choose $A = (0 : 0 : 1), B = (1 : 0 : 1), C = (0 : 1 : 1)$ in such a way that $\angle BAC$ is that right angle. In that case, we recover the tangent function (Definition 4.6) to measure angles.

The last step is recovering a measure for lengths. We already can compare parallel lines (that is always possible in an affine space) but if we were to know a square, then we could choose $A = (0 : 0 : 1), B = (1 : 0 : 1), C = (0 : 1 : 1), D = (1 : 1 : 1)$ to be the vertices of this square. Then we have coordinates that reflect the plane independent of any perspective. All lengths can still be scaled by a uniform factor, so to truly recover a coordinate system one needs to know one particular length in the plane.

5.4. Cross-ratio

In Euclidean geometry, length is the most powerful attribute to relate shapes to each other. Areas, volumes and angles are primarily measured using lengths. The goal in geometries that stem from a Euclidean plane is often to measure the actual lengths of the Euclidean plane to which they are related.

Consider the city of Delft as a plane (i.e. as a constellation of interrelated places and shapes of lines and areas) and bring it in comparison with a map of the city. On such a map absolute lengths have no physical significance: the fact that the distance between my house and the university is 10 cm on the map does not tell me anything. Relative lengths, however, do have physical significance: longer distances on the map mean longer distances in the city. If one adds a scale to relate the map to the city (like 1 cm : 100 m), then one can use that scale to measure any distance in the actual city based solely on its distance on the map.

A geometry where absolute lengths have no significance but relative lengths do is called a similarity geometry (this name stems from the fact that similar figures remain similar under uniform scaling, while congruences change).

Suppose that we want to measure lengths in a projective plane, like a photo of the city of Delft, then relative lengths are not sufficient. The fact that a tree in the picture might take up more pixels than a building does not tell you that the tree is actually longer, maybe the building is just far away. Yet, if we see a picture, then we can infer such data and do not think that the building is small just because it takes up relatively little space on a picture. There must be some information that reflects this, there must be some measurement that we make that characterises lengths even under projection, that remains invariant under projections. Ratios of lengths between two line segments do not remain the same under projections, however, there is a *ratio of ratios* that does remain the same. It is called the cross-ratio. Mathematically, it is defined using lengths of oriented line segments (we symbolise the length by the oriented line segment):

$$(A, B; C, D) = \frac{\overline{AC}/\overline{BC}}{\overline{AD}/\overline{BD}} \tag{5.1}$$

It is a characteristic quantity of directed line segments between four points on a line. The following identities immediately follow from the definition:

$$(A, B; C, D) = (B, A; D, C) = (C, D; A, B) \tag{5.2}$$

Let us demonstrate that cross-ratios are indeed invariant under projections. For this, we also introduce the cross-ratio for four concurrent[1] lines (cf. with duality, which will be introduced in chapter 6).

[1]Lines through the same point.

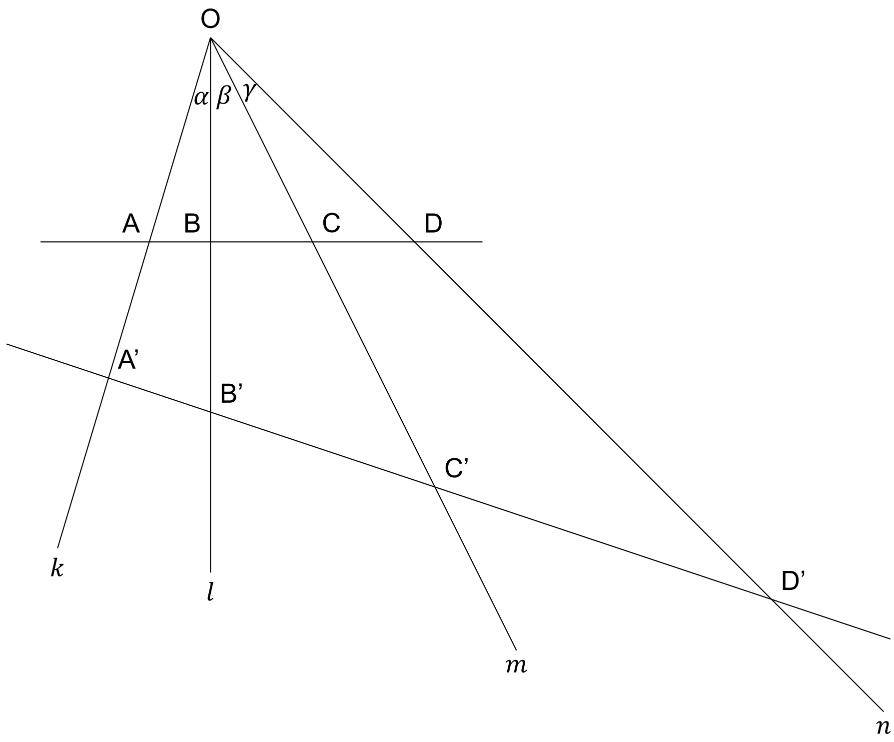

Figure 5.7: Cross-ratio

The cross-ratio for four concurrent lines is defined as:

$$(k, l; m, n) = \frac{\sin(\angle AOC)/\sin(\angle BOC))}{\sin(AOD))/\sin(\angle BOD))} \tag{5.3}$$

Like the cross-ratio for segments, these angles are signed. If A is on the other side of C then we take the sine of the angle greater than two right angles, which is minus the sine of the angle smaller than two right angles). Note the similarity of all terms with those of the cross-ratio for points. We will demonstrate that these two cross-ratios are equal. Then, it will follow immediately that the cross-ratio is constant under projections, as the cross-ratio for concurrent lines is independent of any particular projection.

