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Learning Controllers From Data With Stability Guarantees

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Abstract

Direct data-driven control approaches learn controllers from data without identifying an accurate model of the system. These methods are especially favorable for complex system dynamics as identifying a sufficiently accurate model can be difficult and time-consuming. Using Willems *et al.*'s fundamental lemma, a data-based closed-loop system representation is derived under a feedback control law. This data-based closed-loop representation lays the foundation for developing data-driven controllers and analyzing closed-loop performance for both linear and nonlinear systems.

Key Points

- Derive data-based closed-loop representations for continuous-time linear and nonlinear dynamical systems under state feedback control laws.
- Design data-driven state feedback stabilizers for linear and nonlinear systems using the data-based closed-loop representations.
- Compute an inner estimation of the region of attraction using data for the nonlinear closed-loop systems under the designed data-driven control laws.

Introduction

Most conventional control methods rely on a sufficiently accurate model to rigorously guarantee control performances. As such sufficiently accurate models are never readily available for control designs, system identification techniques are usually used to establish a model from data. This is often called the indirect data-driven control, where the system model is firstly identified and then a control law is designed based on the identified model. With the increasing complexity of the system dynamics to be controlled, identifying an accurate model can be difficult and time-consuming. Moreover, as system identification often aims at establishing a model as accurate as possible, the identified model may not be suitable for applying existing control techniques. As a result, direct data-driven control approaches, which synthesize controllers directly from data without identifying an accurate model, have been attracting more and more attentions recently.

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Direct data-driven control has been investigated via various approaches, such as direct adaptive control (Åström and Wittenmark, 1989), neural network based control (Werbos, 1989), iterative feedback tuning (Hjalmarsson *et al.*, 1998), virtual reference feedback tuning (Campi and Savaresi, 2006), and intelligent PID control (Fliess and Join, 2009). Recently, direct data-driven control has been attracting more attentions, such as the data-driven stabilization and linear quadratic regulator (LQR) designs in De Persis and Tesi (2020) and van Waarde *et al.* (2020), data-based predictive control in Coulson *et al.* (2019) and Berberich *et al.* (2021), data-driven distributed control of multiple systems in Celi *et al.* (2021) and Lederer *et al.* (2022), and data-driven safety control in Hewing *et al.* (2019) and Yang *et al.* (2020). Surveys on data-driven control can be found in works such as Hou and Wang (2013), Hou and Xiong (2019) and Martin *et al.* (2023).

Prominent challenges in direct data-driven control include describing the system dynamics using data and rigorously guaranteeing the control performance. In this chapter, we will show how to overcome these challenges using a data-based closed-loop representation via Willems *et al.*'s fundamental lemma (Willems *et al.*, 2005) and the Lyapunov method. Based on Willems *et al.*'s fundamental lemma, a closed-loop system representation can be derived using only data. Then, by enforcing Lyapunov stability conditions on the data-based closed-loop dynamics, data-driven stabilizers are characterized via data. We will apply this approach to data-driven stabilization of linear systems and then extend the result to data-driven stabilization of nonlinear systems by writing the nonlinear dynamics into linear-like forms. The linear-like forms can be obtained via Taylor's expansion of the unknown functions and the nonlinear systems are stabilized in a neighborhood of the origin. Based on the designed nonlinear data-driven controller, we show that an inner estimation of the region of attraction (RoA) can also be derived using data. The data-based conditions characterizing the controllers and RoA are in the forms of linear matrix inequalities (LMIs) or sum of squares (SOS) conditions, which are computationally efficient.

The core of the direct data-driven control approach presented in this chapter is the data-based closed-loop representation derived using Willems *et al.*'s fundamental lemma. Willems *et al.*'s fundamental lemma indicates that the trajectories generated by a linear time-invariant (LTI) system can be represented by a single trajectory as long as the system is sufficiently excited. Originally proposed for the behavioral framework, Willems *et al.*'s fundamental lemma was used by De Persis and Tesi (2020) for deriving a data-based closed-loop representation of a discrete-time LTI system under state feedback. The data-based closed-loop representation plays an essential role in De Persis and Tesi (2020) for data-driven stabilization, linear quadratic regulation, and robust stabilization. Besides De Persis and Tesi, 2020, the data-driven closed-loop representation has also been used for various linear data-driven control problems. For instance, Berberich *et al.* (2020) considered robust data-driven state feedback control of discrete-time LTI systems, Rueda-Escobedo *et al.* (2022) addressed data-driven control of linear discrete-time delay systems, De Persis and Tesi (2021) investigated the LQR design via noisy data, Shafai *et al.* (2022) stabilized positive linear systems via data, and Nortmann and Mylvaganam (2023) developed a data-driven control approach for linear time-varying systems. By writing nonlinear systems into linear-like forms, Guo *et al.* (2022a) and Luppi *et al.* (2022) designed nonlinear data-driven controllers using Lyapunov methods, Nejati *et al.* (2022) developed data-driven safety controllers via control barrier certificates, De Persis *et al.* (2023b) proposed a data-driven control scheme via nonlinearity cancellation, Verhoek *et al.* (2023) investigated the data-driven linear parameter-varying (LPV) control of nonlinear systems, and Wang *et al.* (2023) considered data-driven control of piecewise affine systems. The data-based closed-loop representation has also been used to design output feedback controllers, such as Dai *et al.* (2023) for data-driven control via input-output data, Hu and Liu (2023) for data-driven output feedback control of switched linear systems, and De Jong *et al.* (2023) for data-driven model-reference control. Considering networked systems, De Persis *et al.* (2023a) investigated data-driven event-triggered control and Liang *et al.* (2023) studied data-driven cooperative output regulation of linear multi-agent systems. The data-based closed-loop representation has also shown its great potential in real-life control applications, such as investigated by Alattas *et al.* (2021) for MEMSs gyroscopes, by Scarpa *et al.* (2022) for snake robot locomotion, and by Lan (2023) for vehicle platoons.

The rest of the chapter is organized as follows. In Section "Data-Based Closed-Loop System Representation and Preliminaries", we present the data-based closed-loop representation of continuous-time LTI systems using finite experimental data. Some preliminary facts are also given for the subsequent data-driven control design and closed-loop analysis. The data-driven stabilizer designs for LTI systems are presented in Section "Data-Driven Control of Linear Systems". In Section "Data-Driven Control of Nonlinear Systems", we show that using Taylor's expansion, one can write nonlinear systems into a linear-like form and extend the linear data-driven control design to the nonlinear case. Based on the designed nonlinear data-driven stabilizers, Section "Data-Based Region of Attraction Estimation" presents data-based RoA estimation for the resulting closed-loop systems. Illustrative examples are presented in Section "Illustrative Example" to verify the performance of the data-driven control designs. Finally, Section "Summary and Future Directions" summarizes the chapter and discusses about future research directions.

Notation. Throughout the chapter, $A\succ(\succeq)0$ denotes that matrix A is positive (semi-)definite, and $A\prec(\preceq)0$ denotes that matrix A is negative (semi-)definite. For vectors $a, b\in\mathbb{R}^n$, $a\leq b$ means that $a_i\leq b_i$ for all $i=1,\ldots,n$. $\|\cdot\|$ denotes the Euclidean norm.

Data-Based Closed-Loop System Representation and Preliminaries

Data-Based Closed-Loop Linear System Representation

In this section, we present the data-based closed-loop representation of LTI systems introduced by De Persis and Tesi (2020). Using Willems *et al.*'s fundamental lemma, one can represent the system dynamics of a linear time-invariant system via input-output data

collected from one or more experiments. As this chapter addresses data-driven *state feedback* stabilization, we discuss using input-state experimental data for the control law design.

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^m$ is the control input. The true system matrices A and B are unknown. We suppose that the system is controllable.

