Master of Science Thesis

Adaptive Synchronization over Uncertain Multi-Agent Systems

A distributed homogenization-based approach

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Delft Center for Systems and Control

Delft University of Technology

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The undersigned hereby certify that they have read and recommend to the Faculty of Mechanical, Maritime and Materials Engineering for acceptance a thesis entitled **"Adaptive Synchronization over Uncertain Multi-Agent Systems"** by **Ilario Antonio Azzollini** in partial fulfillment of the requirements for the degree of **Master of Science** in **Systems and Control**.

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Abstract

A challenging task in synchronization of multi-agent systems is steering the network towards a coherent solution when the dynamics of the constituent systems are heterogeneous and residing in a possibly large uncertainty set. In this situation, synchronization can be achieved via adaptive protocols (with adaptive feedback gains, or adaptive coupling gains, or both). However, as state-of-the-art synchronization methods adopt a distributed observer architecture, they require to communicate extra local observer variables among neighbors, in addition to the neighbors' states (or outputs). The distinguishing feature of this work is to show that synchronization in heterogeneous and uncertain multi-agent systems is possible without the need for any distributed observer. This can be achieved by designing adaptive distributed synchronization protocols, based on homogenization reasoning. Specifically, ideal gains are defined (feedback and coupling gains) that could lead all the heterogeneous agents to a desired homogeneous behavior and thus synchronization. However, since these gains are unknown in view of the unknown dynamics, we design adaptive laws for these gains that lead the agents toward synchronization. In this thesis, different control protocols are designed to address both leaderless and leader-follower synchronization of linear multi-agent systems. The protocols are then extended to achieve synchronization over a special class of nonlinear multi-agent systems, whose units are modeled as Kuramoto-like systems. Throughout this work, convergence of the synchronization error to zero is shown via Lyapunov analysis, and numerical examples demonstrate the effectiveness of the proposed protocols.

Acknowledgements

First of all, I would like to thank my supervisor dr. Simone Baldi for his guidance over the last year. I will never thank you enough for having pushed me since the beginning, and for giving me the possibility to work with Prof. Kosmatopoulos and then attending the European Control Conference to present our works. I also want to say that this year was really important for me, in order to decide what I wanted to do next. I am looking forward to the new adventure.

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Table of Contents

List of Figures

Acronyms

LTI Linear Time-Invariant

SISO Single-Input Single-Output

MIMO Multi-Input Multi-Output

MRAC Model Reference Adaptive Control

MRC Model Reference Control

SPR Strictly Positive Real

MKY Meyer-Kalman-Yakubovich

Chapter 1

Introduction

In this chapter, we first present the motivation of the research carried out in this thesis, driven by the fact that ubiquitous heterogeneity and the uncertainty make adaptive approaches very relevant in distributed control of multi-agent systems. Then, the research questions and main contributions of this work are given, and finally, the chapter is concluded with the outline of the thesis.

1-1 Motivation of the research

In general, cooperative control studies the dynamics of systems linked to each other by a communication graph. The graph represents the allowed information flow between the systems, sometimes also referred to as agents. The objective of cooperative control is to devise control protocols for the individual agents that guarantee a desired collective behavior for the whole network of agents. In cooperative systems, any control protocol must be *distributed* in the sense that it must respect the prescribed graph topology. That is, the control protocol for each agent is allowed to depend only on information about that agent and about its neighbors in the graph. The communication restrictions imposed by graph topologies can severely limit what can be accomplished by local distributed control protocols at each agent. In fact, the graph topological properties complicate the design of synchronization controllers and result in intriguing behaviors that do not occur in single-agent, centralized [Michailidis et al., 2017], or decentralized feedback control systems [Lewis et al., 2013].

In recent years, coordination of multi-agent systems has been studied by different scientific communities, motivated by its applicability to biology [Xie et al., 2016], energy systems [Meng and Li, 2017], autonomous vehicles [Harfouch et al., 2017], and many other fields. Because of the ubiquitous nature of cooperative control, there is no common agreement in the literature in terms of notation and terminology. In order to have a clear overview of all the notable research in this field, it is of interest to introduce the problems referred to as *cooperative output regulation*, *consensus* and *synchronization*. It will soon be clear that these three distinct problems are closely related to each other.

Cooperative output regulation: A central problem in control theory is that of controlling the output of a system in order to asymptotically track some prescribed reference signals while simultaneously rejecting potential exogenous disturbances. In particular, *output regulation* refers to the class of control problems that arise when these references and disturbances are generated as solutions of an autonomous dynamical system, known as the *exosystem*. As originally shown in [Francis and Wonham, 1976] for linear systems, if perfect tracking and/or rejection is sought in presence of uncertainties in the plant, then the closed-loop system must necessarily embed an internal model of the exogenous signals. The fundamentals of the so-called internal model-based design methods for both linear and nonlinear systems can be studied taking [Huang, 2004] and [Isidori et al., 2012] as reference. In the last years, a lot of effort has been put into extending these techniques in a distributed manner, i.e. the goal being to control the output of a multi-agent system to achieve a desired collective behavior such as asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances. This problem is referred to as the *cooperative output regulation problem*. A wide range of multi-agent coordination missions can be formulated as a cooperative output regulation problem [Su and Huang, 2012a]. The main idea behind cooperative output regulation is that the systems can be divided into two groups: the first group of systems can access the signals generated by the exosystem, while the second cannot. As a result, the regulation problem cannot be solved by the decentralized approach, and distributed (or cooperative) control techniques must be devised.

Synchronization and consensus: In [Li et al., 2010] it was shown that, even if there are historical differences, synchronization and consensus can be treated in a unified way. When multiple systems agree on the value of a variable of interest, they are said to have reached consensus (on that variable). When this variable of interest is their state (or output), this network of systems is said to be synchronized. Therefore, at least from a control perspective, the terms consensus and synchronization can be used interchangeably. Literature has distinguished among two types of synchronization (or consensus): *leaderless*, in which synchronization towards the same evolution which is unknown a priori emerges from the negotiation process taking place on the network [Isidori et al., 2014, Azzollini et al., 2018, Baldi et al., 2018a]; and *leader-follower*, in which the network is steered to some desired and a priori known solution using a limited set of leader nodes [Wieland et al., 2011, Abdessameud et al., 2017, Baldi, 2018]. Considering the latter, if a single autonomous leader is present, connected only to some agents of the network that can directly measure its signals, the setting is the same as the one of cooperative output regulation. An established way to solve the synchronization problem is indeed to formulate it in a cooperative output regulation framework. As a matter of fact, in [Wieland et al., 2011], it was shown that an internal model requirement is necessary and sufficient for synchronizability of a network to the behavior of an autonomous exogenous system, denoted as exosystem. This means that the well-known internal model principle [Francis and Wonham, 1976] can be used to solve synchronization problems. Motivated by this result, synchronization protocols were designed for both linear [Seyboth et al., 2015, Ding, 2017] and nonlinear networks [Isidori et al., 2014].

After this first overview it can be concluded that when a leader is present, *in terms of the goal to be achieved*, the output synchronization problem, the consensus on outputs problem, and the cooperative output regulation problem, are closely related to each other. On the other hand, one clear distinction can be found *in the way each problem is solved*. Recall that the problem is that of making a network of systems (sometimes referred to as follower agents) to follow the behavior of a leader exosystem (sometimes referred to as leader agent). For each system, a local controller must be designed, such that the control action is computed only using own information and neighbors information (i.e. the control must be distributed). In cooperative output regulation, the classical output regulation theory is always used, meaning that the local controller for each system in the network is made of a stabilizer (feedback action) and a regulator (feedforward action), computed by using the well-known internal model-based techniques [Huang, 2004, Isidori et al., 2012]. Cooperation arises from solving the problem in a distributed way when not all systems in the network can access the signals of the leader, which is always the case in real-life applications. The main idea is that the systems not directly connected to the exosystem reconstruct the exosystem signals through communication with neighbors, in particular, through the so-called *distributed observer* [Su and Huang, 2012a, Su and Huang, 2012b]. This extra dynamical system is able to "decentralize" the problem, so that the classical single-agent control theory can be used locally for each agent. It has to be noticed that the use of the distributed observer, always entails the communication of extra auxiliary variables (i.e. the state variables of the distributed observer). Although the distributed observer originates from the cooperative output regulation literature, it is an established way of solving the synchronization problem over heterogeneous multi-agent systems [Abdessameud et al., 2014, Feng et al., 2016].

The main motivation for using adaptive control over networks is that the presence of heterogeneity and parametric uncertainty becomes even more relevant when dealing with multi-agent systems made of many and many agents. This need has been already recognized in the literature of pinning control, when dealing with the so called "complex networks" (at least hundreds of agents) [Turci et al., 2014]. The potential of adaptive control, together with the belief that the use of the distributed observer should not be necessary, even when dealing with heterogeneous multi-agent systems, is what drove our research. In particular, in this thesis, we focus our attention on synchronization: (i) both in case of full-state measurement and output measurement, and (ii) both leaderless and leader-follower scenarios.

1-2 State of the art

Initial research on synchronization focused on systems sharing the same (homogeneous) dynamics, linear or nonlinear, possibly with parametric uncertainty. Synchronization of these homogeneous networks has been achieved by adopting either adaptive coupling gains [Li et al., 2013b, Li et al., 2013a, Shafi and Arcak, 2015], or adaptive feedback gains [Li and Ding, 2015, Ding and Li, 2016, Zhou et al., 2008], respectively. In the first case, one exploits the fact that synchronization in stable homogeneous networks can be achieved if the coupling strength is large enough [Wang et al., 2016, Yu et al., 2012]: therefore, adaptive coupling laws, also known as adaptive dynamic protocols, simply increase the coupling strength according to the synchronization error [Li et al., 2013b, Li et al., 2013a, Shafi and Arcak, 2015]. In the second case, static couplings have been used, while a stabilizing feedback gain has been determined in an adaptive way for special classes of homogeneous uncertain systems [Li and Ding, 2015, Ding and Li, 2016, Zhou et al., 2008, Yu et al., 2009]. A more challenging task is that of achieving synchronization when the systems of the network differ from each other, and also their dynamics lie in a possibly large uncertainty set (heterogeneous and uncertain networks). To tackle heterogeneous systems, the main idea is to "cancel" the effect of the heterogeneity through an appropriate design of feedback and/or coupling gains. In the presence of large uncertainty, this cancellation should be adaptive. Adaptive feedback strategies have been mostly explored, namely for unknown linear systems [Gibson, 2016], chaotic systems [Fradkov et al., 2008], systems with unknown identical control directions [Chen et al., 2014], unknown systems in model reference form [Baldi and Frasca, 2018,Baldi et al., 2018c], unknown systems in specific platooning protocols [Harfouch et al., 2017], systems in the Euler-Lagrange form [Abdessameud et al., 2014,Mei et al., 2015,Abdessameud et al., 2017,Feng et al., 2016]: a notable exception is [Ghapani et al., 2016], where a discontinuous protocol with both adaptive feedback and adaptive couplings is implemented. However, differently from some homogeneous approaches that might not require a distributed observer [Li and Ding, 2015, Ding and Li, 2016], all heterogeneous approaches share the need for implementing some form of distributed observer, thus requiring communication of extra variables to reconstruct the leader information. Therefore, relevant questions arise: *what is the simplest distributed adaptive architecture for synchronization of heterogeneous uncertain networks? More specifically, is it possible to get rid of any distributed observer, and reach synchronization by adapting both the feedback and the coupling gains with no further local communication than the neighbors' states (or outputs)?* In this work, we give an answer to these questions.

A substantial part of the synchronization literature treats the study of networks of phase oscillators. The main challenges arise from the fact that the oscillators have nonlinear and coupled dynamics (i.e. neighbors influence each other). In this thesis, in particular, we study the special class of coupled-phase nonlinear oscillators known as Kuramoto oscillators. In the 80's, Kuramoto proposed an exactly solvable model of collective synchronization, which became known as the Kuramoto model [Kuramoto, 1984]. This model has been shown to capture various synchronization phenomena in biological and man-made dynamical systems of coupled oscillators, spanning from flocks of birds and schools of fishes [Couzin et al., 2005], blinks in groups of fireflies [Strogatz, 2003], the utility power grid [Bergen and Hill, 1981], to countless other synchronization phenomena [Okaniwa and Ishii, 2012].

Synchronization research has been first focusing on non-evolving (or non-adaptive) networks of phase oscillators (see [Dorfler and Bullo, 2014] and references therein): it was found that synchronization can emerge in the presence of simple static coupling where neighboring nodes adjust their dynamics proportionally to the mismatch between some output function of their states [Jadbabaie et al., 2003, Olfati-Saber et al., 2007, Wang and Chen, 2002]. Most models have shown that synchronization is favored if the coupling strength is large enough and the spectrum of variety of the oscillators is narrow [Strogatz, 2003]. In this spirit, [Dorfler et al., 2013] provided a threshold of the couplings that brings from incoherence to synchrony: synchronization occurs when the coupling strength dominates the worst-case dissimilarity over the network. However, as we know, real-world networks have uncertain and heterogeneous parameters which might even change with time. Therefore, in place of static couplings, researchers have later been focusing on networks characterized by evolving, adapting couplings which vary in time according to different environmental conditions, leading to the study of evolving (or adaptive) networks [Gross and Blasius, 2008]. In [De Lellis et al., 2008] a set of adaptive (centralized or decentralized) strategies for synchronization and consensus of complex networks is presented: the main limitation of these approaches is that they address networks composed of identical oscillators. On the other hand, it is challenging to consider heterogeneous and uncertain networks of nonlinear oscillators. Indeed, over the last years this has been an hot topic, giving rise to several works [Ren et al., 2014, Ha et al., 2016, Papadopoulos et al., 2017]. Considering the special class of Kuramoto networks, the relevant question we will answer in this work is the following: *Is it possible to design an homogenization-based distributed adaptive protocol for synchronization of heterogeneous uncertain Kuramoto networks? Compared to state-of-the-art approaches based on a distributed observer, is it possible to obtain a lighter architecture?*

1-3 Main contributions

The research goal of this thesis consists in developing novel adaptive control schemes in order to handle heterogeneity and uncertainty in multi-agent systems, while keeping the communication architecture as light and simple as possible. The main contributions are listed in the following:

- **Distributed synchronization of heterogeneous uncertain linear multi-agent systems based on adaptive homogenization reasoning**
  Starting from the fact that synchronization of homogeneous networks is well-known in the literature, we develop novel adaptive distributed protocols for achieving synchronization over heterogeneous and uncertain networks via homogenization. Leaderless and leader-follower synchronization are studied both for the full-state measurement and for the output measurement case.

- **Lyapunov-based adaptation of both the feedback and coupling gains without the need for any distributed observer**
  All the protocols presented in this thesis attain synchronization without the need for a distributed observer. In fact, we succeed in adapting both the feedback and coupling gains via continuous protocols without any extra local communication other than neighbors' states (in the full-state information case) or neighbors' outputs (in the partial-state information case), thus resulting in the simplest communication architecture. The design behind all the proposed protocols is derived based on Lyapunov analysis.

- **Nonlinear extension over networks with heterogeneous uncertain Kuramoto-like units**
  We analyze synchronization capabilities in Kuramoto-like networks whose dynamical features are heterogeneous and unknown and thus synchronization protocols must exhibit co-evolution capabilities that counteract the effect of heterogeneity. In particular, the developed protocols implement a sort of "adaptive feedback linearization", comprising two actions: (i) adaptive estimation and cancellation of the nonlinearities, and (ii) adaptive homogenization to certain desired (linear) homogeneous dynamics.

1-4 Thesis outline

The rest of this thesis is organized as explained in the following, and a schematic outline is provided in Figure 1-1.

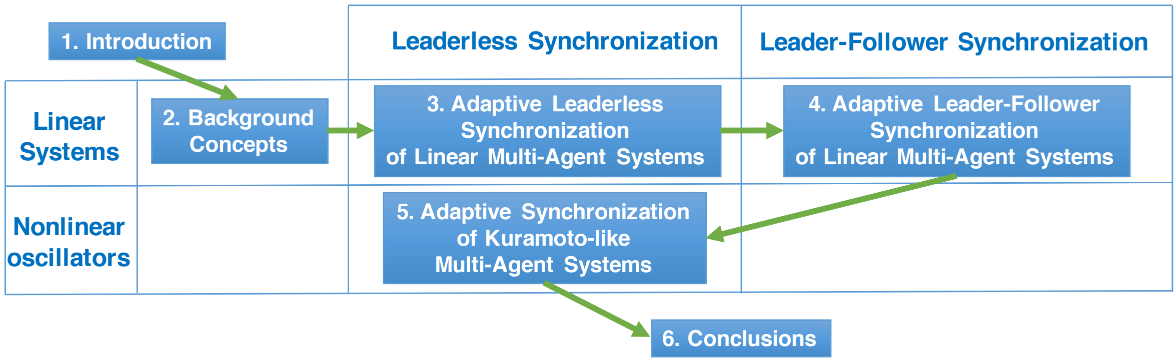

Figure 1-1: Schematic outline of this thesis.

Chapter 2 provides all the needed background theory in order to make this thesis as self-contained as possible. First, the *model reference adaptive control* is treated, then a brief tutorial is presented to explain the needed concepts about *graph theory* and *consensus*. Moreover, in this chapter we present the well-known results on (non-adaptive) *synchronization of homogeneous linear multi-agent systems*, and finally the *distributed observer* architecture.

In Chapter 3 the problem of *leaderless synchronization* of linear (heterogeneous and uncertain) multi-agent systems is solved through the development of adaptive distributed homogenization-based protocols.

In Chapter 4 the synchronization problem of linear (heterogeneous and uncertain) multi-agent systems is solved when *at least one leader* is present in the network.

Chapter 5 deals with extending the leaderless adaptive synchronization results to heterogeneous *Kuramoto-like multi-agent systems* with uncertain dynamics.

Chapter 6 finally draws the main *conclusions* of the work and presents some ideas for *future developments*.

Chapter 2

Background concepts

The purpose of this chapter is to make this thesis as self-contained as possible by explaining the basic theory this work is built upon. First of all, let us introduce the notation.

Notation: A vector signal $x(\cdot)$ is said to belong to $\mathcal{L}_2$ ($x \in \mathcal{L}_2$), if $\int_0^t \|x(\tau)\|^2 \,\mathrm{d}\tau < \infty$, $\forall t \geq 0$. A vector signal $x(\cdot)$ is said to belong to $\mathcal{L}_\infty$ ($x \in \mathcal{L}_\infty$), if $\max_{t\geq 0} \|x(t)\| < \infty$, $\forall t \geq 0$. The transpose of a matrix or of a vector is indicated with X^T and x^T, respectively. The $n \times n$ identity matrix is denoted by I_n. If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

where a_{ij} are the entries of matrix A. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be negative definite if, for every non-zero vector $x \in \mathbb{R}^n$, it results $x^T M x < 0$. The sign function of a real number $r \in \mathbb{R}$ is defined as $sgn(r) = 1$ if $r > 0$, $sgn(r) = 0$ if $r = 0$, and $sgn(r) = -1$ if $r < 0$. If the matrix $\Lambda = [\lambda_{ij}]$ is a square matrix $\in \mathbb{R}^{N\times N}$, then the entries λ_{ii} are called diagonal entries. A square matrix is called diagonal if all non-diagonal entries are zero, and can be denoted by means of the diagonal operator $\Lambda = \mathrm{diag}(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{NN})$. The all-ones N-vector is defined as $\mathbf{1}_N = \mathrm{col}(1, 1, \ldots, 1)$. In the same way we define the all-zeros $(N \times n)$-vector $\mathbf{0}_{Nn} = \mathrm{col}(0, 0, \ldots, 0)$.