Proposition 5.1. *If the concurrent lines k, l, m, n are intersected by another line in the points A, B, C, D respectively, then $(A, B; C, D) = (k, l; m, n)$.*

Proof. The result follows from two different expressions of the area of a triangle.[2] For any triangle with vertices G, F, H and a height h with respect to GF as the base: $Area(GFH) = h \cdot GH/2 = GF \cdot FH \cdot \angle GFH/2$. In particular, for our triangles:

$$\begin{aligned}
h \cdot AC/2 &= OC \cdot OA \cdot \sin(\angle COA)/2 \\
h \cdot BC/2 &= OC \cdot OB \cdot \sin(\angle COB)/2 \\
h \cdot AD/2 &= OD \cdot OA \cdot \sin(\angle DOA)/2 \\
h \cdot BD/2 &= OD \cdot OB \cdot \sin(\angle DOB)/2
\end{aligned}$$

[2]We have not introduced the axioms of area for Euclidean geometry nor the sine function, but the validity of these formulas will be known to all.

Where h is the height of the triangle which is the same for all triangles, therefore:

$$CA/CB = \frac{OA \cdot \sin(\angle COA)}{OB \cdot \sin(\angle COB)} \tag{5.4}$$

$$DA/DB = \frac{OA \cdot \sin(\angle DOA)}{OB \cdot \sin(\angle DOB)} \tag{5.5}$$

$$\tag{5.6}$$

And:

$$(A, B; C, D) = \frac{AC/BC}{AD/BD} = \frac{\sin(\angle AOC)/\sin(\angle BOC))}{\sin(AOD))/\sin(\angle BOD))} = (k, l; m, n) \tag{5.7}$$

In this proof, we disregarded the signs of length segments and angles, as the triangle formulas for area are not defined in terms of oriented line segments or angles greater than two right angles. One can see that changing the orientation of a segment in the cross-ratio corresponds exactly to one sign change in the cross-ratio for lines. Therefore, even for oriented segments they are equal. □

Now it immediately follows that the cross-ratio is independent of the particular line that cuts k, l, m, n and it is therefore invariant under projections. Let us end this section on cross-ratios by giving an application of cross-ratios and homogeneous coordinates: reconstructing lengths from an image.

Example 5.1 (Reconstructing lengths from an image)**.** Let us apply our knowledge of cross-ratio to measure the height of the 12^{th} floor of EWI, the beautiful building in which I grew up mathematically. Why the 12^{th} floor? The most fun would of course be to calculate the height of the whole building based solely on tape measures. Unfortunately, the precision range for such a calculation is about 30 meters, due to the massive differences in height between the known and the unknown, and the pixel resolution of the picture. The height of the 12^{th} floor is available in the literature, hence it was a good candidate because it enables verification of the calculation.

Figure 5.8: EWI building at the TU Delft

Consider the following measurements in pixels:

$$\overline{AB} : 183px,\ \overline{AC} : 2751px,\ \overline{AD} : 3954px \tag{5.8}$$

And the following measurements in meters:[3]

$$\overline{AB} : 1.75m,\ \overline{AD} : 90m \tag{5.9}$$

We can calculate the cross-ratio of the segments in pixels:

$$(A,B;C,D) = \frac{\overline{AC}/\overline{BC}}{\overline{AD}/\overline{BD}} = \frac{2751/(2751-183)}{3954/(3954-183)} \approx 1.022 \tag{5.10}$$

As cross-ratios are invariant under projections, they should remain invariant under the taking of an image. We can also express the same cross-ratio using the lengths in meters:

$$(A,B;C,D) = \frac{x/(x-1.75)}{90/(90-1.75)} \tag{5.11}$$

A simple algebraic calculation now gives the unknown $\overline{AC}$ in meters:

$$\frac{\overline{AC}/(\overline{AC}-1.75)}{90/(90-1.75)} \approx 1.022 \tag{5.12}$$

$$\overline{AC} \approx (\overline{AC}-1.75) \times 1.042 \tag{5.13}$$

$$\overline{AC} \approx \frac{1.75 \times 1.042}{1.042-1} \approx 43.42 \tag{5.14}$$

Our calculation gives that the height of the 12th floor is approximately 43.4 meters. The actual height is 42.5 meters,[4] so our accuracy is about 2%.

[3] $\overline{AB}$ was measured with a tape measure, $\overline{AD}$ can be found via the Delft municipality (Gemeente Delft, n.d.).

[4] (Raji et al., 2020)

6

Axiomatic Projective geometry

Now that we have a better grasp on the nature of projective spaces and how they might arise from three-dimensional position-space, let us study projective geometry synthetically based on axioms, without any reference to an outside three-dimensional geometry.

6.1. Projective axioms of incidence

Like with Hilbert's Euclidean axioms, we will start with the most fundamental axioms, those of incidence.

Axiom *(PI.1)*. *For any two distinct points, there is a unique line incident with them.*

Axiom *(PI.2)*. *For any two lines, there is a unique point incident with them.*

Axiom *(PI.3)*. *There exist four points so that no three are incident with a single line.*

(PI.1) is exactly the same as *(I.1)* of Hilbert; *(PI.3)* is a stronger version of Hilbert's non-degeneracy axiom *(I.3)*. The one that stands out is of course *(PI.2)*, which is not valid in a Euclidean plane. There, this would be true for non-Parallel lines, but parallel lines do not determine a unique point. In projective geometry, however, we consider the horizon as a line and we see it reflected in the axioms here.

Proposition 6.1. *There exist four lines so that no three are incident with a single point.*

Proof. Suppose A, B, C, D are the points of *(PI.3)*. Then consider the lines $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{CD}, \overleftrightarrow{AD}$. All are defined by two points and no point is used to define three lines. No line can be incident with any of the two points that do not define it because then three points would lie on a single line, contradicting *(PI.3)*. Therefore, these are four lines so that no three are incident with a single point. □

Why did we prove this? There is something special going on with *(PI.1)* versus *(PI.2)* and *(PI.3)* versus Proposition 6.1, in both cases it is the same statement with the terms 'point' and 'line' interchanged! This duality of points and lines is a very elegant property of projective spaces and it will enable us to transform any proposition about points and lines into one about lines and points. So whenever you prove one theorem in a projective plane, you have in fact proven two theorems!