Collect data from E experiments where $E \geq 1$ and denote the dataset as

$$\mathcal{DS} := \{\dot{x}(t_k), x(t_k), u(t_k), k = 0, 1, \ldots, T-1\} \tag{2}$$

for some integer $T \geq 1$. Arrange the collected data into data matrices

$$X_0 := \begin{bmatrix} x(t_0) & x(t_1) & \cdots & x(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{n \times T} \tag{3a}$$

$$X_1 := \begin{bmatrix} \dot{x}(t_0) & \dot{x}(t_1) & \cdots & \dot{x}(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{n \times T} \tag{3b}$$

$$U_0 := \begin{bmatrix} u(t_0) & u(t_1) & \cdots & u(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{m \times T}. \tag{3c}$$

The derivative data $\dot{x}(t_k)$, $k = 1, \ldots, T-1$ can be approximated using methods such as numerical differentiation. For instance, using the forward difference approximation gives

$$\dot{x}_i(t_k) = \frac{x_i(t_{k+1}) - x_i(t_k)}{t_{k+1} - t_k} + e_i(t_k), i = 1, \ldots, N,$$

where the approximation error $e_i(t_k)$ is proportional to $t_{k+1} - t_k$ and can be treated as noise. Later in this section, we will discuss how to handle measurement noise.

Under the dynamics (1), we can write that

$$X_1 = AX_0 + BU_0. \tag{4}$$

Designing the state feedback control law as $u = Kx$, one is able to derive a data-based closed-loop system representation as shown in the following lemma.

Lemma 1 (Data-based closed-loop representation). *Consider system (1) with the dataset $\mathcal{DS}$ and a state feedback control law $u = Kx$. For any matrices $K \in \mathbb{R}^{n \times m}$ and $G \in \mathbb{R}^{T \times n}$ such that*

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G, \tag{5}$$

the closed-loop system has a data-based representation

$$\dot{x} = \Psi x \tag{6}$$

where $\Psi := X_1 G$.

Proof. The closed-loop representation is obtained from the equality that

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \overset{(5)}{=} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \overset{(4)}{=} X_1 G. \tag{7}$$

This completes the proof.

Note that the closed-loop matrix Ψ only depends on data and the design variable G. Lemma 1 gives the data-based closed-loop representation when the datasset $\mathcal{DS}$ is noiseless. In the case where noisy data is considered, a data-based closed-loop representation in can be derived in the same manner.

Consider again linear system (1). Suppose that the state measurements and the actual applied input are subject to noise. Specifically, the noisy data matrices are

$$\widehat{X}_0 := X_0 + D_{X_0}, \quad \widehat{X}_1 := X_1 + D_{X_1}, \quad \widehat{U}_0 := U_0 + D_{U_0} \tag{8}$$

where D_{X_0}, D_{X_1}, and D_{U_0} represent the unknown noise matrices affecting the state, state derivative, and the input, respectively. We note that $\widehat{X}_1$ and $\widehat{X}_0$ are the noisy measurements, and $\widehat{U}_0$ is the uncertain actual input in the experiment(s).

Substituting (8) into the data relation $X_1 = AX_0 + B\widehat{U}_0 + D_0$ gives

$$\widehat{X}_1 = A\widehat{X}_0 + BU_0 + \widehat{D}_0 \tag{9}$$

where $\widehat{D}_0 := D_{X_1} - AD_{X_0} + BD_{U_0}$. Using this noisy data-relation, one can also derive a data-based representation of (1).

Lemma 2 (Data-based closed-loop representation with noisy data). *Consider system (1) with the noisy data matrices (8) and a state feedback control law $u = Kx$. For any matrices $K \in \mathbb{R}^{n \times m}$ and $G \in \mathbb{R}^{T \times n}$ such that*

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ \widehat{X}_0 \end{bmatrix} G, \tag{10}$$

the closed-loop system has a data-based representation

$$\dot{x} = \widehat{\Psi} x \tag{11}$$

where $\widehat{\Psi} := \left(\widehat{X}_1 - \widehat{D}_0\right) G$.

Proof. Using the noisy data relation (9), one can write the closed-loop system matrix as

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \overset{(12)}{=} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ \widehat{X}_0 \end{bmatrix} G \overset{(11)}{=} \left(\widehat{X}_1 - \widehat{D}_0\right) G. \tag{12}$$

This completes the proof.

We note that for the noisy data case, the closed-loop matrix $\widehat{\Psi}$ depends on the data, the design variable G, as well as the unknown noisy term $\widehat{D}_0$.

Lemmas 1 and 2 have shown that for continuous-time LTI systems, the closed-loop system dynamics under the state feedback control law $u = Kx$ can be represented using input-state data only. Similar data-based representations can also be derived for discrete-time systems as presented by De Persis and Tesi (2020).

The data-based closed-loop representations imply that by choosing a suitable matrix G, one can enforce stability properties on the closed-loop dynamics Ψ. This is the main idea of the data-driven control approach developed by De Persis and Tesi (2020) which is used for controlling linear and nonlinear systems in the subsequent sections of this chapter.

Preliminaries

For developing data-driven controllers and analyzing the resulting closed-loop systems, we present some preliminary facts that are used in the subsequent sections.

Petersen's lemma

We observe that for the noisy data case, the data-based system dynamics $\widehat{\Psi}$ contains the uncertain term $\widehat{D}_0$. To design a control law capable of handling the uncertainty, we will employ Petersen's lemma introduced by Petersen (1987) for a robust data-driven control design.

Lemma 3 (*Petersen's lemma*, Bisoffi *et al.*, 2022). *Consider matrices* $\mathcal{G} = \mathcal{G}^\top \in \mathbb{R}^{n\times n}$, $\mathcal{M} \in \mathbb{R}^{n\times m}$, $\mathcal{M} \neq 0$, $\mathcal{N} \in \mathbb{R}^{p\times n}$, $\mathcal{N} \neq 0$, *and a set* F *defined as*

$$F = \{\mathcal{F} \in \mathbb{R}^{m\times p} : \mathcal{F}^\top \mathcal{F} \preceq \overline{\mathcal{F}}\}$$

where $\overline{\mathcal{F}} = \overline{\mathcal{F}}^\top \succeq 0$. *Then, for all* $\mathcal{F} \in F$

$$\mathcal{G} + \mathcal{M}\mathcal{F}\mathcal{N} + \mathcal{N}^\top \mathcal{F}^\top \mathcal{M}^\top \preceq 0$$

holds if and only if there exists $\mu > 0$ *such that*

$$\mathcal{G} + \mu \mathcal{M}\mathcal{M}^\top + \mu^{-1} \mathcal{N}^\top \overline{\mathcal{F}} \mathcal{N} \preceq 0.$$

Sum of squares relaxation

For data-driven control of nonlinear systems, we derive stability conditions that are in the form of positive conditions of multivariable polynomials. To efficiently solve these conditions, we will use the SOS technique to obtain tractable sufficient conditions.

Definition 1 (*SOS polynomial matrix*, Chesi, 2010). $M : \mathbb{R}^n \to \mathbb{R}^{\sigma\times\sigma}$ is an SOS polynomial matrix if there exist $M_1, \ldots, M_k : \mathbb{R}^n \to \mathbb{R}^{\sigma\times\sigma}$ such that

$$M(x) = \sum_{i=1}^{k} M_i(x)^\top M_i(x) \;\forall\, x \in \mathbb{R}^n. \tag{13}$$

It is straightforward to see that if a matrix $M(x)$ is an SOS polynomial matrix, then it is positive semi-definite, *i.e.*, $M(x) \succeq 0\ \forall x \in \mathbb{R}^n$. Relaxing the positive polynomial conditions into SOS polynomial conditions makes the conditions tractable and easily solvable by common software.