2-1 Model Reference Adaptive Control for SISO linear systems

Model Reference Adaptive Control (MRAC) is one of the main approaches to adaptive control. The control architecture of the *direct* MRAC approach is shown in Figure 2-1. The *reference model* is designed by the user to generate the desired trajectory y_m, which is in general a filtered version of r. The goal is to make the output y_p of the *plant* to follow the output of the reference model, that means we want the tracking error $e_1 = y_p - y_m$ to go to zero. The plant is

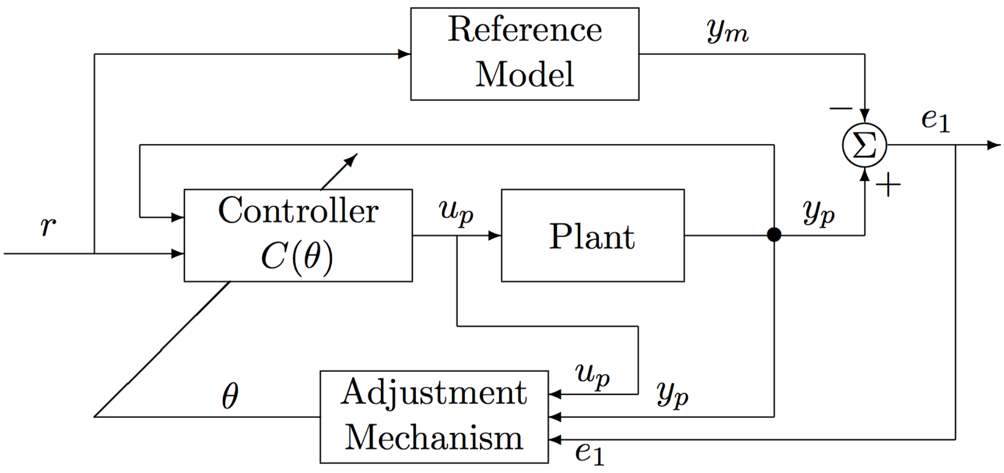

Figure 2-1: Structure of direct MRAC scheme [Ioannou and Sun, 2012].

assumed to be a linear system whose parameters are unknown. The closed-loop plant is made of an ordinary feedback control law that contains the plant and a parameterized *controller* $C(\theta)$ and an *adjustment mechanism* that generates the controller parameter estimates $\theta(t)$ on-line (in view of the unknown parameters of the plant). This is called a *direct* adaptive approach as we want to directly estimate the controller parameters θ, without estimating the unknown plant parameters. In this scenario, the reference model basically describes the input-output properties we want for the closed-loop system.

In the following, the basics of model reference control will be explained. For a more in depth analysis the interested reader is referred to [Ioannou and Sun, 2012, Chapter 6].

2-1-1 The model reference control problem with known plant

Let us start with explaining the reference model following problem, or Model Reference Control (MRC), that is the non-adaptive version of MRAC, as shown in Figure 2-2. In this

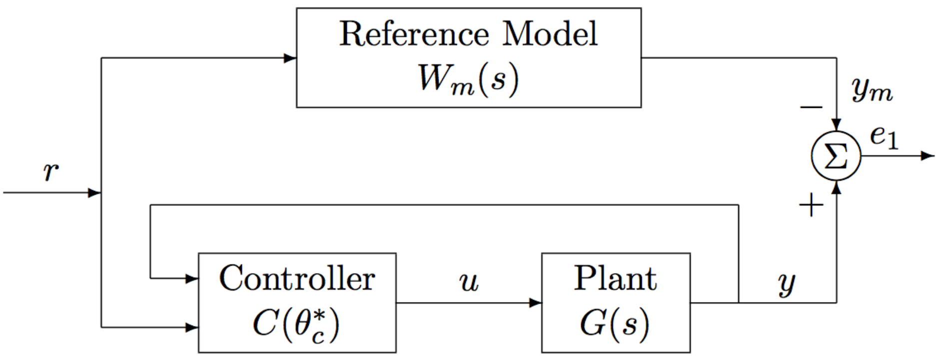

Figure 2-2: Structure of MRC scheme [Ioannou and Sun, 2012].

scenario, we have a good understanding of the plant and the adjustment mechanism is not needed: the resulting controller will not need to embed any adaptive laws.

Full-state measurement case

Let us consider the n^{th} order Linear Time-Invariant (LTI) single-input[1] plant

$$\dot{x} = Ax + bu, \quad x(0) = x_0 \tag{2-1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, and as a consequence $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. From here till the end of the thesis, the time index t will be omitted when obvious. The pair (A, b) is controllable and perfectly known. In addition, the state x is measurable and available for control.

The control objective is to find a control law u such that all the signals in the closed-loop plant are bounded and the plant state x follows the state of the reference model given by

$$\dot{x}_m = A_m x_m + b_m r, \quad x_m(0) = x_{m0} \tag{2-2}$$

where $x_m \in \mathbb{R}^n$ is the state, $r \in \mathbb{R}$ is the reference signal, and therefore $A_m \in \mathbb{R}^{n \times n}$ and $b_m \in \mathbb{R}^n$. The matrix A_m is a Hurwitx matrix, and r is bounded.

The proposed control law is

$$u = -k^{*T} x + l^* r \tag{2-3}$$

where $k^* \in \mathbb{R}^n$ and $l^* \in \mathbb{R}$, that leads to the closed-loop system

$$\dot{x} = (A - bk^{*T})x + bl^* r. \tag{2-4}$$

Now, if we choose the feedback and feedforward gains k^* and l^* such that the following *matching conditions* hold

$$\begin{cases} A - bk^{*T} = A_m \\ bl^* = b_m \end{cases} \tag{2-5}$$

then we have that $x(t) - x_m(t) \to 0$ exponentially fast.

Remark 2-1-1: [The structural issue] *Given matrices A, b, A_m, b_m with arbitrary structure, no k^*, l^* may exist to satisfy the matching conditions* (2-5)*. From a practical point of view this means that the problem of choosing the reference model is twofold. Primarily, we choose the reference model corresponding to the desired performance (input-output behavior) we want for our closed-loop, but, A_m, b_m should also be designed such that the matching conditions* (2-5) *admit solution.*

Output measurement case

In this case we consider the LTI SISO plant model as follows

$$\begin{aligned} \dot{x} &= Ax + bu, \quad x(0) = x_0 \\ y &= c^T x \end{aligned} \tag{2-6}$$

[1]As common in adaptive literautre, we will focus on the Single-Input Single-Output (SISO) case. For extension to Multi-Input Multi-Output (MIMO), the interested reader is referred to [Tao, 2014].

where $x \in \mathbb{R}^n$ and A, b, c have appropriate dimensions. Now, only the output y is measurable and available for control. The transfer function representation of the system is

$$y = G(s)u = k_p \frac{Z_p(s)}{R_p(s)} u \tag{2-7}$$

where Z_p, R_p are monic polynomials and k_p is a constant referred to as the *high frequency gain*.

The reference model is

$$\begin{aligned} \dot{x}_m &= A_m x_m + b_m r, \qquad x_m(0) = x_{m0} \\ y_m &= c_m^T x_m \end{aligned} \tag{2-8}$$

where $x_m \in \mathbb{R}^{p_m}$ and $y_m, r \in \mathbb{R}$. Its transfer function representation is

$$W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} = \frac{y_m(s)}{r(s)} \tag{2-9}$$

where Z_m, R_m are monic Hurwitz polynomials.

The control objective is to determine the control law u so that all signals in the closed-loop are bounded and the plant output y tracks the reference model output y_m as close as possible for *any* given reference input $r(t)$ that is a uniformly bounded and piecewise continuous function of time.

The following assumptions are made.

Plant assumptions:

P1 $Z_p(s)$ (numerator) is a known monic Hurwitz polynomial of degree m;

P2 $R_p(s)$ (denominator) is a known monic polynomial of degree n;

P3 the relative degree $n^* = n - m$ of $G(s)$ is known;

P4 the high frequency gain k_p is also known.

Reference model assumptions:

M1 $Z_m(s), R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n_p$;

M2 the relative degree of $W_m(s)$ is $n_m^* = p_m - q_m = n^*$.

The control objective is again to have an input-output behavior of the controlled system that reflects that of the chosen reference model (i.e. the same transfer function from r to the output), therefore guaranteeing that $y(t) - y_m(t) \to 0$.

A trivial open-loop solution would be

$$u = C(s)r, \qquad C(s) = \frac{k_m}{k_p} \frac{Z_m(s)}{R_m(s)} \frac{R_p(s)}{Z_p(s)} \tag{2-10}$$

leading to the closed-loop system

$$\frac{y(s)}{r(s)} = \frac{k_m}{k_p}\frac{Z_m}{R_m}\frac{R_p}{Z_p}k_p\frac{Z_p}{R_p} = W_m(s). \tag{2-11}$$

This control law is feasible only when $R_p(s)$ is Hurwitz, otherwise may involve zero-pole cancellations outside $\mathcal{C}^-$. In addition, it suffers from the usual drawbacks of open-loop control.

The proposed feedback control law is

$$u = \theta_1^{*T}\frac{\alpha(s)}{\Lambda(s)}u + \theta_2^{*T}\frac{\alpha(s)}{\Lambda(s)}y + \theta_3^* y + c_0^* r \tag{2-12}$$

where

- $\begin{cases} \alpha(s) \triangleq [s^{n-2}, s^{n-3}, \ldots, s, 1] & \text{for } n \geq 2 \\ \alpha(s) \triangleq 0 & \text{for } n = 1 \end{cases}$;
- $c_0^*, \theta_3^* \in \mathbb{R}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{n-1}$ are constant parameters to be designed;
- $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$ that contains Z_m as a factor

  $$\Lambda(s) = \Lambda_0(s)Z_m(s) \tag{2-13}$$

  where, as a consequence, Λ_0 is monic, Hurwitz and of degree $n_0 = n - 1 - q_m$.

A schematic representation of the closed-loop is given in Figure 2-3, which results in

$$G_c(s) = \frac{c_0^* k_p Z_p \Lambda_0 Z_m}{[(\Lambda - \theta_1^{*T}\alpha)R_p - k_p Z_p(\theta_2^{*T}\alpha + \theta_3^*\Lambda)]} = \frac{y(s)}{r(s)}. \tag{2-14}$$

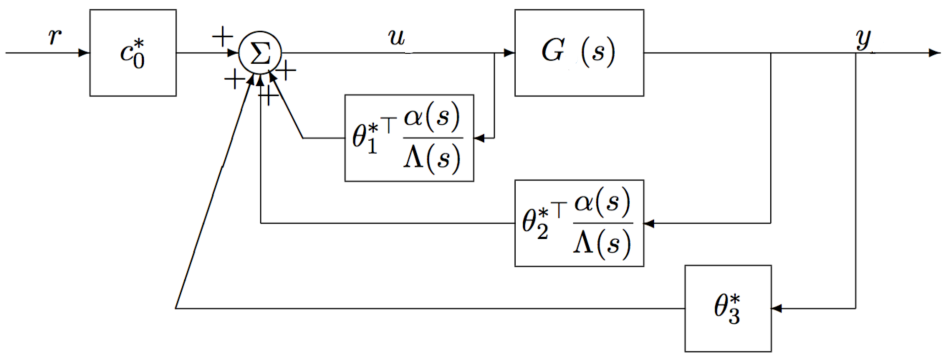

Figure 2-3: MRC architecture using controller (2-12) [Ioannou and Sun, 2012].

This means that the controller parameters in (2-12) must be chosen such that the following equality holds

$$G_c(s) = W_m(s) \Rightarrow \frac{c_0^* k_p Z_p \Lambda^2}{\Lambda[(\Lambda - \theta_1^{*T}\alpha)R_p - k_p Z_p(\theta_2^{*T}\alpha + \theta_3^*\Lambda)]} = k_m\frac{Z_m}{R_m} \tag{2-15}$$

that can be simplified using $\Lambda(s) = \Lambda_0(s)Z_m(s)$, resulting in

$$\frac{c_0^* k_p Z_p \Lambda_0 Z_m}{[(\Lambda - \theta_1^{*T}\alpha)R_p - k_p Z_p(\theta_2^{*T}\alpha + \theta_3^*\Lambda)]} = k_m \frac{Z_m}{R_m} \tag{2-16}$$

and finally leading to the more compact *matching conditions*

$$\begin{cases} c_0^* = \dfrac{k_m}{k_p} \\ (\Lambda - \theta_1^{*T}\alpha)R_p - k_p Z_p(\theta_2^{*T}\alpha + \theta_3^*\Lambda) = Z_p \Lambda_0 R_m. \end{cases} \tag{2-17}$$

Lemma 2-1-1: [Ioannou and Sun, 2012, Lemma 6.3.1] *Let the degrees of $R_p, Z_p, \Lambda, \Lambda_0$ be as specified in assumptions P1, P2, M1. Then the solution to the matching conditions* (2-17) *always exists.*

Remark 2-1-2: [Relation with pole placement control] *MRC can be viewed as a special case of a general pole placement scheme where the desired closed-loop characteristic equation is given by*

$$Z_p(s)\Lambda_0(s)R_m(s) = 0 \tag{2-18}$$

plus, the high frequency gain is changed from k_p to k_m (by using c_0^*).

For completeness, let us show a state-space realization of controller (2-12), which is given by

$$\begin{aligned} \dot{\omega}_1 &= F\omega_1 + gu, \quad && \omega_1(0) = 0 \\ \dot{\omega}_2 &= F\omega_2 + gy, \quad && \omega_2(0) = 0 \\ u &= \theta^{*T}\omega \end{aligned} \tag{2-19}$$

where $\omega_1, \omega_2 \in \mathbb{R}^{n-1}$, $\theta^* = \left[\theta_1^{*T}, \theta_2^{*T}, \theta_3^*, c_0^*\right]^T$, $\omega = \left[\omega_1^T, \omega_2^T, y, r\right]^T$,

$$F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & -\lambda_{n-4} & \dots & -\lambda_0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2-20}$$

where

$$\Lambda(s) = s^{n-1} + \lambda_{n-2}s^{n-2} + \ldots + \lambda_1 s + \lambda_0 = \det(sI - F) \tag{2-21}$$

so (F, g) is the state-space realization of $\alpha(s)/\Lambda(s)$, i.e. $(sI - F)^{-1}g = \alpha(s)/\Lambda(s)$.

2-1-2 The model reference control problem with unknown plant

The control gains we found for solving the MRC problem, both in the full-state measurement case and in the output measurement case, were chosen such that some matching conditions were satisfied. It should be clear that the values of these gains are *ideal*, as they rely on the perfect knowledge of the plant dynamics. The challenge now is to solve the same problem as before, while assuming the plant dynamics to be unknown. This can be done by properly designing adaptive laws to estimate the ideal control gains.

Full-state measurement case

Let us consider the same problem as in the full-state measurement case of the previous section, but now, we assume the pair (A, b) to be controllable and *unknown*.

Assuming that k^*, l^* in (2-3) exist, that satisfy the matching conditions (2-5) (i.e. that there is sufficient structural flexibility to meet the control objective), the proposed control law is

$$u(t) = -k^T(t)x + l(t)r \tag{2-22}$$

where $k(t), l(t)$ are the estimates of k^*, l^*, respectively, to be generated by appropriate adaptive laws. Also, it is assumed that we know $sgn(l^*)$.

Remark 2-1-3: [Structural flexibility is still a requirement] *Please notice that structural flexibility is needed even in this scenario in which we do not know A and b. If the ideal gains do not exist, the problem cannot be solved. The output-measurement formulation of MRAC will not suffer from this issue.*

The proposed *adaptive solution* is

$$\begin{aligned} \dot{k}^T &= +sgn(l^*)\gamma_1 e^T P b_m x^T \\ \dot{l} &= -sgn(l^*)\gamma_2 e^T P b_m r \end{aligned} \tag{2-23}$$

where $\gamma_1 > 0, \gamma_2 > 0$ are the adaptive gains (parameters to be tuned), and $e = x - x_m$ is the *tracking error*. Finally, $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix that satisfies the Lyapunov equation

$$PA_m + A_m^T P = -Q \tag{2-24}$$

for some $Q = Q^T > 0 \in \mathbb{R}^{n \times n}$.

In order to carry out the stability analysis, let us consider the candidate Lyapunov function

$$V = \underbrace{e^T P e}_{V_1} + \underbrace{\frac{\tilde{k}^T \tilde{k}}{\gamma_1 |l^*|}}_{V_2} + \underbrace{\frac{\tilde{l}^2}{\gamma_2 |l^*|}}_{V_3} \tag{2-25}$$

where

$$\begin{aligned} \tilde{k}(t) &= k(t) - k^* \\ \tilde{l}(t) &= l(t) - l^* \end{aligned} \tag{2-26}$$

are the *estimation errors*.

First of all, the closed-loop system is

$$\begin{aligned} \dot{x} &= Ax + b(-k^T x + lr) \\ &= Ax - bk^T x + blr \end{aligned} \tag{2-27}$$

that can be written as a function of the estimation errors as

$$\begin{aligned} \dot{x} &= Ax - b\tilde{k}^T x - bk^{*T} x + b\tilde{l}r + bl^* r \\ &= \underbrace{(A - bk^{*T})}_{A_m} x + \underbrace{bl^*}_{b_m} r - b\tilde{k}^T x + b\tilde{l}r. \end{aligned} \tag{2-28}$$

The tracking error dynamics result in

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{x}_m \\ &= \left(A_m x + b_m r - b\tilde{k}^T x + b\tilde{l}r\right) - \left(A_m x_m + b_m r\right) \\ &= A_m \underbrace{(x - x_m)}_{e} - b\tilde{k}^T x + b\tilde{l}r \\ &= A_m e - b\tilde{k}^T x + b\tilde{l}r.\end{aligned} \tag{2-29}$$

Now that we have written the error dynamics in a convenient form, we are ready to proceed with the Lyapunov analysis. We have

$$\begin{aligned}\dot{V} &= \underbrace{(2e^T P\dot{e})}_{\dot{V}_1} + \underbrace{\left(\frac{2\tilde{k}^T}{\gamma_1|l^*|}\dot{\tilde{k}}\right)}_{\dot{V}_2} + \underbrace{\left(\frac{2\tilde{l}}{\gamma_2|l^*|}\dot{\tilde{l}}\right)}_{\dot{V}_3} \\ &= \left(2e^T PA_m e - 2e^T Pb\tilde{k}^T x + 2e^T Pb\tilde{l}r\right) + \left(\frac{2\tilde{k}^T}{\gamma_1|l^*|}\dot{\tilde{k}}\right) + \left(\frac{2\tilde{l}}{\gamma_2|l^*|}\dot{\tilde{l}}\right) \\ &= \left(e^T PA_m e + e^T A_m^T Pe - 2e^T Pb\tilde{k}^T x + 2e^T Pb\tilde{l}r\right) + \left(\frac{2\tilde{k}^T}{\gamma_1|l^*|}\dot{\tilde{k}}\right) + \left(\frac{2\tilde{l}}{\gamma_2|l^*|}\dot{\tilde{l}}\right) \\ &= \left(-e^T Qe \underbrace{-2e^T Pb\tilde{k}^T x + 2e^T Pb\tilde{l}r}_{\text{non-definite terms}}\right) + \left(\frac{2\tilde{k}^T}{\gamma_1|l^*|}\dot{\tilde{k}}\right) + \left(\frac{2\tilde{l}}{\gamma_2|l^*|}\dot{\tilde{l}}\right).\end{aligned} \tag{2-30}$$

Analyzing (2-30) we can notice that we have a potentially good negative definite term $-e^T Qe$, and some non-definite terms that we would like to cancel by properly designing the adaptive laws. In this way, we could make the Lyapunov function derivative negative *semi-definite* (as $V = V(e, \tilde{k}, \tilde{l})$). Exploiting the fact that $\dot{\tilde{k}} = \dot{k}$ and $\dot{\tilde{l}} = \dot{l}$ (as k^* and l^* are constant), and using the proposed adaptive laws (2-23), we have

$$\dot{V}_2 = \frac{2\tilde{k}^T}{\gamma_1|l^*|}\dot{\tilde{k}} = 2\tilde{k}^T x \underbrace{\frac{sgn(l^*)}{|l^*|} b_m^T}_{b^T} Pe = 2 \underbrace{(\tilde{k}^T x)}_{\text{scalar}} \underbrace{(b^T Pe)}_{\text{scalar}} \tag{2-31}$$

that cancels the first part of the non-definite terms in (2-30), and in the same way

$$\dot{V}_3 = \frac{2\tilde{l}}{\gamma_2|l^*|}\dot{\tilde{l}} = -2\tilde{l}e^T P \underbrace{\frac{sgn(l^*)}{|l^*|} b_m}_{b} r = -2e^T Pb\tilde{l}r \tag{2-32}$$

that cancels the second part of the non-definite terms in (2-30), leading to

$$\dot{V}(e, \tilde{k}, \tilde{l}) = -e^T Qe \tag{2-33}$$

which is negative semi-definite as desired. This is a typical result in adaptive control and our goal is now to continue the analysis along the lines of Barbalat's Lemma, so as to show convergence of the error to zero, even if we cannot conclude asymptotic stability of the equilibrium $(e, \tilde{k}, \tilde{l}) = (0, 0, 0)$.