6.2. Pappos' Theorem

(PI.1) to *(PI.3)* is not enough to differentiate the projective plane we are after (the one induced by adding a horizon to a Euclidean plane) from other projective planes. Instead of taking very basic statements as additional axioms, we are going to take two *theorems* in the Euclidean plane as axioms; or, to be more precise, their projective variant. The reason for this is to show the deep connection between the geometric axioms and the field axioms. These axioms will have direct implications for which of the field axioms hold in length arithmetic induced from a projective plane. The first of these two axioms (i.e. theorems in Euclidean geometry) is due to Pappos.

Axiom *(Pappos)*. *Six points, lying alternately on two straight lines, form a hexagon whose three pairs of opposite sides meet on a line.*

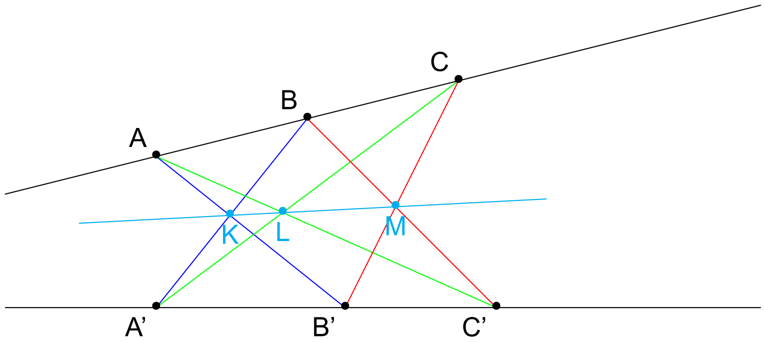

Figure 6.1: Pappos' Theorem

As said, there are projective planes where this proposition does not hold, hence we take it as an axiom for the projective space. Let us use our system of homogeneous coordinates to prove this proposition for a projective real plane.

Theorem 6.2. *(Pappos) is a theorem in the real projective plane $\mathbb{P}^2\mathbb{R}$.*

Proof. We will use homogeneous coordinates to prove Papppos' theorem. With these, we will seek the equation of $\overleftrightarrow{BC}'$ and $\overleftrightarrow{B'C}$ and use them to show that their intersection is also a solution of the equation for the line $\overleftrightarrow{KL}$. Consider the points A, K, A', C. None of these points are collinear, so we can assign homogeneous coordinates:

$$A = (1 : 1 : 1),\ K = (1 : 0 : 0),\ C = (0 : 1 : 0),\ C' = (0 : 0 : 1)$$

Let a point be symbolised as $(x : y : z)$, we can express an equation for a line through two points by taking scalar multiples of each of the coordinates. A parameterisation of $\overleftrightarrow{AK}$ is therefore $(\mu : \lambda) \mapsto (\mu + \lambda : \mu : \mu)$, which can also be expressed as $y = z$. Similarly, $\overleftrightarrow{AC}$ is $x = z$, and $\overleftrightarrow{AC'}$ is $x = y$.

Since B' is on $\overleftrightarrow{AK}$, $B' = (p : 1 : 1)$ for some p, B is on $\overleftrightarrow{AC}$ so $B = (1 : q : 1)$ for some q, L is on $\overleftrightarrow{AC'}$ so $y = (1 : 1 : r)$ for some r.

Now consider the following two pairs of three lines.

- $\overleftrightarrow{CL}, \overleftrightarrow{BK}, \overleftrightarrow{B'C'}$, which are concurrent in A'. $\overleftrightarrow{CL}$ is parameterised by $(\mu : \lambda) \mapsto (\lambda : \lambda + \mu : \lambda r)$, which can also be written as $z = rx$, Similarly $\overleftrightarrow{BK}$ is $y = qz$ and $\overleftrightarrow{B'C'}$ is $x = py$.
- $\overleftrightarrow{BC'}, \overleftrightarrow{B'C}, \overleftrightarrow{KL}$. They are concurrent if and only if Pappos' theorem holds. The equation for $\overleftrightarrow{BC'}$ is $y = qx$, similarly for $\overleftrightarrow{B'C}$ we have $x = pz$ and for $\overleftrightarrow{KL}$ we have $z = ry$.

The first triplet passes through the point A' by assumption, so $z = rx = r(py) = p(r(qz)) = prqz$ which means $prq = 1$. K, L, M are collinear if and only if the second pair of lines pass through one point. This would be the case if $y = qx = q(pz) = q(p(ry))$, which means if $qpr = 1$. Multiplication in $\mathbb{R}$ is commutative, $prq = qpr$, so K, L, M are collinear and Pappos' theorem holds. □

We should make note of one thing here, our proof of Pappos' theorem by means of homogeneous coordinates essentially relied on the commutativity (and associativity) of $\mathbb{R}$. The other steps were just selecting points in general position and inferring lines, both these steps are possible for all projective planes. I will not get ahead of myself here, but this indicates the deep connection between the field axioms on the one hand and our projective plane axioms on the other.

6.3. Desargues' Theorem

The second theorem is due to Desargues, like Pappos' theorem it is a theorem in Euclidean geometry but there are projective planes that do not satisfy it. Desargues' theorem is actually implied by Pappos' proposition so it is not strictly necessary to take it as an axiom. The proof of this fact is due to Hessenberg,[1] but it is a long proof and we do not need it for our purposes hence we neither state it nor

[1](Hessenberg, 1905)

rely on it. There are projective planes, however, that do satisfy Desargues' theorem, but not Pappos. Moreover, Desargues' theorem will also play a role in relating the geometric axioms to the field axioms. Let us start with a definition and after that state the axiom.