Positivstellensatz

To estimate the RoA of a nonlinear system, we will use the Positivstellensatz to characterize polynomials that are positive on a semialgebraic set. Based on Stengle's Positivstellensatz presented by Stengle (1974) and Chesi (2010), we derive the following result.

Lemma 4 *Let φ_1 and φ_2 be polynomials in x. If there exist SOS polynomials s_1 and s_2 in x such that*

$$-(s_1\varphi_1(x)+s_2\varphi_2(x)+x^\top x) \text{ is } SOS \ \ \forall x\in\mathbb{R}^n \tag{14}$$

then the set inclusion condition

$$\{x\in\mathbb{R}^n : \varphi_1(x)\geq 0, x\neq 0\}\subseteq\{x\in\mathbb{R}^n : \varphi_2(x)<0\} \tag{15}$$

holds.

Detailed proof of this lemma can be found in Guo *et al.* (2024).

Data-Driven Control of Linear Systems

In Section "Data-Based Closed-Loop Linear System Representation", we have shown how to represent the closed-loop dynamics of continuous-time LTI systems using the dataset $\mathcal{DS}$. This section will present how to seek a control gain K that stabilizes the origin of system (1).

Data-Driven Control of Linear Systems Without Noise

Consider the linear system (1) and the dataset $\mathcal{DS}$. Lemma 1 derives the closed-loop dynamics under the state-feedback $u=Kx$ as $\Psi=X_1G$ for any K and G satisfying (5). To find a stabilizing gain K, one can search for a matrix G that makes Ψ a Hurwitz matrix, that is, all the eigenvalues of Ψ have strictly negative real parts.

Theorem 1 *Consider system (1) and the dataset $\mathcal{DS}$. For any matrices $P\in\mathbb{R}^{n\times n}$, $P\succ 0$, and $Y\in\mathbb{R}^{T\times n}$ satisfying*

$$X_0Y=P \tag{16a}$$

$$(X_1Y)^\top+X_1Y\preceq -Q \tag{16b}$$

for any fixed $Q\succ 0$, the control law $u=U_0YP^{-1}x$ renders the origin stable for the closed-loop system.

Proof. A closed-loop system $\dot{x}=(A+BK)x$ is stable if and only if there exists $P_0\succ 0$ such that

$$(A+BK)^\top P_0+P_0(A+BK)\prec 0. \tag{17}$$

Letting $G=YP^{-1}$, we write the condition (16b) as

$$(X_1GP)^\top+X_1GP\preceq -Q.$$

Left- and right-multiplying $P_0=P^{-1}$ to both sides of the inequality gives

$$(X_1G)^\top P_0+P_0(X_1G)\preceq -P_0QP_0\prec 0.$$

Under (16a) and the control gain design $K=U_0YP^{-1}=U_0G$, we apply Lemma 1 and obtain the data-based closed-loop representation $\dot{x}=X_1Gx$. Hence, the stability of the origin is guaranteed.

Data-Driven Control of Linear Systems With Noisy Data

Recall that Lemma 2 gives the closed-loop representation of (1) under state-feedback $u=Kx$ as $\dot{x}=\left(\widehat{X}_1-\widehat{D}_0\right)Gx$ using the noisy data matrices (8). The matrix $\widehat{D}_0$ depending on the unknown noise is uncertain which can be handled using robust control techniques. Specifically, we search for a matrix G such that $\left(\widehat{X}_1-\widehat{D}_0\right)G$ is Hurwitz for all $\widehat{D}_0\in\mathcal{D}$, where $\mathcal{D}$ is a known set. The set $\mathcal{D}$ can be obtained using prior knowledge on the considered system or via an over-estimation.

Theorem 2 *Consider system (1) and the noisy data matrices (8). Suppose that $\widehat{D}_0\in\mathcal{D}$ where $\mathcal{D}:=\{D\in\mathbb{R}^{n\times T} : DD^\top\preceq\Delta\Delta^\top\}$ with $\Delta\in\mathbb{R}^{n\times T}$ being a chosen parameter. If there exist matrices $P\in\mathbb{R}^{n\times n}$, $P\succ 0$, $Y\in\mathbb{R}^{T\times n}$, and a scalar $\mu>0$, such that*

$$X_0Y=P \tag{18a}$$

$$\begin{bmatrix} -Q-\left(\widehat{X}_1Y\right)^\top-\widehat{X}_1Y-\mu\Delta\Delta^\top & Y^\top \\ Y & \mu I_T \end{bmatrix}\succeq 0, \tag{18b}$$

for any fixed $Q\succ 0$, the control law $u=U_0YP^{-1}x$ renders the origin stable for the closed-loop system.

Proof. Using Schur complement, we write (18b) as

$$Q+\left(\widehat{X}_1Y\right)^\top+\widehat{X}_1Y+\mu\Delta\Delta^\top+\mu^{-1}Y^\top Y\preceq 0$$

As $\widehat{D}_0 \in \mathcal{D}$, it holds that $\widehat{D}_0\widehat{D}_0^\top \preceq \Delta\Delta^\top$. Using Petersen's lemma (Lemma 3), the above inequality is equivalent to

$$Q + \left(\widehat{X}_1 Y\right)^\top + \widehat{X}_1 Y + Y^\top(-\widehat{D}_0)^\top + (-\widehat{D}_0)Y \succeq 0$$

which can be written as

$$Y^\top\left(\widehat{X}_1 - \widehat{D}_0\right)^\top + (\widehat{X}_1 - \widehat{D}_0)Y \succeq -Q.$$

Letting $G = YP^{-1}$, we rewrite the above inequality as

$$PG^\top\left(\widehat{X}_1 - \widehat{D}_0\right)^\top + \left(\widehat{X}_1 - \widehat{D}_0\right)GP \succeq -Q.$$

Left- and right-multiplying $P_0 = P^{-1}$ to both sides of the inequality, one has that

$$\left((\widehat{X}_1 - \widehat{D}_0)G\right)^\top P_0 + P_0\left((\widehat{X}_1 - \widehat{D}_0)G\right) \succeq P_0 Q P_0 \prec 0.$$

By the condition (18a) and the designed control law $u = U_0 YP^{-1}x$, the closed-loop system dynamics is $\dot{x} = \left(\widehat{X}_1 - \widehat{D}_0\right)Gx$. Therefore, the (18b) guarantees the stability of the closed-loop system.

Data-Driven Control of Nonlinear Systems

The previous section has shown that by making the data-based closed-loop system dynamics Hurwitz, one can design state feedback stabilizers for continuous-time LTI systems. In fact, a linear-like data-based closed-loop representation can be derived for nonlinear systems based on Taylor's expansion. In this section, we will present how to stabilize the origin of continuous-time nonlinear systems via first-order and high-order approximations, respectively.

First-Order Approximation

Consider the nonlinear system

$$\dot{x} = f(x, u) \tag{19}$$

where the state $x \in \mathbb{R}^n$ and the input $u \in \mathbb{R}^m$. Suppose that f is a smooth function and $f(0, 0) = 0$. Then, one can obtain the first-order approximation of (19) as

$$\dot{x} = Ax + Bu + R(x, u) \tag{20}$$

where $R(x, u)$ denotes the unknown remainder containing high-order terms and

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x,u)\,=\,(0,0)}, B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x,u)\,=\,(0,0)}.$$

We observe that the first-order approximation (20) contains a linear part in the same form as (1), and thus a similar approach can be applied for stabilizing (20) using the dataset $\mathcal{DS}$. First, in the same spirit of Lemma 2, we show that the system (19) has a data-based closed-loop representation.

Lemma 5 *Consider system (19) with the dataset $\mathcal{DS}$ and a state feedback control law $u = Kx$. For any matrices $K \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{T\times n}$ satisfying (5), the closed-loop system has a data-based representation*

$$\dot{x} = \Psi x + R(x, Kx) \tag{21}$$

where $\Psi := (X_1 - R_0)G$ and $R_0 := [R(x(t_0), u(t_0))\ R(x(t_1), u(t_1))\ \cdots\ R(x(t_{T-1}), u(t_{T-1}))]$.