It follows from (2-25) and (2-33) that V is a Lyapunov function for the error system

$$\begin{aligned} \dot{e} &= A_m e - b\tilde{k}^T x + b\tilde{l}r \\ \dot{\tilde{k}}^T = \dot{k}^T &= +sgn(l^*)\gamma_1 e^T P b_m x^T \\ \dot{\tilde{l}} = \dot{l} &= -sgn(l^*)\gamma_2 e^T P b_m r \end{aligned} \tag{2-34}$$

and we have that $e, \tilde{k}, \tilde{l} \in \mathcal{L}_\infty$ and $e \in \mathcal{L}_2$. As $e = x - x_m$ and $x_m \in \mathcal{L}_\infty$, we also have $x \in \mathcal{L}_\infty$ and $u \in \mathcal{L}_\infty$. Therefore all signals in the closed-loop are bounded. Now, from the first equation in (2-34) we have $\dot{e} \in \mathcal{L}_\infty$, which, together with $e \in \mathcal{L}_2$, implies that $e(t) \to 0$ as $t \to \infty$ [Ioannou and Sun, 2012, Lemma 3.2.6].

Remark 2-1-4: [Convergence to the ideal gains] *From the analysis above, it should be clear that we cannot guarantee $\tilde{k} \to 0$ and $\tilde{l} \to 0$, even if $e \to 0$. This is a typical result in direct adaptive control approaches. We have $\tilde{k} \to 0$ and $\tilde{l} \to 0$ only if the reference signal r is sufficiently rich (persistency of excitation condition).*

Output measurement case for plants with unitary relative degree

Let us consider the same problem as in the output measurement case of the previous section, but now with the following assumptions.

Plant assumptions:

P1.A $Z_p(s)$ (numerator) is an **unknown** monic Hurwitz polynomial of known degree m;

P2.A $R_p(s)$ (denominator) is an **unknown** monic polynomial of known degree n;

P3.A The relative degree $n^* = n - m$ of $G(s)$ **is** $= \mathbf{1}$;

P4.A The high frequency gain k_p is **unknown but with known sign**.

Reference model assumptions:

M1.A $Z_m(s), R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n_p$;

M2.A The relative degree of $W(s)$ is $n_m^* = p_m - q_m = \mathbf{n}^* = \mathbf{1}$;

M3.A In addition, $W_m(s)$ is designed to be **Strictly Positive Real (SPR)** [Ioannou and Sun, 2012, Lemma 3.5.4].

Because the parameters of the plant are unknown, the ideal controller parameter vector θ^* in (2-19) cannot be calculated from the matching equations (2-17). Let us propose the control law

$$u_p = \theta^T(t)\omega \tag{2-35}$$

where $\theta(t)$ is the estimate of θ^*, to be generated by an appropriate adaptive law.

The *adaptive solution* is

$$\begin{cases} \dot{\omega}_1 & = F\omega_1 + gu, \quad \omega_1(0) = 0 \\ \dot{\omega}_2 & = F\omega_2 + gy, \quad \omega_2(0) = 0 \\ \dot{\theta} & = -sgn(c_0^*)\gamma\epsilon\omega \end{cases} \tag{2-36}$$

where $\gamma > 0$ is the adaptive gain (parameter to be tuned), and $\epsilon = y - y_m$ is the tracking error. Please notice that the assumption of knowing $sgn(c_0^*)$ is equivalent to assumption P4.A (because of the matching conditions (2-17)). This solution guarantees that all signals in the closed-loop are bounded and the tracking error $\epsilon \to 0$ as $t \to \infty$ for any reference input $r \in \mathcal{L}_\infty$. Moreover, if r is sufficiently rich of order $2n$, we also have $\theta(t) \to \theta^*$. The proof is not shown as it is not crucial for understanding the results presented in this thesis, but the interested reader is referred to [Ioannou and Sun, 2012, Section 6.4.1].

2-2 Brief tutorial on graph theory

We consider networks of dynamical systems (also referred to as agents or nodes), which are linked to each other via a *communication graph*, that describes the allowed information flow. We say that system i has a *directed* connection to system j if the second can receive information from the first. When the information can flow in both directions, the connection is said to be *undirected*. In a communication graph, a special role is played by the *leader* node, which is a system (typically indicated as system 0) that does not receive information from any other system in the network. The communication graph describing the allowed information flow between all the systems, *leader excluded*, is completely defined by the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ is a finite nonempty set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of pairs of nodes, called edges. To include the presence of the leader in the network we define $\bar{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}, \mathcal{T}\}$, where $\mathcal{T} \subseteq \mathcal{V}$ is the set of those nodes, called *target nodes*, which receive information from the leader. Figure 2-4 provides a simple schematic example of communication graph, where

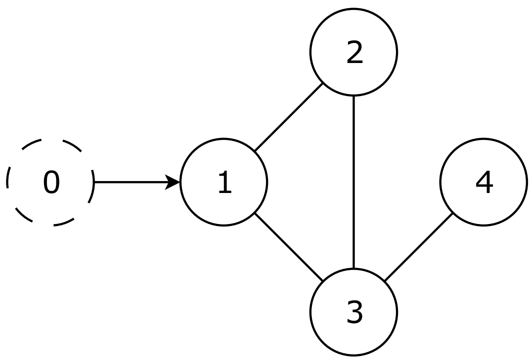

Figure 2-4: Example of communication graph.

we have

$$\begin{aligned}
&\mathcal{V} = \{1, 2, 3, 4\}, \\
&\mathcal{E} = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (3, 4), (4, 3)\}, \\
&\mathcal{T} = \{1\}.
\end{aligned} \tag{2-37}$$

Two square matrices are instrumental to find many useful properties of a communication graph: the *adjacency matrix* $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ and the *Laplacian matrix* $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$. Specifically, the adjacency matrix of an undirected communication graph is defined as $a_{ii} = 0$ and $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, where $i \neq j$; the Laplacian matrix is defined as $l_{ii} = \sum_j a_{ij}$ and $l_{ij} = -a_{ij}$, if $i \neq j$. The adjacency and the Laplacian matrices corresponding to the example in Figure 2-4 are

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

In addition, we use a square diagonal matrix, the *target matrix* $\mathcal{M} = [m_{ij}] \in \mathbb{R}^{N\times N}$, to describe the directed communication of the leader with the target nodes. Specifically, the target matrix is defined as $m_{ii} = 1$ if $i \in \mathcal{T}$ and $m_{ii} = 0$ otherwise. In the example of Figure 2-4, we have $\mathcal{M} = \text{diag}(1,0,0,0)$. An undirected graph $\mathcal{G}$ is said to be *connected* if, taken any arbitrary pair of nodes (i, j) where $i, j \in \mathcal{V}$, there is a path that leads from i to j. Please, notice that the graph $\mathcal{G}$ in Figure 2-4 is undirected and connected. Finally, let us define the *leader-follower topology matrix* as $\mathcal{B} = \mathcal{L} + \mathcal{M}$. When $\mathcal{L}$ is the Laplacian matrix of an undirected and connected graph, $\mathcal{B}$ is positive definite by construction. For instance, for our example of Figure 2-4 we have

$$\mathcal{B} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{2-38}$$

2-3 State synchronization of linear homogeneous multi-agent systems

The results on synchronization of homogeneous networks that are presented in this section are well-known in the literature, and this work is built using these results as starting point.

A network of homogeneous LTI single-input agents of order n is considered in this section:

$$\dot{x}_i = A_0 x_i + b_0 u_i, \qquad i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\} \tag{2-39}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}$ is the input, and we assume full-state measurement. Time index t is usually omitted when obvious. Please notice that, by definition of homogeneous multi-agent system, all the agents in the network share the same dynamics (A_0,b_0), which are assumed to be kwown.

Assuming that the graph $\mathcal{G}$ of the network is undirected and connected, we want to find a distributed state-feedback strategy (i.e. exploiting only state measurements from neighbors) for the control input u_i such that the agents reach consensus on their states, or in other words, the network state synchronizes to the same behavior, i.e. $x_i - x_j \to 0, \forall i, j \in \mathcal{V}$.

The proposed consensus protocol to solve the problem is

$$u_i = f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{2-40}$$

where a_{ij} are the entries of the adjacency matrix associated to the network communication graph $\mathcal{G}$, and the question is how to design $f \in \mathbb{R}^n$ to guarantee synchronization. It is important to notice that the control action (2-40) is distributed. Indeed, for each agent i, it is driven by the sum of disagreement measures between agent i and only its neighbors (due to the presence of the entries of the adjacency matrix).

It is convenient to define the local synchronization error as

$$e_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{2-41}$$

as it is computable by each agent i respecting the topology of the communication graph, and we have that the synchronization problem is solved when $e_i \to 0 \ \forall i \in \mathcal{V}$. It is now clear that the consensus protocol is basically driven by the synchronization error, as (2-40) can be rewritten in the form $u_i = f^T e_i$, where the gain f should be properly tuned. Let us assume that the order of the agents is $n = 1$, meaning they have scalar dynamics. The synchronization error for the overall network can be written as

$$e = \mathcal{L}x \qquad \text{with} \begin{cases} e & = [e_1, e_2, \dots, e_N]^T \\ x & = [x_1, x_2, \dots, x_N]^T \end{cases} \tag{2-42}$$

where $\mathcal{L}$ is the Laplacian matrix associated to the communication graph $\mathcal{G}$. On the other hand, if the order of the agents is $n > 1$, the overall network synchronization error must be necessarily written in the more general form

$$e = (\mathcal{L} \otimes I_n)x \qquad \text{with} \begin{cases} e & = [e_1^T, e_2^T, \dots, e_N^T]^T \\ x & = [x_1^T, x_2^T, \dots, x_N^T]^T \end{cases} \tag{2-43}$$

where $\otimes$ denotes the Kronecker product. The most useful properties of the Kronecker product, some of which will be used in the proofs, are:

- $A \otimes (B + C) = A \otimes B + A \otimes C$;
- $(A + B) \otimes C = A \otimes C + B \otimes C$;
- $(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$ with $k \in \mathbb{R}$;
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;
- $(A \otimes B)^T = A^T \otimes B^T$;
- $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$ that is called *mixed product* (the matrix products AC and BD must be possible).

Example 2-3-1 - Synchronization error and Kronecker product

Let us consider the simplest undirected and connected network we can think of, as in Figure 2-5. System 1 can send information to 2 and also system 2 can send information to 1.

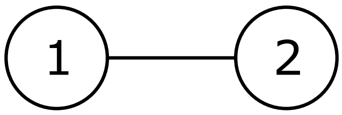

Figure 2-5: Example 2-3-1: communication graph.

We can directly write the adjacency matrix where we have $a_{12} = a_{21} = 1$ and zeros on the diagonal by definition:

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2-44}$$

Now, in order to write the Laplacian matrix, we need to find the *degree of connection* of each agent in the network. This is a trivial case where both agents are connected to only one other agent in the network (in particular, to each other). Using this information we find that the diagonal entries of the Laplacian are $l_{11} = 1$ and $l_{22} = 1$. The off-diagonal elements of the Laplacian are simply the opposite of the corresponding entries of the adjacency (by definition). The Laplacian matrix results in

$$\mathcal{L} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{2-45}$$

It is important to notice that, independently of the order (state dimension) of the agents in the network, both the adjacency and Laplacian matrices are $\in \mathbb{R}^{N \times N}$, where N is the number of agents. Let us assume that the systems have scalar dynamics. The local state-synchronization errors (2-41) result in

$$\begin{aligned} e_1 &= x_1 - x_2 \\ e_2 &= x_2 - x_1 \end{aligned} \tag{2-46}$$

that can be written for the overall network as

$$\begin{aligned} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{e} &= \begin{bmatrix} x_1 - x_2 \\ x_2 - x_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \mathcal{L}x. \end{aligned} \tag{2-47}$$

Let us now consider the agents to have order $n = 2$. The local state-synchronization errors would not be scalar anymore, in particular, showing all the entries we have

$$\begin{aligned}\underbrace{\begin{bmatrix}e_{11}\\ e_{12}\end{bmatrix}}_{e_1} &= \underbrace{\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}}_{x_1} - \underbrace{\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}}_{x_2}\\ \underbrace{\begin{bmatrix}e_{21}\\ e_{22}\end{bmatrix}}_{e_2} &= \underbrace{\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}}_{x_2} - \underbrace{\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}}_{x_1}\end{aligned} \tag{2-48}$$

that lead to the overall synchronization error

$$\begin{aligned}\underbrace{\begin{bmatrix}\begin{bmatrix}e_{11}\\ e_{12}\end{bmatrix}\\ \begin{bmatrix}e_{21}\\ e_{22}\end{bmatrix}\end{bmatrix}}_{e} &= \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\underbrace{\begin{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}\\ \begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}\end{bmatrix}}_{x}\\ &= \begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\\ \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} & \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}\\ \begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}\end{bmatrix}\\ &= \begin{bmatrix}1\cdot I_2 & -1\cdot I_2\\ -1\cdot I_2 & 1\cdot I_2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}\\ \begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}\end{bmatrix}\\ &= \begin{bmatrix}l_{11}\cdot I_2 & l_{12}\cdot I_2\\ l_{21}\cdot I_2 & l_{22}\cdot I_2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}\\ \begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}\end{bmatrix} = (\mathcal{L}\otimes I_2)x = (\mathcal{L}\otimes I_n)x\end{aligned} \tag{2-49}$$

Controller (2-40) applied to (2-39) leads to the closed-loop network

$$\dot{x}_i = A_0 x_i + b_0 f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j), \qquad i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\}. \tag{2-50}$$

The following result, that can be found for instance in [Li et al., 2010], allows us to design f to achieve synchronization for the homogeneous dynamics in (2-50).

Proposition 2-3-1: [Homogeneous network state synchronization] The homogeneous network (2-50), under the assumption that the graph $\mathcal{G}$ is undirected and connected, synchronizes if

$$A_0 + \lambda_i b_0 f^T \text{ is Hurwitz, } \forall i \in \mathcal{V}/\{1\} \tag{2-51}$$

with λ_i's, $i \in \mathcal{V}/\{1\}$, are the non-zero eigenvalues of the Laplacian.

Proof: The overall homogeneous network (2-50) can be written in the more compact form

$$\begin{aligned}\dot{x} &= (I_N \otimes A_0)x + (I_N \otimes b_0 f^T)(\mathcal{L}\otimes I_n)x\\ &= (I_N \otimes A_0 + \mathcal{L}\otimes b_0 f^T)x\end{aligned} \tag{2-52}$$

or as a function of the synchronization error

$$\dot{x} = (I_N \otimes A_0)x + (I_N \otimes b_0 f^T)e. \tag{2-53}$$

We can now write the overall error dynamics as

$$\begin{aligned}\dot{e} &= (\mathcal{L} \otimes I_n)\dot{x} \\ &= (\mathcal{L} \otimes I_n)[(I_N \otimes A_0)x + (I_N \otimes b_0 f^T)e] \\ &= (\mathcal{L} \otimes I_n)(I_N \otimes A_0)x + (\mathcal{L} \otimes I_n)(I_N \otimes b_0 f^T)e \\ &= (\mathcal{L} I_N \otimes I_n A_0)x + (\mathcal{L} \otimes b_0 f^T)e \\ &= (I_N \mathcal{L} \otimes A_0 I_n)x + (\mathcal{L} \otimes b_0 f^T)e \\ &= (I_N \otimes A_0)(\mathcal{L} \otimes I_n)x + (\mathcal{L} \otimes b_0 f^T)e \\ &= [(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e.\end{aligned} \tag{2-54}$$

Let us now consider the Lyapunov function candidate

$$V = e^T (I_N \otimes P)e \tag{2-55}$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Then we have

$$\begin{aligned}\dot{V} &= 2e^T (I_N \otimes P)\dot{e} \\ &= 2e^T (I_N \otimes P)[(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e \\ &= 2e^T (I_N \otimes PA_0 + \mathcal{L} \otimes Pb_0 f^T)e\end{aligned} \tag{2-56}$$

Now, since the graph is undirected and connected we know, e.g. from [Li et al., 2010], that there exists a unitary matrix $\mathcal{U} \in \mathbb{R}^{N \times N}$, where $\mathcal{U} = [\frac{1}{\sqrt{N}}\mathbf{1}_N \ \mathcal{U}_2]$ with $\mathcal{U}_2 \in \mathbb{R}^{N \times (N-1)}$ ($\mathcal{U}^T\mathcal{U} = \mathcal{U}\mathcal{U}^T = I$), such that $\mathcal{U}^T \mathcal{L} \mathcal{U} = \text{diag}(0, \lambda_2, \dots, \lambda_N) \triangleq \Lambda$. This can be used to define the transformation $e = (\mathcal{U} \otimes I_n)\bar{e}$. Moreover let $\bar{e} = [\bar{e}_1^T, \bar{e}_2^T, \dots, \bar{e}_N^T]^T$. It is easily checked that

$$\begin{aligned}\bar{e}_1 &= \left(\frac{1}{\sqrt{N}}\mathbf{1}_N \otimes I_n\right) e \\ &= \left(\frac{1}{\sqrt{N}}\mathbf{1}_N \otimes I_n\right) (\mathcal{L} \otimes I_n)x = \mathbf{0}_{Nn}.\end{aligned}$$

The defined transformation can be applied to (2-56), obtaining

$$\begin{aligned}\dot{V} &= 2e^T (I_N \otimes PA_0 + \mathcal{L} \otimes Pb_0 f^T)e \\ &= 2\bar{e}^T (\mathcal{U}^T \otimes I_n^T)(I_N \otimes PA_0 + \mathcal{L} \otimes Pb_0 f^T)(\mathcal{U} \otimes I_n)\bar{e} \\ &= 2\bar{e}^T (\mathcal{U}^T I_N \otimes I_n PA_0 + \mathcal{U}^T \mathcal{L} \otimes I_n Pb_0 f^T)(\mathcal{U} \otimes I_n)\bar{e} \\ &= 2\bar{e}^T (\mathcal{U}^T I_N \mathcal{U} \otimes I_n PA_0 I_n + \mathcal{U}^T \mathcal{L} \mathcal{U} \otimes I_n Pb_0 f^T I_n)\bar{e} \\ &= 2\bar{e}^T (I_N \otimes PA_0 + \Lambda \otimes Pb_0 f^T)\bar{e} \\ &= \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P]\bar{e}_i\end{aligned} \tag{2-57}$$

which is negative definite if

$$P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P < \mathbf{0}, \qquad \forall i \in \mathcal{V}/\{1\} \tag{2-58}$$

which is equivalent to (2-51) and completes the proof.