Definition 6.1. Two triangles are *in perspective* from a point if the lines through corresponding vertices are concurrent with a single point.

Axiom *(Desargues)*. *If two triangles are in perspective from a point, then their pairs of corresponding sides meet in collinear points.*

Like with Pappos, we will prove that this is a theorem in the real projective plane using homogeneous coordinates.

Theorem 6.3. *(Desargues) is a theorem in the real projective plane $\mathbb{P}^2 R$.*

Proof. Let O be the point of perspective, let $\triangle ABC$ be a triangle and let $\triangle A'B'C'$ be a perspective of $\triangle ABC$ from O. We assign the following coordinates:

$$O = (1 : 1 : 1),\ A = (1 : 0 : 0),\ B = (0 : 1 : 0),\ C = (0 : 0 : 1)$$

Then we get the equations $\overleftrightarrow{OA} : y = z$, $\overleftrightarrow{OB} : x = z$,$\overleftrightarrow{OC} : x = y$ and that for some p, r, q:

$$A' = (1 + p : 1 : 1),\ B' = (1 : 1 + q : 1),\ C' = (1 : 1 : 1 + r)$$

Now we can derive a parameterisation and an equation for the lines of the corresponding sides:

$$\overleftrightarrow{A'C'} : (\mu : \lambda) \mapsto (\lambda(1 + p) + \mu : \lambda + \mu : \lambda + \mu(1 + r))$$
$$\overleftrightarrow{AC} : y = 0$$

Their intersection is when $\lambda = -\mu$ in the point $(p : 0 : -r)$. Using this same method we obtain that the intersection of $\overleftrightarrow{A'B'}$ with $\overleftrightarrow{AB}$ is $(p : -q : 0)$, and $\overleftrightarrow{B'C'}$ with $\overleftrightarrow{BC}$ is $(0 : q : -r)$.
These three points lie on the line $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 0$, which proves Desargues' theorem. □

We can now demonstrate the power of duality.

Corollary 6.4 (Converse Desargues)**.** *If the corresponding sides of a triangle meet in collinear points, then the triangle is in perspective from a single point.*

Proof. The consequence in Desargues' theorem is the dual of the condition: The dual of the definition of two triangles being in perspective is that the points where corresponding vertices intersect, which is exactly what Desargues' theorem states. If *(Desargues)* holds in a projective plane, the theorem then follows immediately from the principle of duality. (This does rely on the fact that the duality principle is not violated by further projective axioms.) □

Corollary 6.5 (Scissors theorem)**.** *If $ABCD$ and $A'B'C'D'$ are quadrilaterals with vertices alternately on two sides, and if all intersections of $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$, $\overleftrightarrow{AD}$ and $\overleftrightarrow{A'D'}$ lie on a line M, then the intersection of $\overleftrightarrow{CD}$ and $\overleftrightarrow{C'D'}$ also lies on M.*[2]

It is not hard to prove this by considering the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ ($\overleftrightarrow{A'D'}$ and $\overleftrightarrow{B'C'}$) and applying Desargues to the resulting triangles.

6.4. Axioms of cyclic order

The second category of Hilbert's axioms are the axioms of order. In chapter 5 we discussed the nature of projective lines: they are like Euclidean lines with an extra horizon point added. If one continues in both directions indefinitely, one will reach the same horizon point. This horizon point is not a special point from the projective perspective. We, therefore, do not have a concept of betweenness on a projective line, there are two ways to get from one point to another on a line. One can get from one point to another in the 'regular way' (from the Euclidean perspective) and by going through the horizon

[2] adapted from (Stillwell, 2005, p.126), a complete proof can be found there as well.

point. The projective lines are topologically circular. We, therefore, do have something called cyclic order.

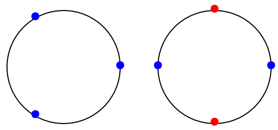

Figure 6.2: It is impossible to establish an order of three points on a circle, but two points can separate two others

On a circle, one cannot order three points, as every point is in between the other two. But with four points, there does exist an order: we can divide the four points into two pairs and these two pairs will always lie in between each other. We formalise this with the following axioms:[3]

Axiom *(CO.1)*. *Let A, B, C, D be distinct points on a line, $AB\sigma CD$ is equivalent to $CD\sigma AB$ and $BA\sigma CD$.*

Axiom *(CO.2)*. *Let A, B, C, D be distinct points on a line, then one and only one of the relations $AB\sigma CD$, $AC\sigma BD$, $AD\sigma BC$ holds.*

Axiom *(CO.3)*. *If $AC\sigma BD$ and $AD\sigma CE$, then $AD\sigma BE$.*

Axiom *(CO.4)*. *There exists a line incident with at least four points.*

Axiom *(CO.5)*. *Let A, B, C, D be distinct points on a line ℓ, and A', B', C', D' their projections on ℓ' from some centre S, then $AC\sigma BD$ implies $A'C'\sigma B'D'$.*

Properly speaking *(CO.4)* is an incidence axiom, but we state it as part of the axioms of circular order because it excludes a degenerate space relevant to circular order. We will not spend a lot of time on these axioms, but let us draw attention to two important observations.

Notice the intimate relation that these axioms have to the cross-ratio. The cross-ratio was defined using four points. *(CO.1)* establishes that we can interchange the two pairs, like with the cross-ratio. Because the cyclic axioms do not state anything about quantities, they also do not distinguish between $AB\sigma CD$ and $BA\sigma CD$. This is exactly like the fact that betweenness does not distinguish between the endpoints of a line. *(CO.5)* also establishes, not the quantitative invariance of the cross-ratio, but the 'topological invariance', i.e. the axiom does not state anything about quantitative invariance but the points will preserve a relative order.