We note that in (21), $R(x, u)$ only contains high-order terms that converge to zero faster than x in a neighborhood of the origin, *i.e.*, $\lim_{x\to 0}\frac{|R(x,Kx)|}{|x|} = 0$. Therefore, by seeking a matrix G that makes Ψ Hurwitz, the closed-loop system is locally asymptotically stable.

Theorem 3 *Consider system (19) and the dataset $\mathcal{DS}$. Suppose that $R_0 \in \mathcal{R}$ where $\mathcal{R} := \{R \in \mathbb{R}^{n\times T} : RR^\top \preceq \Delta\Delta^\top\}$ with $\Delta \in \mathbb{R}^{n\times T}$ being a chosen parameter. If there exist matrices $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, $Y \in \mathbb{R}^{T\times n}$, and a scalar $\mu > 0$, such that*

$$X_0 Y = P \tag{22a}$$

$$\begin{bmatrix} -Q - (X_1 Y)^\top - X_1 Y - \mu\Delta\Delta^\top & Y^\top \\ Y & \mu I_T \end{bmatrix} \succeq 0, \tag{22b}$$

for any fixed $Q \succ 0$, the control law $u = U_0 YP^{-1}x$ renders the origin locally asymptotically stable for the closed-loop system.

If the measurements in the experiment are corrupted by noise, one can still derive a data-based first-order approximation of (19). In particular, the remainder $R(x, Kx)$ in (20) will be replaced by a term $\widehat{R}$ that accounts for uncertainties including the noise and the

approximation error. By estimating a known bound of $\widehat{R}$ during the experiment(s), one can still apply **Theorem 3** for a data-driven stabilizer design.

We note that **Theorems 1–3** characterize data-driven stabilizers using LMIs which are computationally sufficient and can be easily solved by common software.

High-Order Approximation

One can also use Taylor's expansion to obtain a polynomial approximation of the nonlinear system (19). Consider the input-affine system

$$\dot{x} = f(x) + g(x)u \tag{23}$$

where the state $x \in \mathbb{R}^n$ and the input $u \in \mathbb{R}^m$. The functions $f(x)$ and $g(x)$ are smooth and $f(0) = 0$, $g_{ij}(0) \neq 0$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.

Define $\alpha = (\alpha_1, \ldots, \alpha_\sigma)$ as a σ-tuple of nonnegative integers and

$$|\alpha| = \alpha_1 + \cdots + \alpha_\sigma,$$

$$\alpha! = \alpha_1! \cdots \alpha_\sigma!,$$

$$\partial^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial z_1^{\alpha_1} \cdots \partial z_\sigma^{\alpha_\sigma}},$$

$$z^\alpha = z_1^{\alpha_1} \cdots z_\sigma^{\alpha_\sigma}$$

for any $z = [z_1 \cdots z_\sigma]^\top \in \mathbb{R}^\sigma$. Using Taylor's expansion, we write each element $f_i(x)$ in $f(x)$ and $g_{ij}(x)$ in $g(x)$ for $i = 1,\ldots,n$, $j = 1,\ldots,m$ respectively as

$$f_i(x) = \sum_{|\alpha| \leq r_f} \frac{\partial^\alpha f_i(0)}{\alpha!} x^\alpha + R_{f_i}(x),$$

$$g_{ij}(x) = g_{ij}(0) + R_{g_{ij}}(x)$$

where $R_{f_i}(x)$ and $R_{g_{ij}}(x)$ are remainders containing high-order terms. Define $Z(x) \in \mathbb{R}^{r_Z}$ as a vector of monomials in x having degree from 1 to r_f. We write (23) into the linear-like form

$$\dot{x} = AZ(x) + Bu + R(x,u) \tag{24}$$

where $R(x,u) := \begin{bmatrix} R_1(x,u) & R_2(x,u) & \cdots & R_n(x,u) \end{bmatrix}^\top$ with

$$R_i(x,u) := R_{f_i}(x) + \sum_{j=1}^{m} R_{g_{ij}}(x) u_j,$$

and A and $B = g(0)$ are unknown constant matrices. We assume that $g(0) \neq 0$ in this chapter. More generally, one can include state-dependent terms in the approximation of $g(x)$ and refer to the approach developed in Guo *et al.* (2022a) for designing data-driven controllers.

Using the dataset $\mathcal{DS}$, one can evaluate the value of vector $Z(x)$ at each sampling time of the experiment, which is denoted as

$$Z_0 := \begin{bmatrix} Z(x(t_0)) & Z(x(t_1)) & \cdots & Z(x(t_{T-1})) \end{bmatrix}. \tag{25}$$

In what follows, we simultaneously seek a Lyapunov function and stabilize the origin of (24). Design the Lyapunov function as

$$V(x) := \widehat{Z}(x)^\top P^{-1} \widehat{Z}(x)$$

where $P \succ 0$ and $\widehat{Z}(x)$ is a vector of monomials in x, which has smaller size and lower degree than the vector $Z(x)$. Specifically, the $p \times 1$ vector $\widehat{Z}(x)$ is such that

$$Z(x) = H(x)\widehat{Z}(x) \tag{26}$$

for some non-unique polynomial matrix $H : \mathbb{R}^n \to \mathbb{R}^{r_Z \times p}$. We design the state feedback controller as $u = K(x)\widehat{Z}(x)$. Using $\widehat{Z}(x)$ instead of $Z(x)$ for the Lyapunov function and control law offers more flexibility in the control design.

Using the dataset $\mathcal{DS}$, we obtain the data-based closed-loop representation of (24).

Lemma 6 *Consider system (24) with the dataset $\mathcal{DS}$ and a state feedback control law $u = K(x)\widehat{Z}(x)$, where $\widehat{Z}(x)$ is a chosen vector of monomials in x. For any matrices $K(x) \in \mathbb{R}^{m \times n}$ and $G(x) \in \mathbb{R}^{T \times n}$ such that*

$$\begin{bmatrix} K(x) \\ H(x) \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G(x) \tag{27}$$

for some matrix $H(x)$ satisfying (26), the closed-loop system has a data-based representation

$$\dot{x} = \Psi(x)\widehat{Z}(x) + R\Big(x, K(x)\widehat{Z}(x)\Big) \tag{28}$$

where $\Psi(x) := (X_1 - R_0)G(x)$ *and* $R_0 := [R(x(t_0), u(t_0))\ R(x(t_1), u(t_1))\ \cdots\ R(x(t_{T-1}), u(t_{T-1}))]$.

Proof. For the linear-like system (24), the data matrices satisfy the relation

$$X_1 = AZ_0 + BU_0 + R_0. \tag{29}$$

The close-loop dynamics is

$$\begin{aligned}\dot{x} &= AH(x)\widehat{Z}(x) + BK(x)\widehat{Z}(x) + R\Big(x, K(x)\widehat{Z}(x)\Big)\\ &= \begin{bmatrix} B & A \end{bmatrix}\begin{bmatrix} K(x) \\ H(x) \end{bmatrix}\widehat{Z}(x) + R\Big(x, K(x)\widehat{Z}(x)\Big)\\ &\overset{(29)}{=} \begin{bmatrix} B & A \end{bmatrix}\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}G(x)\widehat{Z}(x) + R\Big(x, K(x)\widehat{Z}(x)\Big)\\ &\overset{(31)}{=} (X_1 - R_0)G(x)\widehat{Z}(x) + R\Big(x, K(x)\widehat{Z}(x)\Big).\end{aligned}$$

The proof is thus complete.