2-4 The distributed observer

The distributed observer was born in the last decade in the context of the cooperative output regulation literature, and became very soon used also to solve consensus and synchronization problems. Recall that the cooperative output regulation problem refers to the problem of making a network of systems to follow the behavior of a leader exosystem, while also rejecting possible disturbances. Cooperation arises from the fact that not all systems in the network can access the signals of the leader and the idea is that the systems not directly connected to the exosystem can reconstruct the exosystem signals through communication with neighbors [Su and Huang, 2012a, Su and Huang, 2012b]. As already emphasized in the previous section, the main issue in cooperative control over multi-agent systems is indeed that the control protocols must be distributed. The distributed observer has the ability to reconstruct the reference to be tracked for all the agents in the network. Once the reconstructed reference signal is available all over the network, the problem is divided in many local subproblems in which each agent has to solve a classical single-agent tracking problem.

The problem setting is therefore the following. We want each agent in the network to track a reference signal, which is generated by the following autonomous system, called exosystem

$$\begin{aligned} \dot{v} &= Sv, \qquad v(0) = v_0 \\ r &= Rv. \end{aligned} \tag{2-59}$$

where $v \in \mathbb{R}^q$ is the state and $r \in \mathbb{R}$ is the output (single-output case for simplicity), that is the reference to be tracked by the agents in the network. The matrices S and R are assumed to be known, while the initial conditions v_0 are unknown. System (2-59) is the autonomous leader of the network and it is not in general connected to all the other agents.

Example 2-4-1 - Cooperative output regulation scenario

Let us consider the simplest possible setting as shown in Figure 2-6.

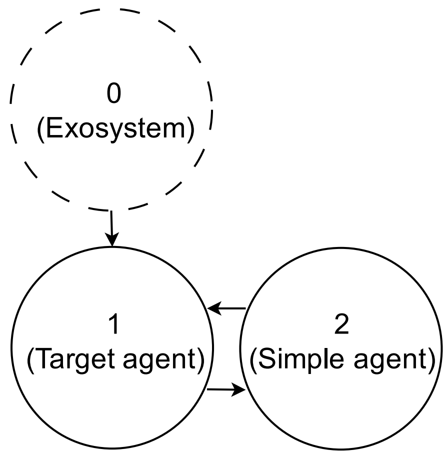

Figure 2-6: Example 2-4-1: communication graph with leader exosystem.

The exosystem is denoted as agent 0 and only the target agent 1 has access to its signals. The directed communication between 0 and 1 simply refers to the fact that in general the exosystem, being an autonomous system, is not influenced by the target agents and does not receive any information from them. Finally, an undirected communication is present between agent 1 and 2, and it should be clear that system 2 does not have access to the leader signals. In order to solve the tracking problem, agent 2 will have therefore to reconstruct the reference through communication with agent 1 (or more in general, with its neighbors).

Now we can introduce the distributed observer

$$\begin{aligned} \dot{\eta}_i &= S\eta_i + \mu\left[\left(\sum_{j=1}^{N} a_{ij}(\eta_j - \eta_i)\right) + m_{ii}(v - \eta_i)\right], \qquad \eta_i(0) = \eta_{i0} \\ \hat{r}_i &= R\eta_i, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i \in \mathcal{V} \end{aligned} \tag{2-60}$$

where η_i and $\hat{r}_i$ are the local estimates of v and r respectively, $\mu > 0$ is a gain that can be tuned, a_{ij} are the non-diagonal entries of the adjacency matrix $\mathcal{A}$ (recall that $a_{ij} = 1$ if agent i and agent j are neighbors), while m_{ii} are the diagonal entries of the target matrix $\mathcal{M}$ (recall that $m_{ii} = 1$ if agent i can receive information from the leader). The distributed observer (2-60) is able to locally estimate v for each agent i using only neighbors information. It can be easily proven [Su and Huang, 2012a] that $\eta_i \to v$ and therefore $\hat{r}_i \to r$, $\forall i \in \mathcal{V}$. Moreover, μ can be tuned to regulate the convergence rate. Then, for each system in the network, a local controller can be designed for solving a classical single-agent tracking problem based on the reconstructed reference $\hat{r}_i$. This practically means that each local control system is made of the observer plus the controller.

Example 2-4-1 cont'd - The distributed observer architecture

Let us now go deeper in the architecture of Example 2-4-1 by considering Figure 2-7.

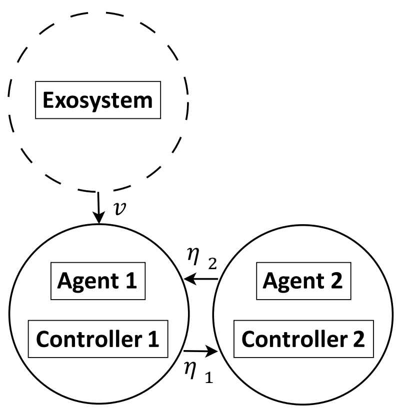

Figure 2-7: Example 2-4-1 cont'd: network architecture.

We have:

- Node "0"

– Exosystem

$$\begin{aligned} \dot{v} &= Sv, \qquad v(0) = v_0 \\ r &= Rv \end{aligned} \tag{2-61}$$

- Node "1"
  - System 1

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1, \qquad x_1(0) = x_{10} \\ y_1 &= C_1 x_1 \end{aligned} \tag{2-62}$$

  - Controller 1
    *

$$\begin{aligned} \dot{\eta}_1 &= S\eta_1 + \mu \left[(\eta_2 - \eta_1) + (v - \eta_1) \right], \qquad \eta_1(0) = \eta_{10} \\ \hat{r}_1 &= R\eta_1 \end{aligned} \tag{2-63}$$

    *

$$u_1 = f_1(y_1, \hat{r}_1) \qquad \text{such that } y_1 - \hat{r}_1 \to 0 \tag{2-64}$$

- Node "2"
  - System 2

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u_2, \qquad x_2(0) = x_{20} \\ y_2 &= C_2 x_2 \end{aligned} \tag{2-65}$$

  - Controller 2
    *

$$\begin{aligned} \dot{\eta}_2 &= S\eta_2 + \mu \left[(\eta_1 - \eta_2) \right], \qquad \eta_2(0) = \eta_{20} \\ \hat{r}_2 &= R\eta_2 \end{aligned} \tag{2-66}$$

    *

$$u_2 = f_2(y_2, \hat{r}_2) \qquad \text{such that } y_2 - \hat{r}_2 \to 0 \tag{2-67}$$

where we know that the distributed observer architecture guarantees $\hat{r}_1 \to r$ and $\hat{r}_2 \to r$, and the control actions u_1 and u_2 can be designed by using any classical single-agent reference tracking technique we may like.

Now, let us change perspective to understand how this dynamical system is used in the synchronization literature. The distributed observer (2-60) can be seen as a network that reaches synchronization. In particular, it can be seen as a virtual homogeneous network in parallel to the real network, as it respects the same communication graph. We know that this homogeneous network will synchronize, and we can even regulate the synchronization rate by tuning μ. The idea behind synchronization protocols for heterogeneous networks using the distributed observer is the following: if each agent of the real network can reach synchronization with its corresponding agent in the parallel virtual network, then the real heterogeneous network will obviously synchronize. In a way, the distributed observer can be seen as a “reference homogeneous network” that we want the real network to follow.

2-5 Concluding remarks

The main idea that motivates this thesis is that the distributed observer may not be needed and it could be possible to design adaptive distributed controllers that are able to directly solve the problem.

After this introductory chapter, we precisely know how to reach synchronization over homogeneous networks by means of distributed consensus protocols. The question is: can we exploit these results in some way to achieve synchronization over heterogeneous (and uncertain) networks? The idea of an adaptive homogenization-based approach originates from this question. If we find a controller that forces the heterogeneous network to behave like an homogeneous one, then we already know how to synchronize the resulting closed-loop homogeneous network. In particular, the intuition that lies behind all the protocols that will be presented in this work, is that some kind of extension of model reference adaptive control could be developed for imposing an homogeneous network as a “reference”. This homogenization-based design could be an efficient and elegant solution to the problem of synchronizing networks of heterogeneous and uncertain agents.

Chapter 3

Leaderless synchronization over heterogeneous uncertain networks

In this chapter we tackle the problem of synchronizing a network of heterogeneous linear systems which are not dynamically coupled, i.e. the dynamics of an agent is not influenced by the dynamics of any other agent in the network. The problem has to be solved by designing distributed controllers, one for each system, which can compute the control action based on both own and neighbors information (respecting the communication graph). In particular, neighboring agents can exchange their states or outputs on-line, and this knowledge can be used by each system to understand whether local synchronization is reached or not. Obviously, when each agent has reached local synchronization with its neighbors, we can infer global synchronization of the whole network.

3-1 Problem formulation

A network of LTI SISO heterogeneous systems with unknown dynamics is considered in this chapter

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i u_i \\ y_i &= c_i^T x_i, \qquad i \in \mathcal{V} \triangleq \{1, 2, \dots, N\} \end{aligned} \tag{3-1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}$ is the input, and $y_i \in \mathbb{R}$ is the output. The triple (A_i, b_i, c_i) is unknown with matrices of appropriate dimensions, and possibly $A_i \neq A_j$, $b_i \neq b_j$ and $c_i \neq c_j$, $i \neq j$, $i, j \in \mathcal{V}$ (uncertain heterogeneous multi-agent systems). The equivalent transfer function form of (3-1) is

$$y_i = k_i \frac{Z_i(s)}{R_i(s)} u_i, \quad i \in \mathcal{V}. \tag{3-2}$$

Analogously, the triple (k_i, Z_i, R_i) is unknown with $R_i(s)$ being monic polynomials of order n, $Z_i(s)$ being monic polynomials of order $q < n$, k_i being constants referred to as the high-frequency gains.

The following connectivity assumption is made.
Assumption 3-1-1: *The graph $\mathcal{G}$ of the network is undirected and connected.*

The following problem is considered in this chapter:
Problem 3-1-1: [Adaptive leaderless synchronization] *Consider a network of uncertain heterogeneous agents* (3-1) *satisfying Assumption 3-1-1. Find a distributed state-feedback (resp. output-feedback) strategy (i.e. exploiting only measurements from neighbors) for the control inputs u_i such that, without any knowledge of the entries of A_i, b_i, and c_i, the network synchronizes to the same behavior, i.e. $x_i - x_j \to 0$ (resp. $y_i - y_j \to 0$), $\forall i, j$.*

3-2 Adaptive state synchronization

Two results are now given which are instrumental to solving Problem 3-1-1, assuming the full states x_i's are available for measurement.

Proposition 3-2-1: [Homogenization via reference model] Consider the following reference model

$$\dot{x}_m = A_0 x_m + b_0 u \tag{3-3}$$

with $x_m \in \mathbb{R}^n$. If there exist a family of vectors $k_i^* \in \mathbb{R}^n$ and a family of scalars l_i^* (with known sign) such that the following matching conditions are satisfied

$$\begin{cases} A_i + b_i k_i^{*T} = A_0 \\ l_i^* b_i = b_0 \end{cases} \tag{3-4}$$

then, there exist ideal controllers (one for each agent i)

$$u_i^* = k_i^{*T} x_i + l_i^* f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{3-5}$$

with $f \in \mathbb{R}^n$ to be designed, which lead to the following dynamics

$$\dot{x}_i = A_0 x_i + b_0 f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i \in \mathcal{V}. \tag{3-6}$$

Proof: The proof directly follows from applying the control input (3-5) to agent (3-1), and using (3-4).

Proposition 3-2-2: [Homogeneous network state synchronization] We already know from Proposition 2-3-1 that the homogeneous network (3-6) synchronizes if

$$A_0 + \lambda_i b_0 f^T \text{ is Hurwitz}, \quad \forall i \in \mathcal{V}/\{1\} \tag{3-7}$$

where λ_i's, $i \in \mathcal{V}/\{1\}$, are the non-zero eigenvalues of the Laplacian.

Proof: The proof was already shown after Proposition 2-3-1.

Remark 3-2-1: [Need for adaptation] *Since A_i, b_i are unknown, the ideal control* (3-5) *cannot be implemented to solve Problem 3-1-1. Therefore, some adaptation mechanisms must be devised to estimate the unknown ideal gains in Proposition 3-2-1, by exploiting only measurements from neighbors.*

3-2-1 Main result

The following state synchronizing protocol is proposed

$$u_i(t) = k_i^T(t)x_i + l_i(t)f^T \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) \tag{3-8}$$

where k_i and l_i are the (time-dependent) estimates of k_i^* and l_i^*, respectively. The following synchronization result holds.

Theorem 3-2-1: *Under Assumption 3-1-1, the uncertain heterogeneous network* (3-1)*, controlled using the synchronizing protocol* (3-8) *and the following adaptive laws*

$$\begin{aligned} \dot{k}_i^T &= -sgn(l_i^*)\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T Pb_0 x_i^T \\ \dot{l}_i &= -sgn(l_i^*)\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T Pb_0 f^T e_i \end{aligned} \tag{3-9}$$

with adaptive gain $\gamma > 0$, reaches synchronization provided that the matrix P and the vector f are chosen such that

$$P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P < \mathbf{0},\ \forall i \in \mathcal{V}/\{1\}. \tag{3-10}$$

Proof: The closed-loop network formed by (3-1) and (3-8) is given by

$$\dot{x}_i = (A_i + b_i k_i^T)x_i + l_i b_i f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{3-11}$$

which can be rewritten as a function of the estimation errors,

$$\dot{x}_i = (A_0 + b_i \tilde{k}_i^T(t))x_i + (b_0 + \tilde{l}_i(t) b_i) f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) \tag{3-12}$$

where $\tilde{k}_i(t) = k_i(t) - k_i^*$ and $\tilde{l}_i(t) = l_i(t) - l_i^*$. By defining for compactness

$$\begin{aligned} B_k(t) &= \text{diag}(b_1 \tilde{k}_1^T(t), \ldots, b_N \tilde{k}_N^T(t)) \\ B_l(t) &= \text{diag}(\tilde{l}_1(t) b_1 f^T, \ldots, \tilde{l}_N(t) b_N f^T) \end{aligned} \tag{3-13}$$

the closed-loop for the overall network can be written as

$$\dot{x} = (I_N \otimes A_0 + B_k(t))x + (I_N \otimes b_0 f^T + B_l(t))e. \tag{3-14}$$

Recalling, from (2-43), that the overall synchronization error is $e = (\mathcal{L} \otimes I_n)x$, the error dynamics are

$$\dot{e} = [(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e + (\mathcal{L} \otimes I_n)(B_k(t)x + B_l(t)e). \tag{3-15}$$

The adaptive laws (3-9) arise from considering the Lyapunov function candidate $V = V_1 + V_2 + V_3$, where

$$V_1 = e^T(I_N \otimes P)e, \quad V_2 = \sum_{i=1}^{N} \frac{\tilde{k}_i^T(t)\gamma^{-1}\tilde{k}_i(t)}{|l_i^*|}, \quad V_3 = \sum_{i=1}^{N} \frac{\tilde{l}_i(t)\gamma^{-1}\tilde{l}_i^T(t)}{|l_i^*|}. \tag{3-16}$$

Then we have

$$\dot{V}_1 = 2e^T(I_N \otimes P)[(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e + 2e^T(I_N \otimes P)[(\mathcal{L} \otimes I_n)(B_k(t)x + B_l(t)e)] \tag{3-17}$$

and following the same procedure as in (2-57):

$$\begin{aligned} \dot{V}_1 = & \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P]\bar{e}_i + \\ & + 2\sum_{i=1}^{N} \tilde{k}_i^T(t) x_i b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) + \\ & + 2\sum_{i=1}^{N} \tilde{l}_i(t) e_i^T f b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) \end{aligned} \tag{3-18}$$

Moreover, by using (3-9) we have:

$$\begin{aligned} \dot{V}_2 &= -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \tilde{k}_i^T(t) x_i b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) \\ \dot{V}_3 &= -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \tilde{l}_i(t) e_i^T f b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) \end{aligned} \tag{3-19}$$

leading to:

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\ &= \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P]\bar{e}_i \end{aligned} \tag{3-20}$$

which is negative semi-definite provided that condition (3-10) holds. Using standard Lyapunov arguments we can prove boundedness of all closed-loop signals and convergence of e to 0. In fact, since $V > 0$ and $\dot{V} \leq 0$, it follows that $V(t)$ has a limit, i.e.,

$$\lim_{t \to \infty} V(e(t), \tilde{\Omega}(t)) = V_\infty < \infty \tag{3-21}$$

where we have collected all parametric errors in $\tilde{\Omega}$. The finite limit implies V, e, $\tilde{\Omega} \in \mathcal{L}_\infty$. In addition, by integrating $\dot{V}$ it follows that for some $Q > 0$

$$\int_0^\infty e^T(\tau)Qe(\tau)\,\mathrm{d}\tau \leq V(e(0), \tilde{\Omega}(0)) - V_\infty \tag{3-22}$$

from which we establish that $e \in \mathcal{L}_2$. Finally, since $\dot{V}$ is uniformly continuous in time (this is satisfied because $\ddot{V}$ is finite), the Barbalat's lemma [Ioannou and Sun, 2012, Lemma 3.2.6] implies $\dot{V} \to 0$ as $t \to \infty$ and hence $e \to 0$, from which we derive $x_i \to x_j$, $\forall i, j$. This concludes the proof.

Remark 3-2-2: [No distributed observer needed, but not for free] *In order to implement* (3-9)*, and in particular the term* $\sum_{j=1}^N a_{ij}(e_i - e_j)$*, it is required to communicate the variable* e_i *among neighbors (extra local information). As we know, synchronization protocols based on distributed observer [Lu and Liu, 2017, Cai et al., 2017] require communication of the observer states. Now, comparing these approaches with* (3-9)*, we see that the proposed disagreement-based protocol is essentially lighter, because it does not require to construct in a distributed manner the observer variables. On the other hand, it is required to exchange both* x_i *and* e_i *among neighbors. In Chapter 4 we will show that in the leader-follower scenario, a simpler communication architecture can be obtained.*

Remark 3-2-3: [Adaptation of both feedback and coupling gains] *It can be noticed from the adaptive protocol* (3-8) *that the vectors* k_i *act as feedback gains, while the scalars* l_i *act as coupling gains. The proposed protocol can therefore adapt both the feedback and the coupling gains. Actually,* (3-8) *can be considered as a node-based protocol (because* l_i *is unique for each node): it is possible to modify* (3-8) *to be edge-based as follows*

$$u_i(t) = k_i^T(t)x_i + f^T \sum_{j=1}^N l_{ij}(t)a_{ij}(x_i(t) - x_j(t)) \tag{3-23}$$

and the corresponding adaptation laws would become

$$\begin{aligned} \dot{k}_i^T &= -sgn(l_i^*)\gamma \left(\sum_{j=1}^N a_{ij}(e_i - e_j) \right)^T Pb_0 x_i^T \\ \dot{l}_{ij} &= -sgn(l_i^*)\gamma \left(\sum_{j=1}^N a_{ij}(e_i - e_j) \right)^T Pb_0 f^T(x_i(t) - x_j(t)) \end{aligned} \tag{3-24}$$

where l_{ij} *would be adapted on each edge separately. This will be properly investigated in the next chapter.*

Remark 3-2-4: [Leaderless vs. leader-follower synchronization] *We want to emphasize that the results we just proved refer to leaderless synchronization, where the state to which the agents will synchronize are in general a priori unknown. Please notice that, since the reference model* (3-3) *simply guarantees the existence of ideal synchronizing gains and does not play the role of a leader, we have proven that the synchronization error converges to zero, but the coherent state is in general a priori unknown. On the other hand, it is not difficult to prove that the addition of a leader in the network, having dynamics* (3-3)*, leads, via the proposed* *protocols, to convergence to the a priori known state of the leader. This will be shown in the simulations in Section 3-4. We would like to anticipate that more convenient adaptive leader-follower synchronizing protocols will be formally proposed in Chapter 4 via an alternative Lyapunov function.*

3-3 Adaptive output synchronization

Two results are now given which are instrumental to solving Problem 3-1-1, in case only the outputs y_i's are available for measurement.