The other important observation is that if we know what the horizon line is, then we can define betweenness. This can simply be done by taking the horizon point as one of the points in the cyclic order, and then the other point in that pair is in between the points of the other pair.

6.5. Projective geometry and conic sections

In chapter 5 we gave an example of a practical application of projective geometry: measuring the heights based on a photo alone. Let us end this chapter by giving an application of projective geometry in theoretical geometry itself.

The most famous textbook of Greek geometry after Euclid's *Elements* is arguably Appolonius' *Conic sections*. In this book, Appolonius studied the intersections of a cone with a plane: the ellipse, parabola and hyperbola.[4] From the viewpoint of projective geometry this is simply taking different planar projections of a cone. It might not come as a surprise then that in projective geometry there is no distinction between ellipses, parabolas or hyperbolas, there is simply a 'conic section.'

If we then identify a line as the horizon line, can we differentiate the different conic sections again? Yes! An ellipse is a conic section that does not cut the horizon line; a parabola has two extremities that get more and more parallel: the parabola intersects the horizon line at exactly one point (it 'touches' it); a hyperbola has two asymptotes and the two branches of a hyperbola both have two extremities corresponding to these asymptotes, it, therefore, cuts the horizon line in two points.

[3]These axioms are adapted from Heyting, 1980.

[4]One can also get degenerate intersections like lines, points or circles, but we disregard those.

This also indicates once more how the projective plane is in a sense 'circular': in a Euclidean plane the branches of the hyperbola go off into opposite directions, but in a projective plane they meet each other at the horizon.

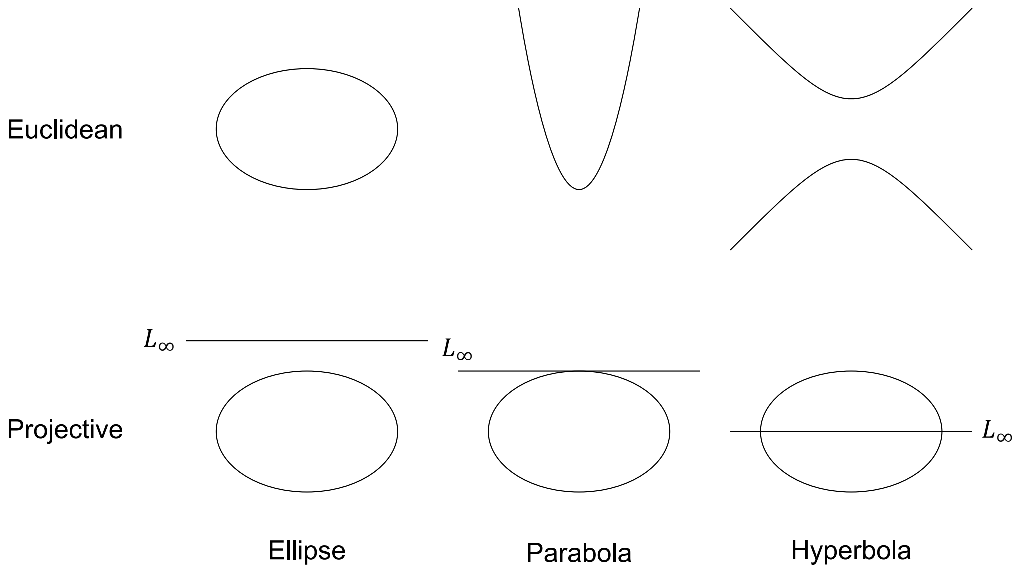

Figure 6.3: Conic sections in a Euclidean and projective plane

Projective geometry greatly simplifies the study of conic sections and makes it possible to reduce many propositions of different conic sections to a single proposition in projective geometry. Historian Kline writes that Desargues derived sixty(!) theorems of Apollonius from a single projective theorem and that Pascal is supposed to have deduced 400(!!!) corollaries from Pascal's hexagon theorem (a more general variant of Pappos' theorem), though no work of Pascal in which this is done has survived.[5]

[5](Kline, 1972, p.300)

7

Projective arithmetic

7.1. Length arithmetic for Projective geometry

In section 3.2 we formed the abstraction (oriented) length and defined arithmetic for it. This was in the context of Euclidean geometry, we cannot do the same in projective geometry because we cannot form such an abstraction as length: congruence relations need not be preserved under projective transformations. What we can do, however, is define arithmetic for oriented line segments on parallel lines such that, if we would know enough information about the projective plane to transform it into a Euclidean plane, then it is consistent with the arithmetic for Euclidean geometry developed in section 3.2. Moreover, the oriented line segments will, like the lengths in section 3.2 form a field.

For Euclidean geometry, we created an arithmetic of oriented lengths to make a coordinate system out of it. We already have a coordinate system for projective geometry, based on three-dimensional Cartesian coordinates. The goal for this arithmetic of projective line segments will not be to create coordinates out of it, but simply to induce a field from projective geometry and to develop an arithmetic that is consistent with that of Euclidean geometry. Let us start with a projective plane Π that satisfies at least *(PI.1)* to *(PI.3)* and *(CO.1)* to *(CO.5)*.

Choosing L_∞

The precondition of such arithmetic is that we have identified a line as the horizon line. It does not make sense to do arithmetic with infinitely long segments. Let us, therefore, start by choosing a line L_∞ in Π.

By identifying a horizon line, we can derive a relation of betweenness for points on a line. Consider any line $\ell \neq L_\infty$. Take the horizon point of this line $\ell_\infty := \ell \cap L_\infty$. Then for arbitrary points $A, B, C \in \ell$, we define $A * C * B$ if $AB\sigma C\ell_\infty$. We can, therefore, form line segments and thus develop arithmetic for line segments.