Similar to the case of first-order approximation, if we make the origin asymptotically stable for $\dot{x} = \Psi(x)\widehat{Z}(x)$ and guarantee that the remainder $R\Big(x, K(x)\widehat{Z}(x)\Big)$ converges to zero faster than $\Psi(x)\widehat{Z}(x)$ in a neighbourhood of the origin, then the origin is locally asymptotically stable for (28).

Theorem 4 *Consider system* (23) *and the dataset* $\mathcal{DS}$. *Suppose that* $R_0 \in \mathcal{R}$ *where* $\mathcal{R} := \{R \in \mathbb{R}^{n\times T} : RR^\top \preceq \Delta\Delta^\top\}$ *with* $\Delta \in \mathbb{R}^{n\times T}$ *being a chosen parameter. If there exist matrices* $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, $Y(x) \in \mathbb{R}^{T\times n}$, *a scalar* $\mu > 0$, *and an SOS polynomial* $\epsilon(x)$, *such that*

$$Z_0 Y(x) = H(x)P \tag{30a}$$

$$\begin{bmatrix} -\dfrac{\partial\widehat{Z}}{\partial x}X_1Y(x) - Y(x)^\top X_1^\top\dfrac{\partial\widehat{Z}}{\partial x}^\top - \epsilon(x)I - \mu\dfrac{\partial\widehat{Z}}{\partial x}\Delta\Delta^\top\dfrac{\partial\widehat{Z}}{\partial x}^\top & Y(x)^\top \\ Y(x) & \mu I_T \end{bmatrix} \text{ is SOS}, \tag{30b}$$

the control law $u = U_0 Y(x)P^{-1}\widehat{Z}(x)$ *renders the origin locally asymptotically stable for the closed-loop system*.

Proof. Using Schur complement, we write the condition (30b) as

$$\frac{\partial\widehat{Z}}{\partial x}X_1Y(x) + Y(x)^\top X_1^\top\frac{\partial\widehat{Z}}{\partial x}^\top + \epsilon(x)I + \mu\frac{\partial\widehat{Z}}{\partial x}\Delta\Delta^\top\frac{\partial\widehat{Z}}{\partial x}^\top + \mu^{-1}Y(x)^\top Y(x) \preceq 0$$

By Petersen's lemma, the above inequality is equivalent to

$$\frac{\partial\widehat{Z}}{\partial x}X_1Y(x) + Y(x)^\top X_1^\top\frac{\partial\widehat{Z}}{\partial x}^\top + \epsilon(x)I + \frac{\partial\widehat{Z}}{\partial x}(-R_0)Y(x) + Y(x)^\top(-R_0)^\top\frac{\partial\widehat{Z}}{\partial x}^\top \preceq 0$$

which can be rearranged as

$$\frac{\partial\widehat{Z}}{\partial x}(X_1 - R_0)Y(x) + Y(x)^\top(X_1 - R_0)^\top\frac{\partial\widehat{Z}}{\partial x}^\top \preceq -\epsilon(x)I. \tag{31}$$

Taking the time derivative of the Lyapunov function $V(x) = \widehat{Z}(x)^\top P^{-1}\widehat{Z}(x)$ along the trajectory of the closed-loop system (28) gives

$$\dot{V}(x) = \widehat{Z}(x)^\top P^{-1}\frac{\partial\widehat{Z}}{\partial x}\Big[(X_1 - R_0)G(x)\widehat{Z}(x) + R\Big(x, K(x, \widehat{Z}(x))\Big)\Big] + \Big[(X_1 - R_0)G(x)\widehat{Z}(x) + R\Big(x, K(x, \widehat{Z}(x))\Big)\Big]^\top\frac{\partial\widehat{Z}}{\partial x}^\top P^{-1}\widehat{Z}(x).$$

Letting $G(x) = Y(x)P^{-1}$, one has that

$$\begin{aligned}\dot{V}(x) &= \widehat{Z}(x)^\top P^{-1}\left(\frac{\partial\widehat{Z}}{\partial x}(X_1 - R_0)Y(x) + Y(x)^\top(X_1 - R_0)^\top\frac{\partial\widehat{Z}}{\partial x}^\top\right)P^{-1}\widehat{Z}(x) + 2\widehat{Z}(x)^\top P^{-1}\frac{\partial\widehat{Z}}{\partial x}R\Big(x, K(x, \widehat{Z}(x))\Big)\\ &\overset{(33)}{\leq} -\epsilon(x)\widehat{Z}(x)^\top P^{-2}\widehat{Z}(x) + 2\widehat{Z}(x)^\top P^{-1}\frac{\partial\widehat{Z}}{\partial x}R\Big(x, K(x, \widehat{Z}(x))\Big).\end{aligned}$$

Note that the remainder $R(x, K(x, \widehat{Z}(x)))$ has a degree of $max\{r_f, 1\}$. As the degrees of $\epsilon(x)$ and $\widehat{Z}(x)$ are chosen by design, we can guarantee that the term $2\widehat{Z}(x)^\top P^{-1}\frac{\partial\widehat{Z}}{\partial x}R(x, K(x, \widehat{Z}(x)))$ is of higher order and thus converges to zero faster than $-\epsilon(x)\widehat{Z}(x)^\top P^{-2}\widehat{Z}(x)$ for all x in a neighborhood of the origin. Moreover, as $\epsilon(x) > 0\ \forall x \neq 0$, we conclude that the origin is locally asymptotically stable for the closed-loop system.

Data-Based Region of Attraction Estimation

The previous section has designed data-driven stabilizers that make the origin locally asymptotically stable for nonlinear systems. In this section, we show how to use data to estimate the RoA for the designed closed-loop system. Specifically, we derive data-driven SOS conditions to determine whether a given sublevel set of the Lyapunov function is an invariant subset of the RoA.

First, we recall the definition of RoA of a given system.

Definition 2 (*Region of attraction*). For the systems $\dot{x}=f(x)$, if for every initial condition $x(t_0)\in\mathcal{W}$, it holds that $\lim_{t\to\infty}x(t)=0$, then $\mathcal{W}$ is a region of attraction of the system with respect to the origin. If there exists a C^1 function $V:\mathbb{R}^n\to\mathbb{R}$ and a positive constant c such that

$$\Omega_c := \{x\in\mathbb{R}^n : V(x)\le c\}$$

is bounded and

$$V(0)=0,\ V(x)>0\ \forall x\in\mathbb{R}^n$$

$$\{x\in\mathbb{R}^n : V(x)\le c, x\neq 0\}\subseteq\{x\in\mathbb{R}^n : \dot{V}(x)<0\},$$

then Ω_c is an invariant subset, or called an estimation, of the RoA.

By the controller design method presented in Section "Data-Driven Control of Nonlinear Systems", the Lyapunov function $V(x)$, and thus the set Ω_c, are available for estimating the RoA. To characterize the set $\{x\in\mathbb{R}^n : \dot{V}(x)<0\}$, we need to bound the remainder using a known function for x in a neighborhood of the origin.

For the data-driven controlled system via first-order approximation, consider a neighborhood of the origin $\mathbb{D}\subseteq\mathbb{R}$ such that for all $x\in\mathbb{D}$, each component in the remainder $R(x,Kx)$ satisfies that

$$|R_i(x,Kx)|\le L_i\|(x,Kx)\|^2 \tag{32}$$

where $L_i>0$ is a known constant.

This bound can be obtained by prior knowledge on the unknown functions. For example, if the gradient of each f_i satisfies the Lipschitz condition with a known bound, the constant L_i can be found using the Lipschitz bound and the designed control gain K. Details on the estimation of Taylor's remainder can be found in works by Folland (1990) and Guo *et al.* (2022b).