Proposition 3-3-1: [Output homogenization via reference model] Consider the following reference model

$$\begin{aligned} \dot{x}_m &= A_0 x_m + b_0 r, \qquad x_m(0) = x_{m0} \\ y_m &= c_0 x_m \end{aligned} \tag{3-25}$$

where $x_m \in \mathbb{R}^{n_m}$ and $y_m, r \in \mathbb{R}$. Its transfer function representation is

$$y_m = k_m \frac{Z_m(s)}{R_m(s)} r \tag{3-26}$$

where $Z_m(s)$ and $R_m(s)$ are monic Hurwitz polynomials of degrees q_m and n_m, respectively. The reference model relative degree is $n_m - q_m = 1$. For simplicity we also choose $n_m = n$.

Considering the transfer function representation of the agents (3-2), in this section we assume they have the same unitary relative degree as the reference model: $n - q = 1$.

If there exist a family of vectors $h_i^* \in \mathbb{R}^{n-1}$, $g_i^* \in \mathbb{R}^{n-1}$ and a family of scalars c_i^*, l_i^* (with $sgn(l_i^*)$ known) such that the following matching conditions are satisfied

$$\begin{cases} (\Lambda(s) - h_i^{*T}\alpha(s))R_i - k_i Z_i(s)(g_i^{*T}\alpha(s) + c_i^*\Lambda(s)) = Z_i(s)\Lambda_0(s)R_m(s) \\ l_i^* = k_m/k_i \end{cases} \tag{3-27}$$

with

$$\begin{cases} \alpha(s) \triangleq [s^{n-2}, s^{n-3}, \dots, s, 1] & \text{for } n \geq 2 \\ \alpha(s) \triangleq 0 & \text{for } n = 1 \end{cases} \tag{3-28}$$

and $\Lambda(s)$ is a monic Hurwitz polynomial of degree $n-1$ which contains Z_m as a factor

$$\begin{aligned} \Lambda(s) &= \Lambda_0(s) Z_m(s) \\ &= s^{n-1} + \mu_{n-2}s^{n-2} + \mu_{n-3}s^{n-3} + \dots + \mu_0 \end{aligned} \tag{3-29}$$

where Λ_0 is to be designed. Then there exist ideal controllers (one for each agent)

$$u_i^* = h_i^{*T}\frac{\alpha(s)}{\Lambda(s)}u_i + g_i^{*T}\frac{\alpha(s)}{\Lambda(s)}y_i + c_i^* y_i + l_i^*\phi\sum_{j=1}^{N} a_{ij}(y_i - y_j) \tag{3-30}$$

with $\phi \in \mathbb{R}$ to be designed, which, leads to the following homogeneous dynamics (in state-space representation)

$$\begin{aligned} \dot{x}_i &= A_0 x_i + b_0 \phi \sum_{j=1}^{N} a_{ij}(y_i - y_j) \\ y_i &= c_0^T x_i, \qquad i \in \mathcal{V} \end{aligned} \tag{3-31}$$

where notice that, similarly to MRAC (Section 2-1-1: output measurement case), we impose the reference model dynamics (A_0, b_0, c_0) for each closed-loop system thanks to the combination of the control input (3-30) and the matching conditions (3-27).

The following result allows us to design ϕ to achieve synchronization for the homogeneous dynamics in (3-31).

Proposition 3-3-2: [Homogeneous network output synchronization] The homogeneous network (3-31) synchronizes if

$$(A_0 + \lambda_i b_0 f c_0^T, b_0, c_0^T) \text{ is SPR}, \quad \forall i \in \mathcal{V}/\{1\} \tag{3-32}$$

where λ_i's, $i \in \mathcal{V}/\{1\}$, are the non-zero eigenvalues of the Laplacian.

Proof: The overall homogeneous network (3-31) can be written in the more compact form

$$\begin{aligned} \dot{x} &= (I_N \otimes A_0 + \mathcal{L} \otimes b_0 \phi c_0^T) x \\ y &= (I_N \otimes c_0^T) x \end{aligned} \tag{3-33}$$

where $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$ and $y = [y_1, y_2, \ldots, y_N]^T$. Let us now define the state and output synchronization errors as

$$\begin{aligned} e_i &= \sum_{j=1}^{N} a_{ij}(x_i - x_j), \qquad e = [e_1^T, e_2^T, \ldots, e_N^T]^T \\ \epsilon_i &= \sum_{j=1}^{N} a_{ij}(y_i - y_j), \qquad \epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]^T. \end{aligned} \tag{3-34}$$

The overall homogeneous network can be now written as a function of the state synchronization error

$$\begin{aligned} \dot{x} &= (I_N \otimes A_0) x + (I_N \otimes b_0 \phi c_0^T) e \\ y &= (I_N \otimes c_0^T) x. \end{aligned} \tag{3-35}$$

The overall error dynamics result in

$$\begin{aligned} \dot{e} &= (\mathcal{L} \otimes I_n) \dot{x} \\ &= (\mathcal{L} \otimes I_n)(I_N \otimes A_0) x + (\mathcal{L} \otimes I_n)(I_N \otimes b_0 \phi c_0^T) e \\ &= \left[(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 \phi c_o^T) \right] e. \end{aligned} \tag{3-36}$$

Now, let us use a similar decomposition as in the proof of Proposition 2-3-1, and consider the Lyapunov function candidate

$$\Upsilon_1 = e^T (I_N \otimes P) e \tag{3-37}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix satisfying the Meyer-Kalman-Yakubovich (MKY) lemma [Ioannou and Sun, 2012, Lemma 3.5.4]

$$\begin{aligned} &P(A_0 + \lambda_i b_0 \phi c_0^T) + (A_0 + \lambda_i b_0 \phi c_0^T)^T P < -Q \\ &P b_0 = c_0, \qquad \forall i \in \mathcal{V}/\{1\}. \end{aligned} \tag{3-38}$$

Then we have

$$\dot{\Upsilon}_1 = \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 \phi c_0^T) + (A_0 + \lambda_i b_0 \phi c_0^T)^T P] \bar{e}_i \tag{3-39}$$

which is negative definite if

$$P(A_0 + \lambda_i b_0 f c_0^T) + (A_0 + \lambda_i b_0 f c_0^T)^T P < \mathbf{0},\ \forall i \in \mathcal{V}/\{1\} \tag{3-40}$$

which is implied by the first in (3-38). The second in (3-38) is not useful now, but it will be for the adaptive laws design. This completes the proof.

3-3-1 Main result

The following output synchronizing protocol is proposed

$$u_i(t) = h_i^T(t)\frac{\alpha(s)}{\Lambda(s)}u_i + g_i^T(t)\frac{\alpha(s)}{\Lambda(s)}y_i + c_i(t)y_i + l_i(t)\phi\sum_{j=1}^{N} a_{ij}(y_i - y_j) \tag{3-41}$$

where h_i, g_i, c_i and l_i are the (time-dependent) estimates of h_i^*, g_i^*, c_i^* and l_i^*, respectively. The following synchronization result holds.

Theorem 3-3-1: *Under Assumption 3-1-1, the uncertain heterogeneous network* (3-1), *controlled using the following distributed adaptive controller*

$$\begin{aligned} &u_i(t) = \theta_i^T(t)\omega_i, \qquad \dot{\theta}_i = -sgn(l_i^*)\gamma\left(\sum_{j=1}^{N} a_{ij}(\epsilon_i - \epsilon_j)\right)\omega_i \\ &\dot{\omega}_{i_1} = F\omega_{i_1} + du_i, \qquad \dot{\omega}_{i_2} = F\omega_{i_2} + dy_i \\ &\theta_i = \begin{bmatrix} h_i^T & g_i^T & c_i & l_i \end{bmatrix}^T \\ &\omega_i = \begin{bmatrix} \omega_{i_1}^T & \omega_{i_2}^T & y_i & \phi\epsilon_i \end{bmatrix}^T \\ &F = \begin{bmatrix} -\mu_{n-2} & -\mu_{n-3} & \cdots & -\mu_0 \\ & I_{n-2} & & 0_{(n-2)\times 1} \end{bmatrix}, \ d = \begin{bmatrix} 1 \\ 0_{(n-2)\times 1} \end{bmatrix} \end{aligned} \tag{3-42}$$

with adaptive gain $\gamma > 0$, reaches synchronization provided that the matrix P and the scalar f are chosen such that condition (3-32) *holds.*

Proof: The proof follows very similar steps as the one of Theorem 3-2-1. The Lyapunov function Υ_1 in (3-37) should be used together with

$$\Upsilon_2 = \sum_{i=1}^{N} \frac{\tilde{\theta}_i^T(t)\gamma^{-1}\tilde{\theta}_i(t)}{|l_i^*|} \tag{3-43}$$

Then we have

$$\dot{\Upsilon}_1 = 2e^T(I_N \otimes P)[(I_N \otimes A_0) + (\mathcal{L} \otimes b_0\phi c_0^T)]e + 2e^T(I_N \otimes P)[(\mathcal{L} \otimes I_n)(B_\theta(t)\omega)] \tag{3-44}$$

where

$$\begin{aligned} B_\theta(t) &= \mathrm{diag}(b_1\tilde{\theta}_1^T(t), \dots, b_N\tilde{\theta}_N^T(t)) \\ \omega &= [\omega_1^T, \omega_2^T, \dots, \omega_N^T]^T \end{aligned} \tag{3-45}$$

and following a similar procedure as in (2-57):

$$\begin{aligned} \dot{\Upsilon}_1 =& \sum_{i=2}^{N} \bar{e}_i^T[P(A_0 + \lambda_i b_0 f c_0^T) + (A_0 + \lambda_i b_0 f c_0^T)^T P]\bar{e}_i + \\ &+ 2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j)\right)^T Pb_0\tilde{\theta}_i^T(t)\omega_i \\ =& \sum_{i=2}^{N} \bar{e}_i^T[P(A_0 + \lambda_i b_0 f c_0^T) + (A_0 + \lambda_i b_0 f c_0^T)^T P]\bar{e}_i + \\ &+ 2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \tilde{\theta}_i^T(t)\omega_i \left(\sum_{j=1}^{N} a_{ij}(\epsilon_i - \epsilon_j)\right) \end{aligned} \tag{3-46}$$

where we have used the second equation in (3-32). Moreover, by using (3-42) we have:

$$\dot{\Upsilon}_2 = -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \tilde{\theta}_i^T(t)\omega_i \left(\sum_{j=1}^{N} a_{ij}(\epsilon_i - \epsilon_j)\right) \tag{3-47}$$

leading to:

$$\begin{aligned} \dot{\Upsilon} &= \dot{\Upsilon}_1 + \dot{\Upsilon}_2 \\ &= \sum_{i=2}^{N} \bar{e}_i^T[P(A_0 + \lambda_i b_0 f c_0^T) + (A_0 + \lambda_i b_0 f c_0^T)^T P]\bar{e}_i \end{aligned} \tag{3-48}$$

which is negative semi-definite provided that condition (3-32) holds. Using standard Lyapunov arguments as in Theorem 3-2-1 we can prove boundedness of all closed-loop signals and convergence of e to 0, from which we derive $\epsilon \to 0$, i.e. $y_i \to y_j$, $\forall i, j \in \mathcal{V}$. This concludes the proof.

3-4 Numerical examples

Simulations using both controllers (3-8)-(3-9) and (3-42) are carried out in the following, considering the graph shown in Figure 3-1, where agent 0 acts as a leader node.

The heterogeneous agents (3-1) are taken as second-order linear systems with unitary relative degree having transfer function representation

$$y_i = \frac{n_{1_i}s + n_{2_i}}{s^2 + d_{1_i}s + d_{2_i}}u_i, \qquad i \in \mathcal{V}. \tag{3-49}$$

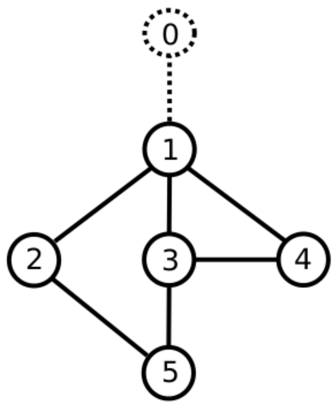

Figure 3-1: The undirected communication graph.

The state-space representation in controllable canonical form

$$\begin{aligned} \dot{x}_i &= \underbrace{\begin{bmatrix} 0 & 1 \\ -d_{2_i} & -d_{1_i} \end{bmatrix}}_{A_i} x_i + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_i} u_i \\ y_i &= \underbrace{\begin{bmatrix} n_{2_i} & n_{1_i} \end{bmatrix}}_{c_i^T} x_i \end{aligned} \tag{3-50}$$

is considered to guarantee that, in the full-state measurement case, the matching conditions (3-4) have a solution. The parameters and initial conditions for each heterogeneous agent (3-50) are reported in Table 3-1. Recall that the agent parameters are unknown to the designer, i.e. the values in Table 3-1 are not used for control design but only for simulation.

Table 3-1: Parameters and initial conditions for the agents

	d_{1_i}	d_{2_i}	n_{1_i}	n_{2_i}	$x_i(0)$
agent #1	1	2	1	1.5	$[0\ 1]^T$
agent #2	0.75	2.5	0.5	1	$[0\ 2]^T$
agent #3	1.25	2	1.25	1	$[0\ 3]^T$
agent #4	0.5	1	0.75	0.75	$[0\ 4]^T$
agent #5	0.75	1	1.5	2	$[0\ 5]^T$

For the *state synchronization* case, the reference model (and leader) is chosen as a harmonic oscillator

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ -(0.7^2) & 0 \end{bmatrix}}_{A_0} x_m + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_0} u \ , \quad x_m(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{3-51}$$

The vector f and matrix P that satisfy condition (3-10) are

$$P = \begin{bmatrix} 1.4082 & 0.2671 \\ 0.2671 & 0.5551 \end{bmatrix}, \quad f^T = \begin{bmatrix} -1 & -1 \end{bmatrix} . \tag{3-52}$$

Finally, the adaptive gain is taken $\gamma = 10$ and all estimated control gains k_i, l_i, are initialized to 0. The resulting adaptive state synchronization is shown in Figure 3-2, with adaptive gains shown in Figure 3-3.

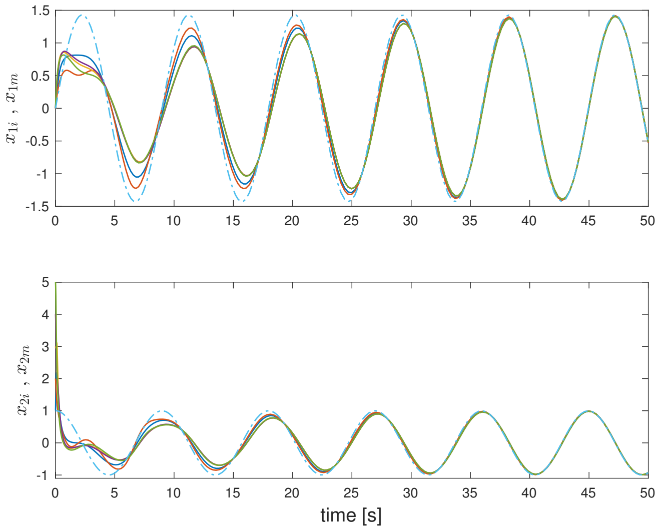

Figure 3-2: Synchronization of the states of each agent i to the leader state using (3-8) and (3-9).

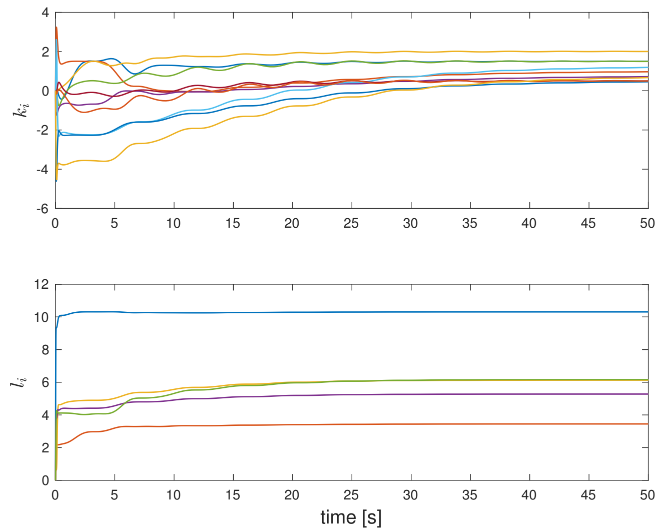

Figure 3-3: Adaptive gains resulting from (3-9).

For the *output synchronization* case, the same parameters and initial conditions as in Table 3-1 are taken. Please notice that the controllable canonical form is in principle not needed anymore. The reference model is chosen again as a harmonic oscillator

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ -(0.7^2) & 0 \end{bmatrix}}_{A_0} x_m + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_0} u$$
$$y_m = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{c_0} x_m, \qquad x_m(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3-53}$$

The scalar ϕ that satisfies conditions (3-32) if $\phi = -1$. The adaptive gain is taken $\gamma = 10$ and all estimated control gains θ_i are initialized to 0. The resulting adaptive output synchronization is shown in Figure 3-4 together with the adaptive gains.

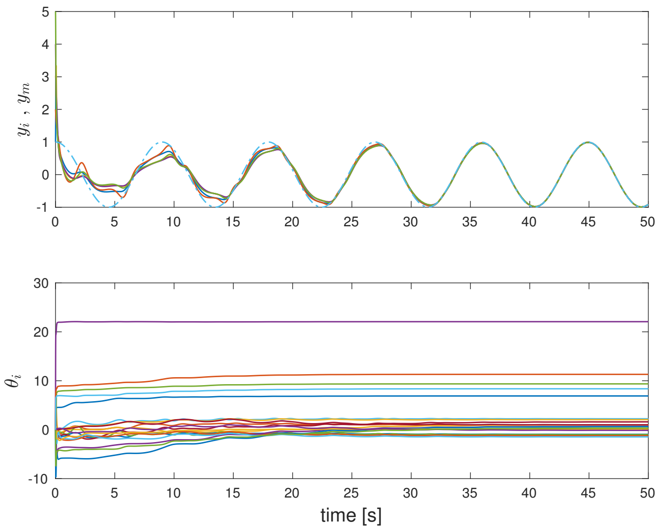

Figure 3-4: Synchronization of the outputs of each agent i to the leader output using (3-42), and corresponding adaptive gains.