We also get a concept of parallelism, two lines are parallel if they cut the same point on L_∞. A class of parallel lines can now simply be defined as all lines that cut a given point on L_∞. We can, therefore, denote it as P_λ, where $\lambda \in L_\infty$. We identify a class P_λ within which we define arithmetic. As with the oriented arithmetic in the Euclidean case, our arithmetic depends on choosing a particular class but any class can be chosen. From now on we will always assume that picking points of lines in P_λ means picking points that are not λ itself.

Choosing a line K, orientation and unit

In defining oriented lengths, we considered the class of parallel lines as a whole. We could in principle do the same, but let us take a different approach: we choose a particular line and a particular orientation point and then we define the arithmetic by means of line segments on it. This will produce the same arithmetic and it will make some demonstrations later on easier.

Choose a line $K \in P_\lambda$. Choose points $O, I \in K$, O will be the orientation point and $\overline{OI}$ the unit line segment with length 1. We define a linear order using betweenness (this also works in the projective case and is not hard to check) and choosing $O < I$. We, therefore, have oriented line segments on K. We bring line segments in P_λ in standard position by translating them to O.

Definition 7.1. The oriented line segment in standard position of an oriented line segment $\overline{AB} \in P_\lambda$, is the line segment on K obtained in the following way. Consider the line $\overleftrightarrow{OA}$ and take the line M through B parallel to $\overleftrightarrow{OA}$ (i.e. the line that cuts it on L_∞). Then the intersection point C of M and K defines $\overline{OC}$, the oriented line segment in standard position.

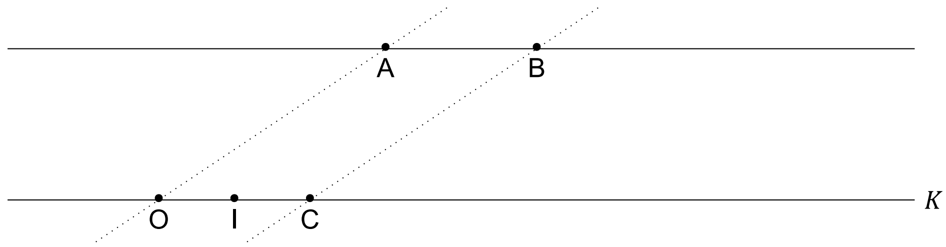

Figure 7.1: Translating a line segment in standard position

Each line segment in P_λ can be associated with a unique line segment in standard position. The idea is of course that, if we were to have Euclidean lengths, then a line in standard position is a congruent line segment (by properties of a parallelogram). We do not have a congruence relationship, however. Let us, therefore, define a quasi-length, based on equivalence with respect to standard position.

Definition 7.2. A *quasi-length* is an attribute $\pm a$ of an oriented line segment $\overline{AB}$ in a class P_λ with the following relation. If $\overline{CD} \in P_\lambda$ has quasi-length $\pm c$, then $\pm a = \pm c$ if and only if $\overline{AB}$ and $\overline{CD}$ have the same standard position.

A quasi-length exists only within a given class of parallel lines and with respect to a line in this class with an orientation point. One can see that a quasi-length is very related to oriented lengths in Euclidean geometry, we do not designate them with the same name to stress that it is not truly a length, it is a *perspective* on a length (and with some more information about the projective space one can identify it with an actual length).

Addition and multiplication

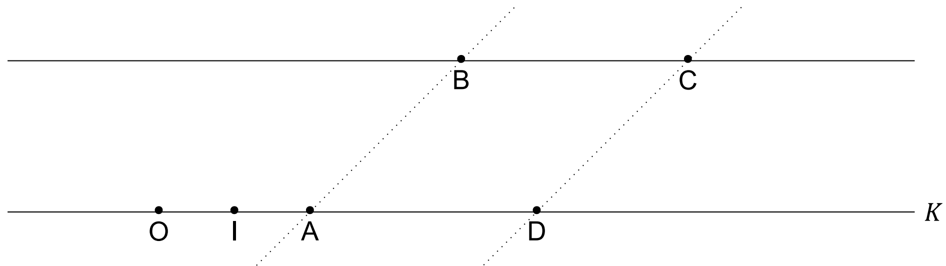

Figure 7.2: Addition of quasi-lengths

Definition 7.3. Addition of two quasi-lengths. Suppose a and b are two quasi-lengths with respect to P_λ. We define the *sum* $a + b$ to be the following quasi-length. Consider the segment $\overline{OA}$ in standard position with quasi-length a and $\overline{BC}$ not on K with quasi-length b. (That such a segment not on K always exists is straightforward). Then apply the same procedure of placing $\overline{CD}$ in standard position but now with respect to A as an origin, not O. That is, we consider the line through $\overline{AB}$, take the line through C that is parallel to it and consider the point D where this line cuts K. Then the quasi-length c of $\overline{OD}$ is the sum of a and b, $a + b := c$.