For the closed-loop system with the controller $u=Kx$ designed via **Theorem 3**, the derivative of the Lyapunov function $V(x)=x^\top P^{-1}x$ satisfies

$$\begin{aligned}\dot{V}(x) &= x^\top P^{-1}\Psi x + x^\top\Psi^\top P^{-1}x + 2x^\top P^{-1}R(x,Kx)\\ &\le -x^\top P^{-1}QP^{-1}x + 2x^\top P^{-1}R(x,Kx).\end{aligned}$$

As we have the bound of the remainder as (32), for all $x\in\mathbb{D}$, there exists a continuous $\rho_i(x)$ for each $i=1,\dots,n$ such that

$$\begin{aligned}R_i(x,Kx)&=\rho_i(x)\|(x,Kx)\|^2,\\ \rho_i(x)&\in[-L_i,L_i].\end{aligned}$$

Define $\rho(x)=[\rho_1(x)\dots\rho_n(x)]^\top$. By the definition of polytopes (Blanchini and Miani, 2008, Definition 3.21), the vector $\rho(x)$ belongs to the polytope

$$\mathcal{L}=\{\varrho : -l\le\varrho\le l\} \tag{33}$$

where $l=\begin{bmatrix}L_1 & \cdots & L_n\end{bmatrix}^\top$. Denote $\mathcal{Q}_i$ as the ith column of P^{-1} and

$$\kappa(x) := \begin{bmatrix}x^\top\mathcal{Q}_1\|(x,Kx)\|^2 & x^\top\mathcal{Q}_2\|(x,Kx)\|^2 & \cdots & x^\top\mathcal{Q}_n\|(x,Kx)\|^2\end{bmatrix}. \tag{34}$$

Then, the derivative of the Lyapunov function satisfies for all $x\in\mathbb{D}$

$$\dot{V}(x)\le -x^\top P^{-1}QP^{-1}x + 2\kappa(x)\rho(x)$$

where $\rho(x)\in\mathcal{L}$. Then, one can derive the data-based RoA estimation for the system (19) using first-order approximation.

Proposition 1 *Consider the controller $u=Kx$ designed by* **Theorem 3** *which renders the origin a locally asymptotically stable equilibrium for (19) with the Lyapunov function $V(x)=x^\top P^{-1}x$. Suppose that for all x in a neighborhood of the origin $\mathbb{D}\subseteq\mathbb{R}$, each component in the remainder $R(x,Kx)$ satisfies that $|R_i(x,Kx)|\le L_i\|(x,Kx)\|^2$ with $L_i>0$ a known constant. Given a $c>0$ such that $\Omega_c=\{x\in\mathbb{R}^n : V(x)\le c\}\subseteq\mathbb{D}$, if there exist SOS polynomials s_{1k},s_{2k} in x, $k=1,\dots,v$ such that*

$$-\left[s_{1k}(c-V(x))+s_{2k}\left(-x^\top P^{-1}QP^{-1}x+2\kappa(x)h_k\right)+x^\top x\right] \tag{35}$$

is SOS, where $\kappa(x)$ is defined as in (34) and h_k are the distinct vertices of the polytope $\mathcal{L}$ defined in (33), then Ω_c is an invariant subset of the RoA of the system $\dot{x}=f(x)+g(x)Kx$ relative to the equilibrium $x=0$.

Similar result can be derived for the high-order approximation. To simplify the analysis, we consider the system (23) with a single input, *i.e.*, $u\in\mathbb{R}$. We also consider that in a neighborhood of the origin $\mathbb{D}\subseteq\mathbb{R}$, the remainders $R_{f_i}(x)$ and $R_{g_i}(x)$ of each

function f_i and g_i are bounded by a known function in x for all $x\in\mathbb{D}$. Then, one can bound the remainder $R_i\left(x,K(x)\widehat{Z}(x)\right) := R_{f_i}(x)+R_{g_i}(x)u$ as

$$|R_i\left(x,K(x)\widehat{Z}(x)\right)|\le L_i\|x\|^{r_f+1}+M_i\sum_{j=1}^{q}\|x\|^{2j} \tag{36}$$

for some known $L_i>0$, $M_i>0$, and a positive integer q. Letting $\tilde{\rho}_i=\max\{L_i,M_i\}$, one can write

$$R_i\left(x,K(x)\widehat{Z}(x)\right)=\tilde{\rho}_i(x)\left(\|x\|^{r_f+1}+\sum_{j=1}^{q}\|x\|^{2j}\right)$$

where $|\tilde{\rho}_i(x)|\le\tilde{\rho}_i\ \forall x\in\mathbb{D}$. Denote $\mathcal{Q}_i(x)$ as the ith column of $P^{-1}\frac{\partial\widehat{Z}(x)}{\partial x}$, and define $\tilde{\rho}(x)=\begin{bmatrix}\tilde{\rho}_1(x) & \cdots & \tilde{\rho}_n(x)\end{bmatrix}^\top$ and

$$\tilde{\kappa}(x)=\left[\widehat{Z}(x)^\top\mathcal{Q}_1(x)\left(\|x\|^{r_f+1}+\textstyle\sum_{j=1}^{q}\|x\|^{2j}\right)\cdots\widehat{Z}(x)^\top\mathcal{Q}_n(x)\left(\|x\|^{r_f+1}+\textstyle\sum_{j=1}^{q}\|x\|^{2j}\right)\right]. \tag{37}$$

The derivative of the Lyapunov function $\widehat{Z}(x)^\top P^{-1}\widehat{Z}(x)$ along the trajectory of the closed-loop system designed in **Theorem 4** satisfies that

$$\dot{V}(x)\le-\epsilon(x)\widehat{Z}(x)^\top P^{-2}\widehat{Z}(x)+2\tilde{\kappa}(x)\tilde{\rho}(x) \tag{38}$$

for all $x\in\mathbb{D}$, where the vector $\tilde{\rho}(x)$ is contained in the polytope

$$\mathcal{H}:=[-\tilde{\varrho}_1,\tilde{\varrho}_1]\times\cdots\times[-\tilde{\varrho}_n,\tilde{\varrho}_n]. \tag{39}$$

Using the above analysis, a data-based RoA estimation for the system (23) having single input can be derived as summarized in the following result.

Proposition 2 *Consider the controller $u=K(x)\widehat{Z}(x)$ designed by **Theorem 4** which renders the origin a locally asymptotically stable equilibrium for (23) with the Lyapunov function $V(x)=\widehat{Z}(x)^\top P^{-1}\widehat{Z}(x)$. Suppose that for all x in a neighborhood of the origin $\mathbb{D}\subseteq\mathbb{R}$, each component in the remainder $R(x,K(x)\widehat{Z}(x))$ satisfies (36). Given a $c>0$ such that $\Omega_c=\{x\in\mathbb{R}^n:V(x)\le c\}\subseteq\mathbb{D}$, if there exist SOS polynomials s_{1k},s_{2k} in x, $k=1,\ldots,v$ such that,*

$$-\left[s_{1k}(c-V(x))+s_{2k}\left(-\epsilon(x)\widehat{Z}(x)^\top P^{-2}\widehat{Z}(x)+2\tilde{\kappa}(x)\tilde{h}_k\right)+x^\top x\right]\ is\ \ SOS \tag{40}$$

where $\tilde{\kappa}(x)$ is defined as in (37) and $\tilde{h}_k$ are the distinct vertices of polytope $\mathcal{H}$ defined in (39), then Ω_c is an invariant subset of the RoA of the system $\dot{x}=f(x)+g(x)K(x)\widehat{Z}(x)$ relative to the equilibrium $x=0$.

In Propositions 1 and 2, we derive SOS conditions for the RoA estimation, which can be easily checked using software such as Matlab. Alternatively, De Persis *et al.* (2023b) estimated the RoA using a mesh method for discrete-time systems. Specifically, a sufficient condition of $V(x^+)-V(x)<0$ is derived using data, and the grids in a compact region are tested to see whether the sufficient condition is satisfied so that the sublevel sets of $V(x)$ can be found as the RoA estimation. We also note that once the controller is computed from data, it is possible to use other data-driven methods to estimate the RoA, see for example Bobiti and Lazar (2018), Colbert and Peet (2018) and Shen *et al.* (2022).