In order to better emphasize that the proposed protocols can guarantee leaderless synchronization (even though in this case it is impossible to define a priori the synchronized behavior), let us cut the leader agent from the network in Figure 3-1 and simulate the network using the protocol (3-42). The result is shown in Figure 3-5, where we see that the network reaches output synchronization to zero. We want to recall that, in general, we do not know anything about the synchronized behavior that the systems will reach. It could also happen that the network synchronizes to an unstable behavior (an example will be shown at the end of Chapter 5). On the other hand, it is known that in the leaderless scenario it could also happen that the agents stabilize each other [Narendra and Harshangi, 2014, Narendra and Harshangi, 2015].

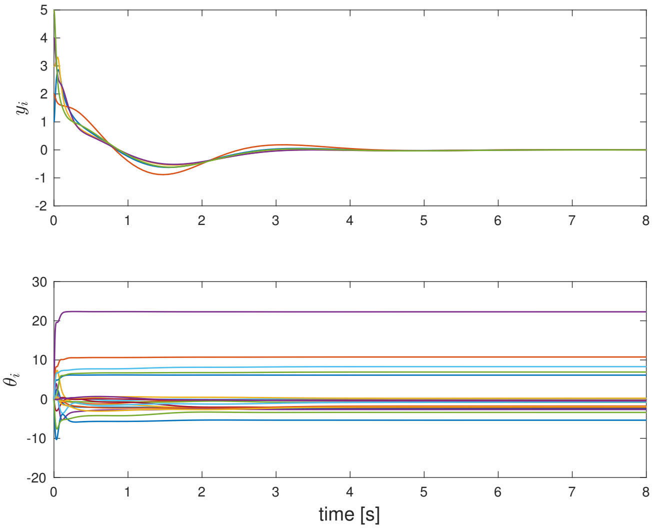

Figure 3-5: Leaderless simulation: synchronization of the outputs of each agent i using (3-42), and corresponding adaptive gains.

3-5 Concluding remarks

The adaptive distributed protocols presented in this chapter guarantee leaderless synchronization over heterogeneous and uncertain networks, without the need for any distributed observer, thus improving on state-of-the-art controllers.

As already presented in the numerical examples, it is often practical to add a leader (even a virtual one) to define a reference trajectory to which we want our network to synchronize. For this reason, in the next chapter we will solve the problem of leader-follower synchronization, meaning that we will include the presence of an autonomous leader in the network from the beginning of the mathematical analysis.

Chapter 4

Leader-follower synchronization over heterogeneous uncertain networks

A network of heterogeneous linear systems is still considered in this chapter. In addition, an autonomous linear leader is present, which generates the reference to be tracked. At least one agent of the network has the leader among its neighbors, meaning that the reference can be used by its local controller. In this setting, knowing that the leader cannot be controlled and will always autonomously generate the reference, we can safely say that if each agent reaches local synchronization with its neighbors, the problem is solved.

4-1 Problem formulation

A network of LTI SISO heterogeneous systems with unknown dynamics is still considered in this chapter

$$\begin{aligned} \dot{x}_i &= A_i x_i + b_i u_i \\ y_i &= c_i^T x_i, \qquad i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\} \end{aligned} \tag{4-1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}$ is the input, and $y_i \in \mathbb{R}$ is the output. The triple (A_i, b_i, c_i) is unknown with matrices of appropriate dimensions, and possibly $A_i \neq A_j$, $b_i \neq b_j$ and $c_i \neq c_j$, $i \neq j$, $i, j \in \mathcal{V}$ (uncertain heterogeneous systems). The equivalent transfer function form of (4-1) is

$$y_i = k_i \frac{Z_i(s)}{R_i(s)} u_i, \quad i \in \mathcal{V}. \tag{4-2}$$

Analogously, the triple (k_i, Z_i, R_i) is unknown with $R_i(s)$ being monic polynomials of order n, $Z_i(s)$ being monic polynomials of order $q < n$, k_i being constants referred to as the high-frequency gains. In addition, consider the leader dynamics

$$\begin{aligned} \dot{x}_0 &= A_0 x_0, \qquad x_0(0) = x_{00} \\ y_0 &= c_0^T x_0 \ , \end{aligned} \tag{4-3}$$

where $x_0 \in \mathbb{R}^{n_0}$ with $n_0 = n$, is the state, $y_0 \in \mathbb{R}$ is the output, and the matrix A_0 and the vector c_0 have appropriate dimensions. The leader information is accessible to the target nodes only.

The following connectivity assumption is made.

Assumption 4-1-1: *The graph $\mathcal{G}$ of the network is undirected and connected, and the leader interacts with at least one system ($\mathcal{T} \neq \emptyset$).*

The following problem is considered:

Problem 4-1-1: [Adaptive leader-follower synchronization] *Consider a network of uncertain heterogeneous systems* (4-1) *and the leader* (4-3)*, satisfying Assumption 4-1-1. Find a state-feedback (resp. output-feedback) adaptive distributed strategy (i.e. exploiting only measurements from neighbors) for the control inputs u_i such that synchronization to the leader dynamics is achieved, i.e. $x_i - x_0 \to 0$ (resp. $y_i - y_0 \to 0$), $\forall i \in \mathcal{V}$.*

4-2 Adaptive state synchronization

Two results are now given which are instrumental to solving Problem 4-1-1, assuming the full states x_i's and x_0 are available for measurement.

Proposition 4-2-1: [Leader-follower state-feedback homogenization] Consider homogeneous dynamics (A_0, b_0), with A_0 being as in (4-3). If there exist a family of vectors $k_i^* \in \mathbb{R}^n$ and a family of scalars $l_i^* \in \mathbb{R}$ such that the following matching conditions are satisfied

$$\begin{cases} A_i + b_i k_i^{*T} = A_0 \\ l_i^* b_i = b_0 \end{cases} \tag{4-4}$$

then, there exist ideal controllers (one for each agent)

$$u_i^* = k_i^{*T} x_i + l_i^* f^T \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + m_{ii}(x_i - x_0) \right) \tag{4-5}$$

with $f \in \mathbb{R}^n$ to be designed, giving the closed-loop dynamics

$$\dot{x}_i = A_0 x_i + b_0 f^T \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) + m_{ii}(x_i - x_0) \right), \quad i \in \mathcal{V}. \tag{4-6}$$

Proof: The proof directly follows from applying the control input (4-5) to system (4-1), and using (4-4).

The following result allows us to design f to achieve synchronization for the homogeneous dynamics in (4-6).

Proposition 4-2-2: [State synchronization of an homogeneous network with leader] The homogeneous network (4-6) synchronizes to the reference state x_0 if

$$\lambda_i A_0 + b_0 f^T \text{ is Hurwitz}, \quad \forall i \in \mathcal{V} \tag{4-7}$$

with λ_i the eigenvalues of the inverse of the leader-follower topology matrix $\mathcal{B}^{-1}$.

Proof: Define $x = [x_1^T, x_2^T, \dots, x_N^T]^T \in \mathbb{R}^{Nn}$ and $x_m = [x_0^T, x_0^T, \dots, x_0^T]^T \in \mathbb{R}^{Nn}$, and the local synchronization error

$$e_i = \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j) \right) + m_{ii}(x_i - x_0) \tag{4-8}$$

where $e = [e_1^T, e_2^T, \dots, e_N^T]^T$ can be written as

$$e = (\mathcal{L} \otimes I_n)x + (\mathcal{M} \otimes I_n)(x - x_m). \tag{4-9}$$

Exploiting the fact that $(\mathcal{L} \otimes I_n)x_m = 0$ [Gibson, 2016], we can write

$$e = (\mathcal{L} \otimes I_n)(x - x_m) + (\mathcal{M} \otimes I_n)(x - x_m) = (\mathcal{B} \otimes I_n)(x - x_m). \tag{4-10}$$

Moreover, the overall homogeneous network dynamics (4-6) can be written in the compact form

$$\dot{x} = (I_N \otimes A_0)x + (\mathcal{B} \otimes b_0 f^T)(x - x_m) = (I_N \otimes A_0)x + (I_N \otimes b_0 f^T)e. \tag{4-11}$$

Positive-definiteness of $\mathcal{B}$ leads to the existence of a unitary matrix $\mathcal{U} \in \mathbb{R}^{N \times N}$ such that $\mathcal{U}^T \mathcal{B}^{-1} \mathcal{U} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N) \triangleq \Lambda$. This can be used to define the transformation $e = (\mathcal{U} \otimes I_n)\bar{e}$ with $\bar{e} = [\bar{e}_1^T, \bar{e}_2^T, \dots, \bar{e}_N^T]^T$ [Li et al., 2010].

We can now write the overall error dynamics, using (4-10) and (4-11)

$$\begin{aligned} \dot{e} =& (\mathcal{B} \otimes I_n)(I_N \otimes A_0)x + (\mathcal{B} \otimes I_n)(I_N \otimes b_0 f^T)e - (\mathcal{B} \otimes I_n)(I_N \otimes A_0)x_m \\ =& [(I_N \otimes A_0) + (\mathcal{B} \otimes b_0 f^T)]e \ . \end{aligned} \tag{4-12}$$

Consider the Lyapunov candidate

$$V_1 = e^T(\mathcal{B}^{-1} \otimes P)e \tag{4-13}$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. We have

$$\begin{aligned} \dot{V}_1 &= 2e^T(\mathcal{B}^{-1} \otimes P)[(I_N \otimes A_0) + (\mathcal{B} \otimes b_0 f^T)]e \\ &= 2\bar{e}^T(\Lambda \otimes PA_0 + I_N \otimes Pb_0 f^T)\bar{e} \\ &= \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0 f^T\right) + \left(\lambda_i A_0 + b_0 f^T\right)^T P \right] \bar{e}_i \end{aligned} \tag{4-14}$$

which is negative definite if

$$\left[P\left(\lambda_i A_0 + b_0 f^T\right) + \left(\lambda_i A_0 + b_0 f^T\right)^T P \right] < \mathbf{0}, \ \forall i \in \mathcal{V} \ . \tag{4-15}$$

This completes the proof.

4-2-1 Main result

The following edge-based state synchronizing protocol is proposed

$$u_i = k_i^T x_i + f^T \left(\sum_{j=1}^{N} l_{ij} a_{ij}(x_i - x_j) + l_{im} m_{ii}(x_i - x_0) \right) \tag{4-16}$$

where k_i is the estimate of k_i^*, while, l_{ij} and l_{im}, are the edge-based estimates of l_i^*. All the estimates are time-dependent, driven by distributed adaptive laws to be designed. In the next Theorem 4-2-1 we propose adaptive laws based on homogenization, which commonly require the knowledge of the sign of l_i^* (denoted with $sgn(l_i^*)$) [Ioannou and Sun, 2012, Chapter 6]. This amounts to having knowledge of the system control direction: if all the systems have the same control direction, e.g. $sgn(l_i^*) = 1 \; \forall i \in \mathcal{V}$, all the subsequent adaptive laws (e.g. (4-17)) will simplify accordingly.

Theorem 4-2-1: [Heterogeneous network state synchronization] Under Assumption 4-1-1, the heterogeneous uncertain network (4-1), controlled using the protocol (4-16) and the adaptive laws

$$\begin{aligned} \dot{k}_i^T &= -sgn(l_i^*)\gamma e_i^T P b_0 x_i^T \\ \dot{l}_{ij} &= -sgn(l_i^*)\gamma e_i^T P b_0 f^T (x_i - x_j) \\ \dot{l}_{im} &= -sgn(l_i^*)\gamma e_i^T P b_0 f^T (x_i - x_0) \end{aligned} \tag{4-17}$$

with adaptive gain $\gamma > 0$, reaches synchronization to the reference state x_0, provided that the matrix P and the vector f are chosen such that condition (4-15) holds.

Proof: The closed-loop formed by (4-1) and (4-16) can be rewritten as a function of the estimation errors

$$\begin{aligned} \dot{x}_i =& A_0 x_i + b_i \tilde{k}_i^T(t) x_i \\ &+ b_0 f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) + b_i f^T \sum_{j=1}^{N} \tilde{l}_{ij}(t) a_{ij}(x_i - x_j) \\ &+ b_0 f^T m_{ii}(x_i - x_0) + b_i f^T \tilde{l}_{im}(t) m_{ii}(x_i - x_0) \end{aligned}$$

where $\tilde{k}_i(t) = k_i(t) - k_i^*$, $\tilde{l}_{ij}(t) = l_{ij}(t) - l_i^*$ and $\tilde{l}_{im}(t) = l_{im}(t) - l_i^*$. By defining for compactness

$$\begin{aligned} B_k(t) =& \mathrm{diag}(b_1 \tilde{k}_1^T(t), \ldots, b_N \tilde{k}_N^T(t)) \\ B_l(t) =& \mathrm{diag}\Big(b_1 f^T \sum_{j=1}^{N} \tilde{l}_{1j} a_{1j}(x_1 - x_j), \ldots, b_N f^T \sum_{j=1}^{N} \tilde{l}_{Nj} a_{Nj}(x_N - x_j) \Big) \\ B_m(t) =& \mathrm{diag}\Big(b_1 f^T \tilde{l}_{1m} m_{11}(x_1 - x_0), \ldots, b_N f^T \tilde{l}_{Nm} m_{NN}(x_N - x_0) \Big) \end{aligned} \tag{4-18}$$

the closed-loop for the overall network can be written as

$$\dot{x} = (I_N \otimes A_0 + B_k(t)) x + (I_N \otimes b_0 f^T) e + B_l(t) + B_m(t) \; .$$

From the synchronization error (4-10), we obtain the error dynamics

$$\dot{e} = [(I_N \otimes A_0) + (\mathcal{B} \otimes b_0 f^T)]e + (\mathcal{B} \otimes I_n)(B_k(t)x + B_l(t) + B_m(t)). \tag{4-19}$$

The adaptive laws (4-17) arise from the Lyapunov candidate $V = V_1 + V_2 + V_3 + V_4$, where V_1 is (4-13), and

$$V_2 = \sum_{i=1}^{N} \frac{\tilde{k}_i^T(t)\gamma^{-1}\tilde{k}_i(t)}{|l_i^*|}, \quad V_3 = \sum_{i=1}^{N} \frac{\tilde{l}_{ij}(t)\gamma^{-1}\tilde{l}_{ij}^T(t)}{|l_i^*|}, \quad V_4 = \sum_{i=1}^{N} \frac{\tilde{l}_{im}(t)\gamma^{-1}\tilde{l}_{im}^T(t)}{|l_i^*|}. \tag{4-20}$$

In fact, following the same procedure as in (4-14), we have

$$\begin{aligned}
\dot{V}_1 =& 2e^T(\mathcal{B}^{-1} \otimes P)[(I_N \otimes A_0) + (\mathcal{B} \otimes b_0 f^T)]e \\
&+ 2e^T(\mathcal{B}^{-1} \otimes P)[(\mathcal{B} \otimes I_n)(B_k(t)x + B_l(t) + B_m(t))] \\
=& \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0 f^T\right) + \left(\lambda_i A_0 + b_0 f^T\right)^T P\right] \bar{e}_i \\
&+ 2\sum_{i=1}^{N} \tilde{k}_i^T(t) x_i b_i^T P e_i \\
&+ 2\sum_{i=1}^{N} \left(\sum_{j=1}^{N} \tilde{l}_{ij}(t) a_{ij}(x_i - x_j)\right)^T f b_i^T P e_i \\
&+ 2\sum_{i=1}^{N} (\tilde{l}_{im} m_{ii}(x_i - x_0))^T f b_i^T P e_i.
\end{aligned} \tag{4-21}$$

Moreover, by using (4-17) we have

$$\begin{aligned}
\dot{V}_2 &= -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \tilde{k}_i^T(t) x_i b_0^T P e_i \\
\dot{V}_3 &= -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} \left(\sum_{j=1}^{N} \tilde{l}_{ij}(t) a_{ij}(x_i - x_j)\right)^T f b_0^T P e_i \\
\dot{V}_4 &= -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|} (\tilde{l}_{im} m_{ii}(x_i - x_0))^T f b_0^T P e_i
\end{aligned}$$

leading to

$$\dot{V} = \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0 f^T\right) + \left(\lambda_i A_0 + b_0 f^T\right)^T P\right] \bar{e}_i$$

which is negative semi-definite provided that condition (4-15) holds. Using standard Lyapunov arguments we can prove boundedness of all closed-loop signals and convergence of e to 0. In fact, since $V > 0$ and $\dot{V} \leq 0$, it follows that $V(t)$ has a limit, i.e.,

$$\lim_{t\to\infty} V(e(t), \tilde{\Omega}(t)) = V_\infty < \infty \tag{4-22}$$

where $\tilde{\Omega}$ collects all parametric errors. The finite limit implies V, e, $\tilde{\Omega} \in \mathcal{L}_\infty$. In addition, by integrating $\dot{V}$ it follows that

$$\int_0^\infty e^T(\tau) Q e(\tau)\, \mathrm{d}\tau \leq V(e(0), \tilde{\Omega}(0)) - V_\infty$$

for some $Q > 0$, from which we establish that $e \in \mathcal{L}_2$. Finally, since $\dot{V}$ is uniformly continuous in time (being $\ddot{V}$ finite), the Barbalat's lemma implies $\dot{V} \to 0$ as $t \to \infty$ and hence $e \to 0$, from which we derive $x_i \to x_0$, $\forall i \in \mathcal{V}$. This concludes the proof.

Remark 4-2-1: [Advances wrt the state of the art] *Similarly to the pinning control literature [Turci et al., 2014, Gibson, 2016] and in contrast with most consensus literature, we describe the network by using the leader-follower topology matrix $\mathcal{B}$ instead of an augmented Laplacian matrix. This leads to the Lyapunov function* (4-13)*, which exploits the invertibility of $\mathcal{B}$ to remove any need for distributed observer. However, in contrast with [Gibson, 2016] (only adaptive feedback gains) or [Turci et al., 2014] (only adaptive coupling gains), here we manage to adapt both sets of gains. Finally, in contrast with [Ghapani et al., 2016], where both feedback and coupling gains are adapted with discontinuous laws, we have not only obtained a continuous protocol, but also removed any need for a distributed observer for the leader velocity.*

Remark 4-2-2: [Asymmetric weights] *While in homogeneous networks the homogeneous dynamics induce symmetry in the couplings [Li et al., 2013b, Shafi and Arcak, 2015], in the proposed approach the heterogeneous dynamics require asymmetric couplings (as illustrated by the matching conditions* (4-4)*). Therefore, the second law in* (4-17) *makes all couplings adaptively evolve so as to "balance" the heterogeneity in the network in a distributed fashion.*

Remark 4-2-3: [Synchronization error and Laplacian eigenvalues] *Two errors have been considered in synchronization problems: (i) the tracking error with the exosystem, as in [Ding, 2017, Ghapani et al., 2016]; (ii) the disagreement error with neighbors, as in [Gibson, 2016, Chen et al., 2014]. Since the former is not locally computable, a distributed observer is mandatory in all heterogeneous networks designs we are aware of. Therefore, we resorted to the latter, which is locally computable, thus avoiding any need for a distributed observer. However, this simpler design is paid in terms of requiring some information of the Laplacian eigenvalues, e.g. P and f must be such that* (4-15) *holds. It has to be remarked that, to the best of the author's knowledge, there exists no adaptive or non-adaptive protocol for heterogeneous networks based on (ii) that can get rid of the information of the Laplacian eigenvalues.*

4-3 Adaptive output synchronization

In line with [Li and Ding, 2015, Ding and Li, 2016], for the output-feedback case we consider relative degree $n - q = 1$ for systems (4-2). Two results are now given which are instrumental to solving Problem 4-1-1, in case only the outputs y_i's and y_0 are available for measurement.