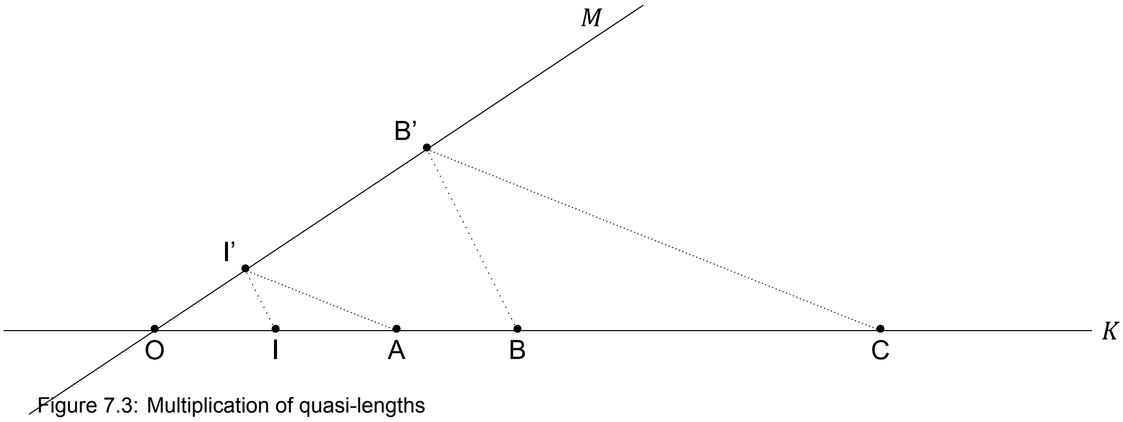

Figure 7.3: Multiplication of quasi-lengths

Definition 7.4. Multiplication of two quasi-lengths. Suppose a and b are two quasi-lengths with respect to P_λ and unit quasi-length 1. We define the *product* $a \times b$ to be the following quasi-length. Consider the following segments in standard position: $\overline{OI}$ with quasi-length 1, $\overline{OA}$ with quasi-length a and $\overline{OB}$ with quasi-length b. Consider a line M through O that is not in P_λ and consider an arbitrary point I' on M. Consider the line $\overleftrightarrow{II'}$ and consider the line parallel to it through B. Call the intersection of M with this line B'. Then consider the line $\overleftrightarrow{I'A}$ and the line parallel to it through B'. Consider the intersection of K with this line in the point C. We call the quasi-length c of $\overline{OC}$ the product of a and b, $a \times b := c$.

Theorem 7.1 (Choice independence)**.** *Multiplication is independent of the choice for M and I'.*

We will give a visual indication of why this is the case, not a formal proof. It can be seen in fig. 7.4 Note that even choosing I'' on the other side of O produces the same C, this is immediately clear from this drawing.

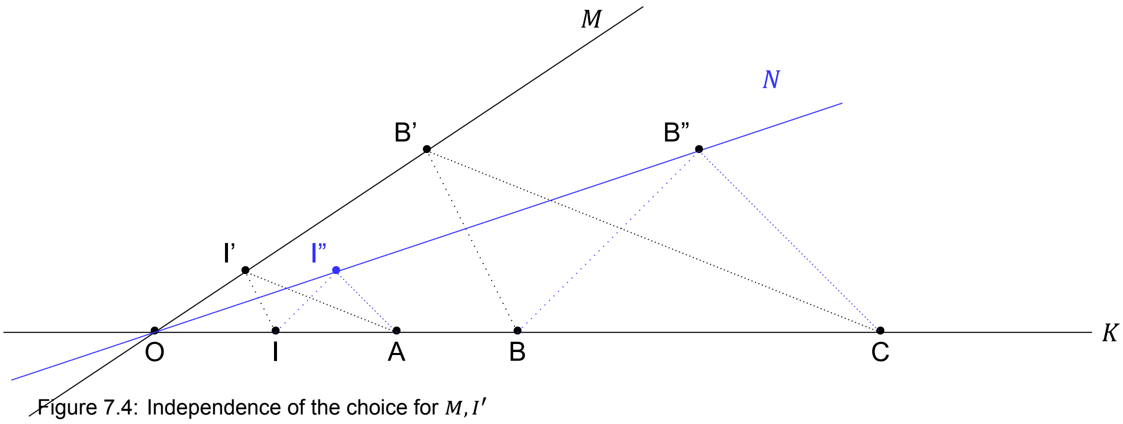

Figure 7.4: Independence of the choice for M, I'

Theorem 7.2. *If $\Pi \setminus L_\infty$ satisfies the Euclidean axioms, then the definition of projective quasi-length arithmetic is equivalent to that of oriented length arithmetic.*

Proof. If $\Pi \setminus L_\infty$ satisfies the Euclidean axioms, then all quasi-lengths are oriented lengths. This follows from the fact that under the Euclidean axioms an oriented line segment is congruent with that segment in standard position and all segments in standard position define unique oriented lengths. That the definition of addition is equivalent is straightforward as it consists of placing the two line segments next to each other. For the definition of multiplication consider M to be perpendicular to K (possible with the Euclidean axioms). The triangles $\triangle OI'A$ and $\triangle OB'C$ in the projective case are exactly those that define multiplication in the case of Euclidean geometry, so the products are also equal. Therefore the two definitions coincide. □

7.2. Pappos and commutativity

Theorem 7.3. *Projective quasi-length arithmetic is commutative if and only if Pappos' theorem holds.*

Proof. We demonstrate the commutativity of multiplication. The proof for addition is exactly the same if we consider the oriented segments both in standard position and on another 'parallel' line. Consider the lines and points in fig. 7.5.