Illustrative Examples

In this section, illustrative examples for data-driven control of linear and nonlinear systems are presented, respectively. In particular, we present simulation results for a batch reactor (Walsh and Ye, 2001) using **Theorem 1** and for an inverted pendulum using **Theorem 3** and Proposition 1. It is noted that the models of the batch reactor and the inverted pendulum are only used for collecting experimental data but not for designing controllers.

Linear Data-Driven Control

Consider the batch reactor presented in Walsh and Ye (2001), which is in the form of (1) with

$$A=\begin{bmatrix}1.38 & -0.2077 & 6.715 & -5.676\\ -0.5814 & -4.29 & 0 & 0.675\\ 1.067 & 4.273 & -6.654 & 5.893\\ 0.048 & 4.273 & 1.343 & -2.104\end{bmatrix},\ B=\begin{bmatrix}0 & 0\\ 5.679 & 0\\ 1.136 & -3.146\\ 1.136 & 0\end{bmatrix}.$$

An experiment is conducted with $x(0)=\begin{bmatrix}0.1 & 0.2 & -0.3 & 0.1\end{bmatrix}^\top$ and $u=\begin{bmatrix}0.1\sin(t) & 0.1\cos(t)\end{bmatrix}^\top$. The experimental data is sampled with fixed sampling period $T_s=0.1$s, and we arrange the first 15 sampled data into the data matrices, *i.e.*, $T=15$. Using CVX (Grant and Boyd, 2020) in MATLAB, we apply **Theorem 1** with $Q=0.5I_4$ and obtain the data-driven control gain K as

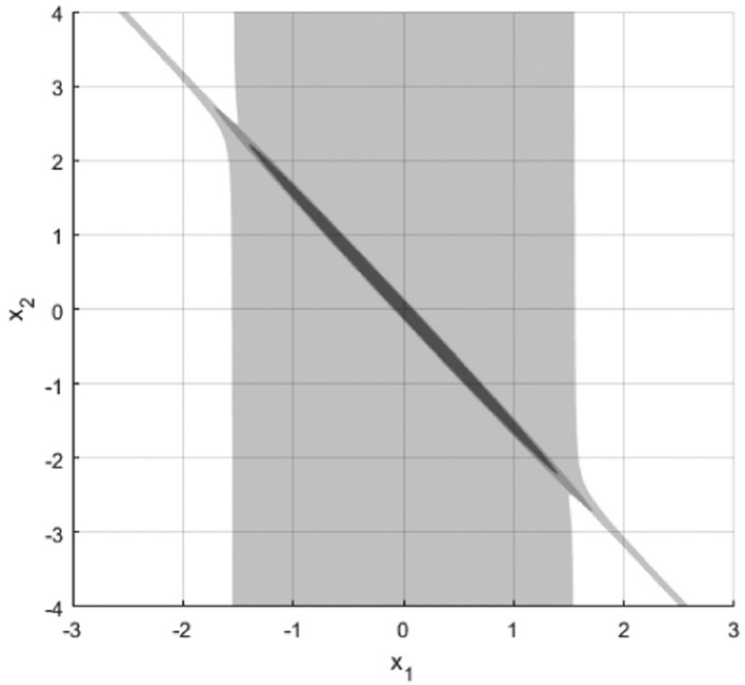


Fig. 1 RoA estimations of the inverted pendulum.

$$K = \begin{bmatrix} 3.0574 & -3.1317 & 1.0413 & -3.2924 \\ 0.4322 & 0.3820 & 0.3541 & 0.8687 \end{bmatrix},$$

which renders $(A + BK)$ Hurwitz, *i.e.*, stabilizes the closed-loop system.

Nonlinear Data-Driven Control

For the nonlinear data-driven control, we consider an inverted pendulum which has the dynamics

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \frac{mgl}{J}\sin(x_1) - \frac{r}{J}x_2 + \frac{l}{J}\cos(x_1)u,$$

where $m = 0.1$, $g = 9.8$, and $r = l = J = 1$. For the experiment, we set $x(0) = [0.1 \quad -0.1]^\top$ and $u = 0.5\sin(t)$. The data is sampled with fixed sampling period $T_s = 0.05$s, and the data matrices is obtained with $T = 15$.

We overestimate the remainder bound and choose that $\Delta\Delta^\top \succeq 10\, RR^\top$. Applying **Theorem 3** with $Q = 0.5I_2$, the data-driven feedback controller is designed as $u = -44.0407x_1 - 27.7346x_2$. For the RoA estimation, we first analyze the bounds of the remainders. From physical considerations, we argue that as the x_1-subsystem for an inverted pendulum is linear, $L_1 = 0$, and we overestimate L_2 as $L_2 = 1.6971$. Applying Proposition 5, we find the largest c that satisfies the SOS conditions is $c = 0.22$.

Fig. 1 illustrates the estimated RoA using different approaches. The light grey area is the RoA obtained by checking every point in an meshed area, the medium grey area is the largest sublevel set of the obtained Lyapunov function contained in the light grey RoA, and the dark grey area is the estimated RoA found by Proposition 1.

Summary And Future Directions

This chapter has presented the data-based closed-loop representation based on Willems *et al.*'s fundamental lemma and how to use the representation for data-driven stabilizer designs. When the data is corrupted by noise, robust data-driven stabilizers are designed using an uncertain closed-loop representation. Writing nonlinear systems into linear-like forms using Taylor's expansion, local stabilization is achieved by data-driven linear or nonlinear state feedback control, and a data-based inner estimation of the RoA is obtained. We also note that many recent works have shown that the data-based closed-loop representation can be used to solve various theoretical and practical data-driven control problems.

The future developments in direct data-driven control will be focused on nonlinear data-driven control with theoretical guarantees, including research topics such as nonlinear control via input-output data, equilibrium-independent control, and optimal nonlinear regulation.

References

Alattas, K.A., Mohammadzadeh, A., Mobayen, S., *et al*., 2021. A new data-driven control system for memss gyroscopes: Dynamics estimation by type-3 fuzzy systems. Micromachines 12 (11), 1390.

Åström, K.J., Wittenmark, B., 1989. Adaptive Control, first ed. Addison-Wesley.

Berberich, J., Koch, A., Scherer, C.W., Allgöwer, F., 2020. Robust data-driven state-feedback design. In: Proceedings of the 2020 American Control Conference (ACC), Denver, CO, USA, pp. 1532–1538.

Berberich, J., Köhler, J., Müller, M., Allgöwer, F., 2021. Data-driven model predictive control with stability and robustness guarantees. IEEE Transactions on Automatic Control 66 (4), 1702–1717.

Bisoffi, A., De Persis, C., Tesi, P., 2022. Data-driven control via Petersen's lemma. Automatica 145, 110537(1)–110537(14).

Blanchini, F., Miani, S., 2008. Set-Theoretic Methods in Control (Systems & Control: Foundations & Applications). Birkhäuser Boston.

Bobiti, R., Lazar, M., 2018. Automated sampling-based stability verification and DOA estimation for nonlinear systems. IEEE Transactions on Automatic Control 63 (11), 3659–3674.

Campi, M., Savaresi, S., 2006. Direct nonlinear control design: The virtual reference feedback tuning (VRFT) approach. IEEE Transactions on Automatic Control 51 (1), 14–27.

Celi, F., Baggio, G., Pasqualetti, F., 2021. Distributed learning of optimal controls for linear systems. In: Proceedings of the 60th IEEE Conference on Decision and Control (CDC), Austin, Texas, pp. 5771–5776.