Proposition 4-3-1: [Leader-follower output-feedback homogenization] Consider the homogeneous dynamics defined by the triple (A_0, b_0, c_0), or, equivalently, by the transfer function (k_0, Z_0, R_0), with A_0 and c_0 as in (4-3), and relative degree $n_0 - q_0 = 1$, where n_0 and q_0 represent the order of R_0 and Z_0, respectively. If there exist a family of vectors $h_i^* \in \mathbb{R}^{n-1}$,

$g_i^* \in \mathbb{R}^{n-1}$ and a family of scalars $c_i^*, l_i^* \in \mathbb{R}$ (with $sgn(l_i^*)$ known) such that the following matching conditions are satisfied

$$\begin{cases} (\Lambda(s) - h_i^{*T}\alpha(s))R_i - k_i Z_i(s)(g_i^{*T}\alpha(s) + c_i^*\Lambda(s)) = Z_i(s)\Lambda_0(s)R_m(s) \\ l_i^* = k_m/k_i \end{cases} \tag{4-23}$$

with

$$\begin{cases} \alpha(s) \triangleq [s^{n-2}, s^{n-3}, \ldots, s, 1] & \text{for } n \geq 2 \\ \alpha(s) \triangleq 0 & \text{for } n = 1 \end{cases} \tag{4-24}$$

and $\Lambda(s)$ is a monic Hurwitz polynomial of degree $n-1$ that contains Z_0 as a factor

$$\begin{aligned} \Lambda(s) &= \Lambda_0(s)Z_0(s) \\ &= s^{n-1} + \mu_{n-2}s^{n-2} + \mu_{n-3}s^{n-3} + \ldots + \mu_0 \end{aligned} \tag{4-25}$$

where $\Lambda_0(s)$ is to be designed. Then, there exists an ideal controller

$$u_i^* = h_i^{*T}\frac{\alpha(s)}{\Lambda(s)}u_i + g_i^{*T}\frac{\alpha(s)}{\Lambda(s)}y_i + c_i^* y_i + l_i^*\phi\left(\sum_{j=1}^{N} a_{ij}(y_i - y_j) + m_{ii}(y_i - y_0)\right) \tag{4-26}$$

with $\phi \in \mathbb{R}$ to be designed, giving the closed-loop dynamics

$$\begin{aligned} \dot{x}_i &= A_0 x_i + b_0\phi\left(\sum_{j=1}^{N} a_{ij}(y_i - y_j) + m_{ii}(y_i - y_0)\right) \\ y_i &= c_0^T x_i, \qquad i \in \mathcal{V}. \end{aligned} \tag{4-27}$$

The following result allows us to design ϕ to achieve synchronization for the homogeneous dynamics in (4-27).

Proposition 4-3-2: [Output synchronization of an homogeneous network with leader] The homogeneous network (4-27) synchronizes if

$$\left(\lambda_i A_0 + b_0 f c_0^T, b_0, c_0^T\right) \text{ is SPR}, \quad \forall i \in \mathcal{V} \tag{4-28}$$

where λ_i's, $i \in \mathcal{V}$, are the eigenvalues of the $\mathcal{B}^{-1}$ matrix.

Proof: The overall homogeneous network (4-27) can be written in the more compact form

$$\begin{aligned} \dot{x} &= (I_N \otimes A_0 + \mathcal{B} \otimes b_0\phi c_0^T)(x - x_m) \\ y &= (I_N \otimes c_0^T)x \end{aligned} \tag{4-29}$$

where $y = [y_1, y_2, \ldots, y_N]^T \in \mathbb{R}^N$. Let us now define the state and output synchronization errors as

$$\begin{aligned} e_i &= \left(\sum_{j=1}^{N} a_{ij}(x_i - x_j)\right) + m_{ii}(x_i - x_0) \\ \epsilon_i &= \left(\sum_{j=1}^{N} a_{ij}(y_i - y_j)\right) + m_{ii}(y_i - y_0) \end{aligned} \tag{4-30}$$

with $e = [e_1^T, e_2^T, \dots, e_N^T]^T$ and $\epsilon = [\epsilon_1, \epsilon_2, \dots, \epsilon_N]^T$. The overall homogeneous network can be now written as

$$\begin{aligned} \dot{x} &= (I_N \otimes A_0)x + (I_n \otimes b_0 \phi c_0^T)e \\ y &= (I_N \otimes c_0^T)x. \end{aligned} \tag{4-31}$$

Recalling that $e = (\mathcal{B} \otimes I_n)(x - x_m)$, the error dynamics result in

$$\begin{aligned} \dot{e} &= (\mathcal{B} \otimes I_n)[(I_N \otimes A_0)x + (I_N \otimes b_0 \phi c_0^T)e - (I_N \otimes A_0)x_m] \\ &= [(I_N \otimes A_0) + (\mathcal{B} \otimes b_0 \phi c_0^T)]e. \end{aligned}$$

Now, let us use a similar decomposition as in Proposition 2-3-1 and consider the Lyapunov candidate

$$\Upsilon_1 = e^T(\mathcal{B}^{-1} \otimes P)e \tag{4-32}$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix satisfying the MKY lemma [Ioannou and Sun, 2012, Lemma 3.5.4]

$$\begin{aligned} &P\left(\lambda_i A_0 + b_0 \phi c_0^T\right) + \left(\lambda_i A_0 + b_0 \phi c_0^T\right)^T P < -Q \\ &P b_0 = c_0, \qquad \forall i \in \mathcal{V}. \end{aligned} \tag{4-33}$$

Then, we have

$$\dot{\Upsilon}_1 = \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0 \phi c_0^T\right) + \left(\lambda_i A_0 + b_0 \phi c_0^T\right)^T P\right] \bar{e}_i$$

which is negative definite if

$$\left[P\left(\lambda_i A_0 + b_0 \phi c_0^T\right) + \left(\lambda_i A_0 + b_0 \phi c_0^T\right)^T P\right] < \mathbf{0},\ \forall i \in \mathcal{V} \tag{4-34}$$

implied by the first in (4-33). This completes the proof.

4-3-1 Main result

The following edge-based output synchronizing protocol is proposed

$$u_i(t) = h_i^T(t)\frac{\alpha(s)}{\Lambda(s)}u_i + g_i^T(t)\frac{\alpha(s)}{\Lambda(s)}y_i + c_i(t)y_i + \phi\left(\sum_{j=1}^{N} l_{ij}(t)a_{ij}(y_i - y_j) + l_{im}(t)m_{ii}(y_i - y_0)\right) \tag{4-35}$$

where h_i, g_i, and c_i are the estimates of h_i^*, g_i^* and c_i^*, respectively, while l_{ij} and l_{im} are the edge-based estimates of l_i^*. The following synchronization result holds.

Theorem 4-3-1: [Heterogeneous network output synchronization] *Under Assumption 4-1-1, the heterogeneous uncertain network* (4-1)*, controlled using the following distributed adaptive* *controller*

$$
\begin{aligned}
&u_i(t) = \theta_i^T(t)\omega_i, \qquad \dot{\theta}_i = -sgn(l_i^*)\gamma\epsilon_i\omega_i \\
&\dot{\omega}_{i_1} = F\omega_{i_1} + du_i, \qquad \dot{\omega}_{i_2} = F\omega_{i_2} + dy_i \\
&\theta_i = \begin{cases} \left[h_i^T \; g_i^T \; c_i \; [l_{ij}]_{j\in\mathcal{N}_i} \; l_{im}\right]^T & \text{if } i \in \mathcal{T} \\ \left[h_i^T \; g_i^T \; c_i \; [l_{ij}]_{j\in\mathcal{N}_i}\right]^T & \text{otherwise} \end{cases} \\
&\omega_i = \begin{cases} \left[\omega_{i_1}^T \; \omega_{i_2}^T \; y_i \; [\phi(y_i - y_j)]_{j\in\mathcal{N}_i} \; \phi(y_i - y_0)\right]^T & \text{if } i \in \mathcal{T} \\ \left[\omega_{i_1}^T \; \omega_{i_2}^T \; y_i \; [\phi(y_i - y_j)]_{j\in\mathcal{N}_i}\right]^T & \text{otherwise} \end{cases} \\
&F = \begin{bmatrix} -\mu_{n-2} & -\mu_{n-3} & \cdots & -\mu_0 \\ & I_{n-2} & & 0_{(n-2)\times 1} \end{bmatrix}, \; d = \begin{bmatrix} 1 \\ 0_{(n-2)\times 1} \end{bmatrix}
\end{aligned}
\tag{4-36}
$$

with adaptive gain $\gamma > 0$*, reaches synchronization to the reference output* y_0*, provided that the scalar* ϕ *is chosen such that condition* (4-28) *holds. The notation* $[v]_{j\in\mathcal{N}_i}$ *is used to indicate row vectors that collect all the components associated to the neighbors of system* i*. Please notice that* u_i *in* (4-36) *is equivalent to* (4-35)*, as* (F, d) *is a state-space realization of* $\alpha(s)/\Lambda(s)$*.*

Proof: The proof follows very similar steps as the one of Theorem 4-2-1. The Lyapunov candidate Υ_1 in (4-32) should be used together with

$$
\Upsilon_2 = \sum_{i=1}^{N} \frac{\tilde{\theta}_i^T(t)\gamma^{-1}\tilde{\theta}_i(t)}{|l_i^*|}. \tag{4-37}
$$

Then, similar with (4-21), we have

$$
\begin{aligned}
\dot{\Upsilon}_1 =& 2e^T(\mathcal{B}^{-1} \otimes P)[(I_N \otimes A_0) + (\mathcal{B} \otimes b_0\phi c_0^T)]e \\
&+ 2e^T(\mathcal{B}^{-1} \otimes P)[(\mathcal{B} \otimes I_n)(B_\theta(t)\omega)]
\end{aligned}
\tag{4-38}
$$

where

$$
\begin{aligned}
B_\theta(t) &= \text{diag}(b_1\tilde{\theta}_1^T(t), \ldots, b_N\tilde{\theta}_N^T(t)) \\
\omega &= [\omega_1^T, \omega_2^T, \ldots, \omega_N^T]^T
\end{aligned}
\tag{4-39}
$$

and, following a similar procedure as in (4-21), we obtain

$$
\begin{aligned}
\dot{\Upsilon}_1 =& \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0\phi c_0^T\right) + \left(\lambda_i A_0 + b_0\phi c_0^T\right)^T P\right]\bar{e}_i \\
&+ 2\sum_{i=1}^{N} e_i^T P b_i \tilde{\theta}_i^T(t)\omega_i \\
=& \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0\phi c_0^T\right) + \left(\lambda_i A_0 + b_0\phi c_0^T\right)^T P\right]\bar{e}_i \\
&+ 2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|}\tilde{\theta}_i^T(t)\omega_i\epsilon_i
\end{aligned}
$$

where we have used the second equation in (4-33). Moreover, from (4-36) we have

$$\dot{\Upsilon}_2 = -2\sum_{i=1}^{N} \frac{sgn(l_i^*)}{|l_i^*|}\tilde{\theta}_i^T(t)\omega_i\epsilon_i$$

leading to

$$\dot{\Upsilon} = \sum_{i=1}^{N} \bar{e}_i^T \left[P\left(\lambda_i A_0 + b_0\phi c_0^T\right) + \left(\lambda_i A_0 + b_0\phi c_0^T\right)^T P\right]\bar{e}_i$$

which is negative semi-definite provided that (4-34) holds. Using standard Lyapunov arguments as in Theorem 4-2-1 we can prove boundedness of all closed-loop signals and convergence of e to 0, from which we derive $\epsilon \to 0$, i.e. $y_i \to y_0$, $\forall i \in \mathcal{V}$. This concludes the proof.

4-4 Numerical examples

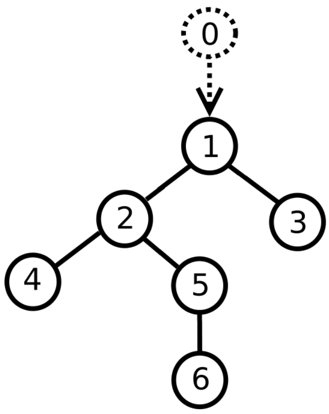

Figure 4-1: The undirected communication graph.

Simulations using controllers (4-16)-(4-17) and (4-36) are carried out on the graph of Figure 4-1, where system 0 is the leader node and system 1 is the only target node. The heterogeneous systems (4-1) are taken as second-order linear systems with relative degree equal to one

$$\begin{aligned} \dot{x}_i &= \underbrace{\begin{bmatrix} 0 & 1 \\ -d_{2_i} & -d_{1_i} \end{bmatrix}}_{A_i} x_i + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_i} u_i \\ y_i &= \underbrace{\begin{bmatrix} n_{2_i} & n_{1_i} \end{bmatrix}}_{c_i^T} x_i \end{aligned} \tag{4-40}$$

where the controllable canonical form is required only for the state-feedback case, while the second equation in (4-40) is used only in the output-feedback case. The parameters and initial conditions for each system (unknown to the designer and used only for simulation) are reported in Table 4-1.

Table 4-1: Parameters and initial conditions for the agents

	d_{1_i}	d_{2_i}	n_{1_i}	n_{2_i}	$x_i(0)$
agent #1	0.75	2.5	0.5	1	$\begin{bmatrix}-0.25 & 1\end{bmatrix}^T$
agent #2	1	2	1	1.5	$\begin{bmatrix}0.25 & -1\end{bmatrix}^T$
agent #3	0.5	1	0.75	0.75	$\begin{bmatrix}-0.5 & 0.5\end{bmatrix}^T$
agent #4	1.25	2	1.25	1	$\begin{bmatrix}0.5 & -0.5\end{bmatrix}^T$
agent #5	1.5	1.5	1	1.25	$\begin{bmatrix}-1 & 0.25\end{bmatrix}^T$
agent #6	0.75	1	1.5	2	$\begin{bmatrix}1 & -0.25\end{bmatrix}^T$

For the *state synchronization* case, the desired homogeneous dynamics are chosen as a harmonic oscillator

$$\dot{x}_0 = \underbrace{\begin{bmatrix} 0 & 1 \\ -(0.8^2) & 0 \end{bmatrix}}_{A_0} x_0 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_0} u \,, \qquad x_0(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The vector f and matrix P that satisfy condition (4-15) are

$$P = \begin{bmatrix} 0.4774 & 0.0641 \\ 0.0641 & 0.5681 \end{bmatrix}, \qquad f^T = \begin{bmatrix} -1 & -10 \end{bmatrix}.$$

Finally, the adaptive gain is $\gamma = 50$ and all estimated gains k_i, l_{ij} and l_{im} are initialized to 0. The resulting adaptive state synchronization is shown in Figure 4-2, with adaptive gains shown in Figure 4-3.

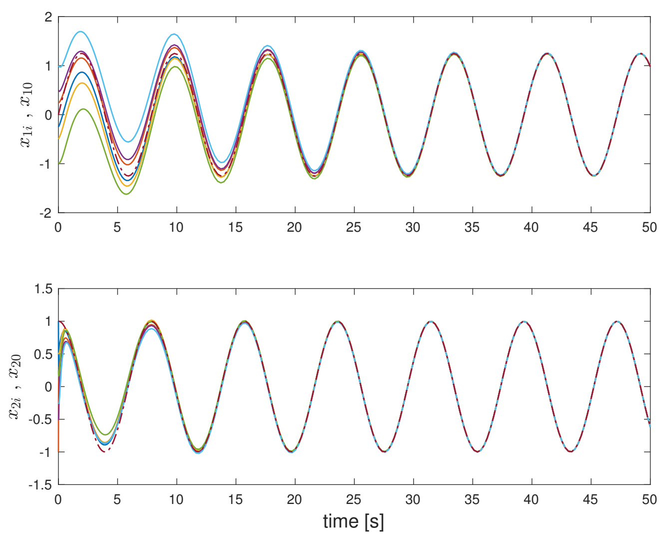

Figure 4-2: Synchronization of the states of each system to the leader reference state using (4-16) and (4-17).

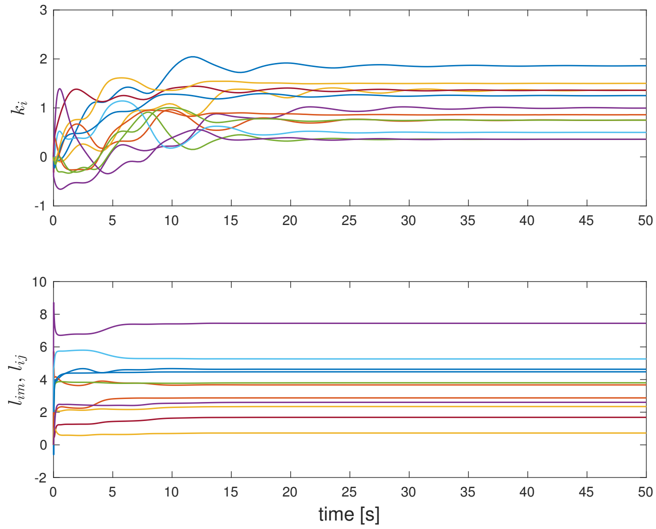

Figure 4-3: Adaptive gains resulting from (4-17).

For the *output synchronization* case, the same parameters and initial conditions as in Table 4-1 are taken. The desired homogeneous dynamics are chosen again as a harmonic oscillator

$$\dot{x}_0 = \underbrace{\begin{bmatrix} 0 & 1 \\ -(0.8^2) & 0 \end{bmatrix}}_{A_0} x_0 + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_0} u$$

$$y_0 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{c_0^T} x_0, \qquad x_0(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

that in transfer function form is $(s+0.64)/(s^2+0.64)$. Therefore we have $F = -0.64$ and $d = 1$. The scalar ϕ that satisfies condition (4-28) is $\phi = -1$. The adaptive gain is taken $\gamma = 50$ and all estimated gains θ_i are initialized to 0. The resulting adaptive output synchronization is shown in Figure 4-4 together with the adaptive gains.

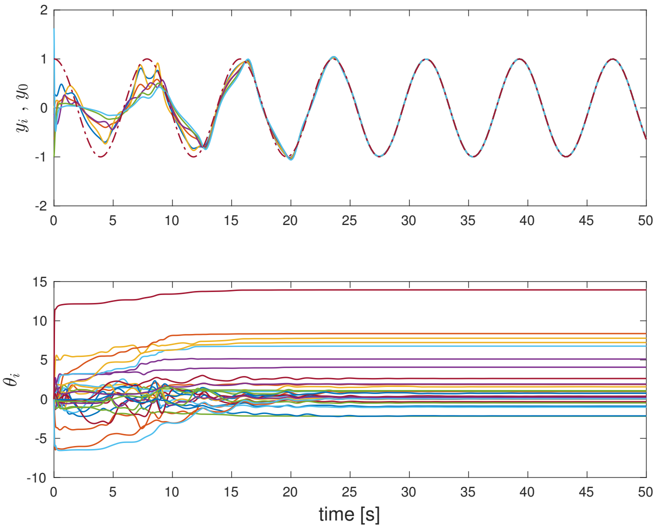

Figure 4-4: Synchronization of the outputs of each system to the leader reference output using (4-36), and corresponding adaptive gains.

4-5 Concluding remarks

The protocols presented in this chapter completely answer the research questions we presented at the beginning of this thesis, by improving on the results achieved in Chapter 3: not only we do not require a distributed observer architecture, but we do not need any extra local communication more than the agents' states or outputs.