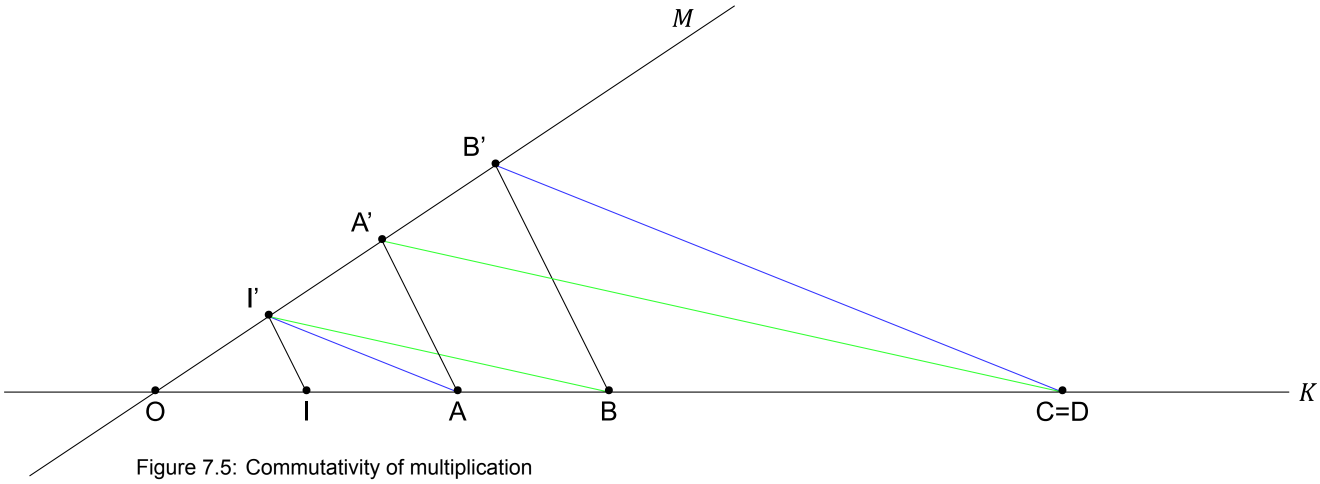

Figure 7.5: Commutativity of multiplication

$\overline{OI}, \overline{OA}, \overline{OB}$ have respective quasi-lengths $1, a, b$; $\overline{OC}$ has quasi-length ab and $\overline{OD}$ has quasi-length ba. The 'parallelism' of the three pairs of colored lines means that they intersect on the line L_∞. We therefore, have six points that form a hexagon, I', A', B' and the three points of intersection on the horizon of the lines drawn parallel in fig. 7.5 (ignoring the line II'). B is on the ray $\overrightarrow{OI}$ by definition and the intersection of the two different lines that determine multiplication lies on this line (i.e the products are equal) if and only if Pappos' theorem holds. □

This also proves the commutativity of Euclidean oriented length arithmetic, because Pappos' theorem is a theorem in Euclidean geometry.

7.3. Desargues and associativity

Theorem 7.4. *Projective quasi-length arithmetic is associative if and only if Desargues' theorem holds.*

Proof. Like with Pappos, we demonstrate the associativity of multiplication. Consider the line and points in fig. 7.6.

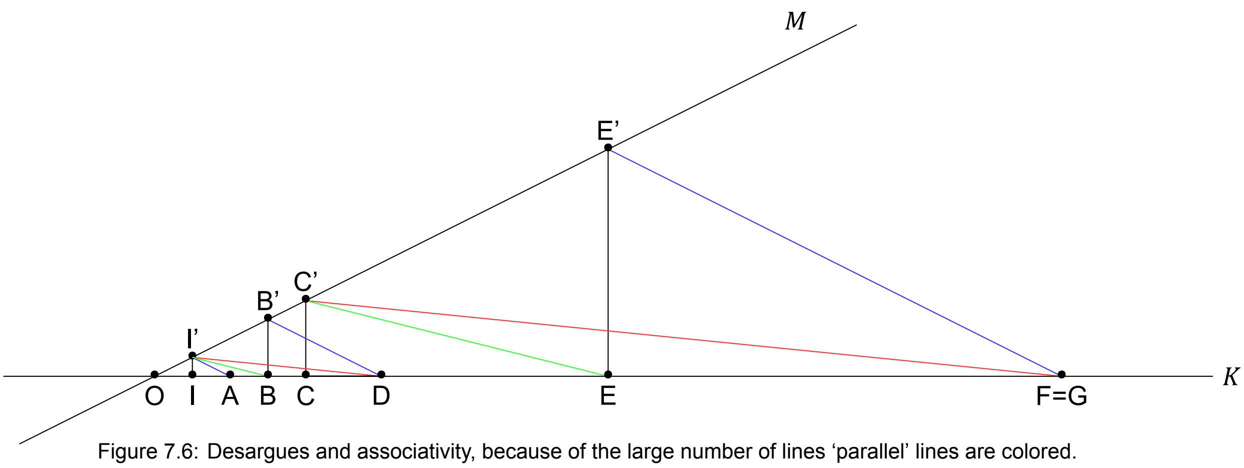

Figure 7.6: Desargues and associativity, because of the large number of lines 'parallel' lines are colored.

$\overline{OI}, \overline{OA}, \overline{OB}$ and $\overline{OC}$ have respective quasi-lengths $1, a, b, c$. $\overline{OD}$ has quasi-length ab, $\overline{OE}$ quasi-length bc; $\overline{OF}$ quasi-length $a(bc)$, and $\overline{OG}$ quasi-length $(ab)c$. The fact that $F = G$ now follows from Corollary 6.5. The two quadrilaterals of importance are $IDB'I'$ and $CFE'C'$. □

Theorem 7.5. *The field of quasi-length arithmetic that is induced from a projective plane where (PI.1) to (PI.3), (CO.1) to (CO.5), (Desargues) and (Dedekind) hold, is isomorphic to* $\mathbb{R}$.

We will not give a proof of this fact because it is long and relies on some results we have not shown (for example, that *(Desargues)* implies *(Pappos)* in the context of *(Dedekind)*). The interested reader can find such proofs in (Heyting, 1980).

7.4. Conclusion of chapters 5 to 7

In the last three chapters we have worked in the projective plane, both as induced from an three-dimensional Euclidean space and synthetically by its characteristic axioms. We have shown that there exists a deep relationship between geometric axioms and algebraic axioms. Pappos' theorem is directly related to the commutativity of length arithmetic and Desargues is directly related to associativity.

This demonstrates that algebra and geometry have deep relationships that go all the way down to the axioms. The usage of algebra greatly expanded the power of geometry and modern geometry fundamentally depends on the power of algebra; geometry on the other hand aids algebra by giving a concrete meaning to algebraic calculations. Let us, therefore, end this thesis with the following words by Joseph-Louis Lagrange:

> "As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection."[1]

[1](Moritz, 1914, p.82)

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