Chesi, G., 2010. LMI techniques for optimization over polynomials in control: A survey. IEEE Transactions on Automatic Control 55 (11), 2500–2510.

Colbert, B.K., Peet, M.M., 2018. Using trajectory measurements to estimate the region of attraction of nonlinear systems. In: Proceedings of the 57th IEEE Conference on Decision and Control, Miami, FL, USA, pp. 2341–2347.

Coulson, J., Lygeros, J., Dörfler, F., 2019. Data-enabled predictive control: In the shallows of the DeePC. In: Proceedings of the 18th European Control Conference, Naples, Italy, pp. 307–312.

Dai, X., De Persis, C., Monshizadeh, N., Tesi, P., 2023. Data-driven control of nonlinear systems from input-output data. In: Proceedings of the 2023 62nd IEEE Conference on Decision and Control (CDC), Marina Bay Sands, Singapore, pp. 1613–1618.

Fliess, M., Join, C., 2009. Model-free control and intelligent PID controllers: Towards a possible trivialization of nonlinear control? IFAC Proceedings Volumes 42 (10), 1531–1550.

Folland, G.B., 1990. Remainder estimates in Taylor's theorem. The American Mathematical Monthly 97 (3), 233–235.

Grant, M., Boyd, S., 2020. CVX: Matlab software for disciplined convex programming, version 2.2 cvxr.com/cvx/ .

Guo, M., De Persis, C., Tesi, P., 2022a. Data-driven stabilization of nonlinear polynomial systems with noisy data. IEEE Transactions on Automatic Control 67 (8), 4210–4217.

Guo, M., De Persis, C., Tesi, P., 2022b. Data-driven stabilizer design and closed-loop analysis of general nonlinear systems via Taylor's expansion. arXiv preprint arXiv:2209.01071.

Guo, M., De Persis, C., Tesi, P., 2024. Data-driven stabilization of nonlinear systems via Taylor's expansion. Hybrid and Networked Dynamical Systems - Modeling, Analysis and Control. Cham: Springer.

Hewing, L., Wabersich, K.P., Menner, M., Zeilinger, M.N., 2019. Learning-based model predictive control: Toward safe learning in control. Annual Review of Control, Robotics, and Autonomous Systems 3, 269–296.

Hjalmarsson, H., Gevers, M., Gunnarsson, S., Lequin, O., 1998. Iterative feedback tuning: Theory and applications. IEEE Control Systems Magazine 18 (4), 26–41.

Hou, Z., Wang, Z., 2013. From model-based control to data-driven control: Survey, classification and perspective. Information Sciences 235, 3–35.

Hou, Z., Xiong, S., 2019. On model-free adaptive control and its stability analysis. IEEE Transactions on Automatic Control 64 (11), 4555–4569.

Hu, K., Liu, T., 2023. Data-driven output-feedback control for unknown switched linear systems. IEEE Control Systems Letters 7, 2299–2304.

De Jong, T.O., Breschi, V., Schoukens, M., Formentin, S., 2023. Data-driven model-reference control with closed-loop stability: The output-feedback case. IEEE Control Systems Letters 7. 2475-1456.

Lan, J., 2023. Data-driven dual-loop control for platooning mixed human-drivenand automated vehicles. IET Intelligent Transport Systems. 1–10.

Lederer, A., Yang, Z., Jiao, J., Hirche, S., 2022. Cooperative control of uncertain multi-agent systems via distributed Gaussian processes. IEEE Transactions on Automatic Control 68 (5), 3091–3098.

Liang, D., Dong, Y., Wang, C., Tian, E., 2023. Data-driven optimal cooperative output regulation of unknown linear discrete-time multi-agent systems. In: Proceedings of the 2023 35th Chinese Control and Decision Conference (CCDC), Yichang, China, pp. 1738–1743.

Luppi, A., De Persis, C., Tesi, P., 2022. On data-driven stabilization of systems with nonlinearities satisfying quadratic constraints. Systems & Control Letters 163, 105206 (1)–105206(11).

Martin, T., Schön, T.B., Allgöwer, F., 2023. Guarantees for data-driven control of nonlinear systems using semidefinite programming: A survey. Annual Reviews in Control 56, 100911.

Nejati, A., Zhong, B., Caccamo, M., Zamani, M., 2022. Data-driven controller synthesis of unknown nonlinear polynomial systems via control barrier certificates. In: Proceedings of the 4th Annual Learning for Dynamics and Control Conference 168, pp. 763–776.

Nortmann, B., Mylvaganam, T., 2023. Direct data-driven control of linear time-varying systems. IEEE Transactions on Automatic Control 68 (8), 4888–4895.

De Persis, C., Postoyan, R., Tesi, P., 2023a. Event-triggered control from data. IEEE Control Systems Letters 7, 2431–2436.

De Persis, C., Rotulo, M., Tesi, P., 2023b. Learning controllers from data via approximate nonlinearity cancellation. IEEE Transactions on Automatic Control 68 (10), 6082–6097.

De Persis, C., Tesi, P., 2020. Formulas for data-driven control: Stabilization, optimality, and robustness. IEEE Transactions on Automatic Control 65 (3), 909–924.

De Persis, C., Tesi, P., 2021. Low-complexity learning of linear quadratic regulators from noisy data. Automatica 128, 109548.

Petersen, I.R., 1987. A stabilization algorithm for a class of uncertain linear systems. Systems & Control Letters 8 (4), 351–357.

Rueda-Escobedo, J.G., Fridman, E., Schiffer, J., 2022. Data-driven control for linear discrete-time delay systems. IEEE Transactions on Automatic Control 67 (7), 3321–3336.

Scarpa, M.L., Pettersen, K. Y., Nortmann, B., Mylvaganam, T., 2022. Data-driven control of planar snake robot locomotion. In: Proceedings of the 61st IEEE Conference on Decision and Control (CDC), Cancun, Mexico, pp. 1823–1828.

Shafai, B., Moradmand, A., Siami, M., 2022. Data-driven positive stabilization of linear systems. In: Proceedings of the 2022 8th International Conference on Control, Decision and Information Technologies (CoDIT), Istanbul, Turkey, pp. 1031–1036.

Shen, Y., Bichuch, M., Mallada, E., 2022. Model-free learning of regions of attraction via recurrent sets. In: Proceedings of the 61st IEEE Conference on Decision and Control, Cancún, Mexico, pp. 4714–4719.
Stengle, G., 1974. A nullstellensatz and a positivstellensatz in semialgebraic geometry. Mathematische Annalen 207, 87–97.
Verhoek, C., Abbas, H.S., Tóth, R., 2023. Direct data-driven lpv control of nonlinear systems: An experimental result. IFAC PapersOnLine 56 (2), 2263–2268.
van Waarde, H.J., Eising, J., Trentelman, H.L., Camlibel, M.K., 2020. Data informativity: A new perspective on data-driven analysis and control. IEEE Transactions on Automatic Control 65 (11), 4753–4767.
Walsh, G., Ye, H., 2001. Scheduling of networked control systems. IEEE Control Systems Magazine 21 (1), 57–65.
Wang, M., Qiu, J., Yan, H., Tian, Y., Li, Z., 2023. Data-driven control for discrete-time piecewise affine systems. Automatica 155, 111168.
Werbos, P.J., 1989. Neural networks for control and system identification. In: Proceedings of the 28th IEEE Conference on Decision and Control, pp. 260–065.
Willems, J., Rapisarda, P., Markovsky, I., De Moor, B., 2005. A note on persistency of excitation. Systems & Control Letters 54 (4), 325–329.
Yang, Y., Vamvoudakis, K.G., Modares, H., 2020. Safe reinforcement learning for dynamical games. International Journal of Robust and Nonlinear Control 30 (9), 1049–8923.