Now, since a big part of the synchronization literature treats coupled nonlinear oscillators, due to their applicability to many real-life phenomena, it would be of interest to see if the proposed protocols could be extended in that direction. In particular, in the next chapter we will treat Kuramoto-like networks.

Chapter 5

Synchronization over Kuramoto-like heterogeneous uncertain networks

In this chapter we deal with an heterogeneous network whose units are modeled as Kuramoto-like systems. Considering a leaderless scenario, we will expand the approach presented in Chapter 3. The main challenge with respect to the previous results lie in the fact that these systems are nonlinear and dynamically coupled.

5-1 Introduction: the Kuramoto-like model

First of all, let us define the *weighted adjacency matrix*, whose entries will indicate the interaction strenght among neighboring oscillators. The adjacency matrix of a weighted undirected graph $\mathcal{K} = [k_{ij}]$ is defined as $k_{ii} = 0$ and $k_{ij} = k_{ji} > 0$ if $(i, j) \in \mathcal{E}$, where $i \neq j$.

Let us consider a network of N oscillators, each characterized by a phase angle θ_i and a natural rotation frequency ω_i. The dynamics of each isolated oscillator would be

$$\dot{\theta}_i = \omega_i, \qquad i \in \mathcal{V} \triangleq \{1, 2, \dots, N\}. \tag{5-1}$$

The interaction topology and coupling strenghts among the oscillators can be modeled by a connected, undirected, and *weighted* graph. Considering the simplest 2π-periodic interaction function between neighboring oscillators, the overall model of coupled phase oscillators reads as

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} k_{ij} \sin(\theta_i - \theta_j), \qquad i \in \mathcal{V}. \tag{5-2}$$

Kuramoto showed that, if we consider a complete interaction graph (where each agent is connected to all the other agents in the network) and uniform weights $k_{ij} = K/N$, obtaining the coupled phase oscillator dynamics

$$\dot{\theta}_i = \omega_i - \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j), \qquad i \in \mathcal{V}, \tag{5-3}$$

then, synchronization occurs if the coupling gain K exceeds a certain threshold $K_{critical}$, function of the distribution of the natural frequencies.

The dynamics (5-3) are nowadays known as the *Kuramoto model* of coupled oscillators, and Kuramoto's original work initiated a broad stream of research.

In this chapter, we will consider a more general (second-order) formulation of coupled phase oscillators model, that is

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \tau_i - \sum_{j=1}^{N} k_{ij}\sin(\theta_i - \theta_j), \qquad i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\}. \tag{5-4}$$

The model (5-4) is referred to as the *coupled oscillators model with inertia*, or *Kuramoto model with inertia*, or *Kuramoto-like model.*

The meaning of the parameters in (5-4) can be examined via the mechanical analogy of mass points in Figure 5-1. After neglecting any collisions, each point, or agent, will move on the

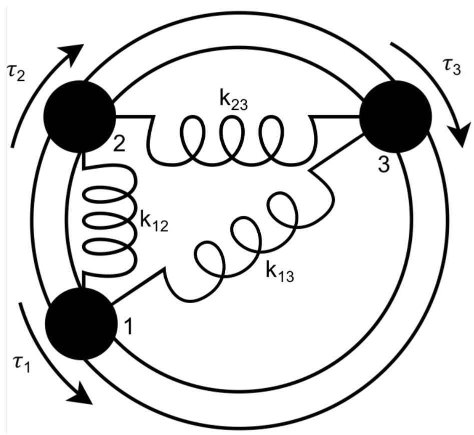

Figure 5-1: Mechanical analogy of a network of three coupled oscillators.

circle describing an angle (or phase, by analogy) θ_i and an angular velocity (or frequency, by analogy) $\dot{\theta}_i$, under the effect of an external driving torque τ_i, an elastic restoring torque $k_{ij}\sin(\theta_i - \theta_j)$ (with $k_{ij} = k_{ji}$), and a viscous damping torque $d_i\dot{\theta}_i$ that is opposite to the direction of motion. All inertial coefficients m_i, damping coefficients d_i and stiffness coefficients k_{ij} have positive values.

In the limit of small masses m_i and uniformly-high viscous damping $D = d_i$, that is $m_i/D \approx 0$, we recover the coupled oscillator dynamics (5-2) from its mechanical analog (5-4) with natural rotation frequencies $\omega_i = \tau_i/D$ and with coupling strenghts k_{ij}/D.

5-2 Problem formulation

The network (5-4) of heterogeneous coupled oscillators with unknown dynamics is considered. All coefficients m_i, d_i, and k_{ij}, have positive but *unknown* value. The external driving torque has two components

$$\tau_i = \omega_i + u_i, \qquad i \in \mathcal{V} \tag{5-5}$$

where u_i is the actual control torque and ω_i is a constant term which is proportional to the natural angular velocity (or frequency, by analogy) of the agent i. After defining the state $x_i = [\theta_i, \dot{\theta}_i]^T$, (5-4) can be rewritten as

$$\dot{x}_i = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{d_i}{m_i} \end{bmatrix}}_{A_i} x_i + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix}}_{b_i} \left(u_i + \omega_i - \sum_{j=1}^{N} k_{ij} \sin(\theta_i - \theta_j) \right). \tag{5-6}$$

The following connectivity assumption is made.

Assumption 5-2-1: *The graph $\mathcal{G}$ of the network is undirected and connected.*

Problem 5-2-1: [Adaptive synchronization] *Consider a network of unknown oscillators* (5-4) *satisfying Assumption 5-2-1. Find a distributed strategy (i.e. exploiting only measurements from neighbors) for the control inputs u_i such that, without any knowledge of the parameters m_i, d_i and k_{ij}, the network synchronizes to the same behavior (i.e. $x_i - x_j \to 0$, $\forall i, j$ in the full-state measurement case.*

5-3 Adaptive state synchronization

Two results are now given which are instrumental to solving Problem 5-2-1.

Proposition 5-3-1: [Homogenization via reference model] For the following reference model

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ a_{21}^* & a_{22}^* \end{bmatrix}}_{A_0} x_m + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m^*} \end{bmatrix}}_{b_0} u \tag{5-7}$$

with $x_m \in \mathbb{R}^2$, there exist a family of vectors $k_i^* \in \mathbb{R}^2$ and a family of scalars $l_i^* > 0$ such that

$$\begin{cases} A_i + b_i k_i^{*T} = A_0 \\ l_i^* b_i = b_0 \end{cases} \tag{5-8}$$

Furthermore, there exist ideal controllers

$$u_i^* = k_i^{*T} x_i + l_i^* f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) + c_i^* + \sum_{j=1}^{N} g_{ij}^* a_{ij} \sin(\theta_i - \theta_j) \tag{5-9}$$

with $c_i^* = -\omega_i$, $g_{ij}^* = k_{ij}$ and $f \in \mathbb{R}^2$ to be designed, which lead to the following dynamics

$$\dot{x}_i = A_0 x_i + b_0 f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j), \qquad i \in \mathcal{V} \tag{5-10}$$

Proof: The proof directly follows from applying the control input (5-9) to agent (5-6), and using (5-8).

The following result, allows us to design f to achieve synchronization for the homogeneous dynamics in (5-10).

Proposition 5-3-2: [Homogeneous network synchronization] The homogeneous network (5-10) synchronizes if

$$A_0 + \lambda_i b_0 f^T \text{ is Hurwitz}, \quad \forall i \in \mathcal{V}/\{1\} \tag{5-11}$$

where λ_i's, $i \in \mathcal{V}/\{1\}$, are the non-zero eigenvalues of the Laplacian, or equivalently if

$$P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P < \mathbf{0}, \; \forall i \in \mathcal{V}/\{1\} \tag{5-12}$$

where $P \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix.

Proof: The proof was already investigated after Proposition 2-3-1.

The aim of the adaptation mechanism in the following section is to make the heterogeneous network converge to the behavior of the homogeneous network in Proposition 5-3-2, estimating the unknown gains by exploiting only measurements from neighbors.

5-3-1 Main result

The following synchronizing protocol is proposed

$$u_i(t) = k_i^T(t)x_i + l_i(t)f^T \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + c_i(t) + \sum_{j=1}^{N} g_{ij}(t)a_{ij}\sin(\theta_i(t) - \theta_j(t)) \tag{5-13}$$

where k_i, l_i, c_i, g_{ij}, are the (time-dependent) estimates of k_i^*, l_i^*, c_i^*, g_{ij}^*, respectively. The following synchronization result holds.

Theorem 5-3-1: *Under Assumption 5-2-1, the heterogeneous Kuramoto network* (5-6)*, controlled using the synchronizing protocol* (5-13) *and the following adaptive laws*

$$\begin{aligned}
\dot{k}_i^T &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T P b_0 x_i^T \\
\dot{l}_i &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T P b_0 f^T e_i \\
\dot{c}_i &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T P b_0 \\
\dot{g}_{ij} &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)^T P b_0 \sin(\theta_i - \theta_j)
\end{aligned} \tag{5-14}$$

with adaptive gain $\gamma > 0$, reaches synchronization provided that the matrix P and the vector f are chosen such that condition (5-12) *holds.*

Proof: The closed-loop network formed by (5-6) and (5-13) is given by

$$\dot{x}_i = (A_i + b_i k_i^T)x_i + l_i b_i f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) + b_i c_i + b_i \sum_{j=1}^{N} g_{ij} a_{ij} \sin(\theta_i - \theta_j) \tag{5-15}$$

which can be rewritten as a function of the estimation errors,

$$\dot{x}_i = (A_0 + b_i \tilde{k}_i^T(t))x_i + (b_0 + \tilde{l}_i(t) b_i) f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) + b_i \tilde{c}_i(t) + b_i \sum_{j=1}^{N} \tilde{g}_{ij}(t) a_{ij} \sin(\theta_i - \theta_j) \tag{5-16}$$

where $\tilde{k}_i(t) = k_i(t) - k_i^*$, $\tilde{l}_i(t) = l_i(t) - l_i^*$, $\tilde{c}_i(t) = c_i(t) - c_i^*$, $\tilde{g}_{ij}(t) = g_{ij}(t) - g_{ij}^*$. By defining for compactness

$$\begin{aligned} B_k(t) &= \text{diag}(b_1 \tilde{k}_1^T(t), \ldots, b_N \tilde{k}_N^T(t)) \\ B_l(t) &= \text{diag}(\tilde{l}_1(t) b_1 f^T, \ldots, \tilde{l}_N(t) b_N f^T) \\ B_c(t) &= \text{diag}(b_1 \tilde{c}_1(t), \ldots, b_N \tilde{c}_N(t)) \\ B_g(t) &= \text{diag}(b_1 \sum_{j=1}^{N} \tilde{g}_{1j}(t) a_{1j} \sin(\theta_1 - \theta_j), \ldots, b_N \sum_{j=1}^{N} \tilde{g}_{Nj}(t) a_{Nj} \sin(\theta_N - \theta_j)) \end{aligned} \tag{5-17}$$

the closed-loop for the overall network can be written as

$$\dot{x} = (I_N \otimes A_0 + B_k(t))x + (I_N \otimes b_0 f^T + B_l(t))e + B_c(t) + B_g(t). \tag{5-18}$$

Recalling that the synchronization error is $e = (\mathcal{L} \otimes I_2)x$, the error dynamics are

$$\dot{e} = [(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e + (\mathcal{L} \otimes I_2)(B_k(t)x + B_l(t)e + B_c(t) + B_g(t)). \tag{5-19}$$

The adaptive laws (5-14) arise from considering the Lyapunov function candidate $V = V_1 + V_2 + V_3 + V_4 + V_5$, where

$$\begin{aligned} &V_1 = e^T (I_N \otimes P) e \\ &V_2 = \sum_{i=1}^{N} \frac{\tilde{k}_i^T(t) \gamma^{-1} \tilde{k}_i(t)}{|l_i^*|} \qquad V_3 = \sum_{i=1}^{N} \frac{\tilde{l}_i(t) \gamma^{-1} \tilde{l}_i^T(t)}{|l_i^*|} \\ &V_4 = \sum_{i=1}^{N} \frac{\tilde{c}_i(t) \gamma^{-1} \tilde{c}_i^T(t)}{|l_i^*|} \qquad V_5 = \sum_{i=1}^{N} \frac{\tilde{g}_{ij}(t) \gamma^{-1} \tilde{g}_{ij}^T(t)}{|l_i^*|}. \end{aligned} \tag{5-20}$$

Then we have

$$\begin{aligned} \dot{V}_1 =& [2e^T (I_N \otimes P)]\dot{e} \\ =& 2e^T (I_N \otimes P)[(I_N \otimes A_0) + (\mathcal{L} \otimes b_0 f^T)]e + 2e^T (I_N \otimes P)[(\mathcal{L} \otimes I_2)(B_k x + B_l e + B_c + B_g)] \end{aligned} \tag{5-21}$$

and following the same procedure as in (2-57):

$$\begin{aligned}\dot{V}_1 =& \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P] \bar{e}_i + \\ &+ 2\sum_{i=1}^{N} \tilde{k}_i^T(t) x_i b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) + \\ &+ 2\sum_{i=1}^{N} \tilde{l}_i(t) e_i^T f b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) + \\ &+ 2\sum_{i=1}^{N} \tilde{c}_i(t) b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) + \\ &+ 2\sum_{i=1}^{N} \left(\sum_{j=1}^{N} \tilde{g}_{ij}(t) \sin(\theta_i - \theta_j) \right) b_i^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)\end{aligned} \tag{5-22}$$

Moreover, by using (5-14) we have:

$$\begin{aligned}\dot{V}_2 &= -2\sum_{i=1}^{N} \frac{1}{|l_i^*|} \tilde{k}_i^T(t) x_i b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) \\ \dot{V}_3 &= -2\sum_{i=1}^{N} \frac{1}{|l_i^*|} \tilde{l}_i(t) e_i^T f b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) \\ \dot{V}_4 &= -2\sum_{i=1}^{N} \frac{1}{|l_i^*|} \tilde{c}_i(t) b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right) + \\ \dot{V}_5 &= -2\sum_{i=1}^{N} \frac{1}{|l_i^*|} \left(\sum_{j=1}^{N} \tilde{g}_{ij}(t) \sin(\theta_i - \theta_j) \right) b_0^T P \left(\sum_{j=1}^{N} a_{ij}(e_i - e_j) \right)\end{aligned} \tag{5-23}$$

leading to:

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5 \\ &= \sum_{i=2}^{N} \bar{e}_i^T [P(A_0 + \lambda_i b_0 f^T) + (A_0 + \lambda_i b_0 f^T)^T P] \bar{e}_i\end{aligned} \tag{5-24}$$

which is negative semi-definite provided that condition (5-12) holds. Using standard Lyapunov arguments we can prove boundedness of all closed-loop signals and convergence of e to 0. In fact, since $V > 0$ and $\dot{V} \leq 0$, it follows that $V(t)$ has a limit, i.e.,

$$\lim_{t\to\infty} V(e(t), \tilde{\Omega}(t)) = V_\infty < \infty \tag{5-25}$$

where we have collected all parametric errors in $\tilde{\Omega}$. The finite limit implies V, e, $\tilde{\Omega} \in \mathcal{L}_\infty$. In addition, by integrating $\dot{V}$ it follows that for some $Q > 0$

$$\int_0^\infty e^T(\tau) Q e(\tau)\, \mathrm{d}\tau \leq V(e(0), \tilde{\Omega}(0)) - V_\infty \tag{5-26}$$

from which we establish that $e \in \mathcal{L}_2$. Finally, since $\dot{V}$ is uniformly continuous in time (this is satisfied because $\ddot{V}$ is finite), the Barbalat's lemma implies $\dot{V} \to 0$ as $t \to \infty$ and hence $e \to 0$, from which we derive $x_i \to x_j$, $\forall i, j$. This concludes the proof.

Remark 5-3-1: [Differences with distributed observer-based architectures] *In order to implement* (5-14)*, and in particular the term* $\sum_{j=1}^{N} a_{ij}(e_i - e_j)$*, it is required to communicate among neighbors the extra variable* e_i*, which is also local information. Communication of extra local variables is often at the core of many synchronization protocols: for example, synchronization based on distributed observer [Lu and Liu, 2017, Cai et al., 2017] requires communication of extra local variables representing the observer states. For comparison purposes, let us consider the same synchronizing protocol* (5-13)*, but this time with the following adaptive version of the distributed observer*

$$\begin{aligned}
\dot{\chi}_i &= A_0 \chi_i + \mu \left(b_0 f^T \sum_{j=1}^{N} a_{ij}(\chi_i - \chi_j) \right) \\
\dot{k}_i^T &= -\gamma (x_i - \chi_i)^T P b_0 x_i^T \\
\dot{l}_i &= -\gamma (x_i - \chi_i)^T P b_0 f^T \sum_{j=1}^{N} a_{ij}(x_i - x_j) \\
\dot{c}_i &= -\gamma (x_i - \chi_i)^T P b_0 \\
\dot{g}_{ij} &= -\gamma (x_i - \chi_i)^T P b_0 \sin(\theta_i - \theta_j)
\end{aligned} \tag{5-27}$$

with adaptive gain $\gamma > 0$ *and distributed observer gain* $\mu > 0$*. Recall that the following intuition lies behind distributed observer-based protocols like* (5-27)*: a virtual homogeneous network in the form* (5-10) *can be constructed in a distributed way, having the same graph as the heterogeneous network. This is the first equation in* (5-27)*. Since Proposition 5-3-2 guarantees synchronization of the virtual homogenous network, the adaptation laws in* (5-27) *can now force each agent in the heterogeneous Kuramoto network to behave as its corresponding agent in the homogeneous network* ($x_i - \chi_i \to 0$)*, therefore also achieving synchronization. Protocol* (5-27) *resembles, with minor modifications, the synchronization protocol adopted in literature for the so-called Euler-Lagrange agents [Feng et al., 2016, Mei et al., 2015]. Now, comparing* (5-27) *with* (5-14)*, we see that the proposed disagreement-based protocol is essentially simpler, because it does not require to construct in a distributed manner the observer variables* χ_i*.*

5-4 Adaptive frequency synchronization

The previous section presented a full-state synchronization: however, in some applications it is of interest to synchronize the only frequency, while the phase may not synchronize. One possibility to achieve this via (5-14) is to introduce a phase error in the form $\theta_i - \theta_j = h(\omega_i - \omega_j)$, with $h > 0$ a design parameter: this resembles the idea of velocity-dependent time headway in platooning [Harfouch et al., 2017]. A simpler alternative is to formulate an output synchronizing protocol as follows

$$u_i(t) = k_i(t)\dot{\theta}_i + l_i(t)\phi \sum_{j=1}^{N} a_{ij}(\dot{\theta}_i(t) - \dot{\theta}_j(t)) + c_i(t) + \sum_{j=1}^{N} g_{ij}(t) a_{ij} \sin(\theta_i(t) - \theta_j(t)) \tag{5-28}$$

where ϕ is to be designed, and k_i, l_i are the estimates of the scalar ideal gains in

$$\begin{cases} -\dfrac{d_i}{m_i} + \dfrac{1}{m_i} k_i^* = \alpha_0 \\ \quad\quad l_i^* \dfrac{1}{m_i} = \beta_0 \end{cases} \tag{5-29}$$

and c_i, g_{ij} are the estimates of c_i^*, g_{ij}^* as before. The following synchronization result holds.

5-4-1 Main result

Theorem 5-4-1: *Define the error*

$$\varepsilon_i = \sum_{j=1}^{N} a_{ij}(\dot{\theta}_i - \dot{\theta}_j), \qquad \varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N]^T \tag{5-30}$$

Under Assumption 5-2-1, the heterogeneous Kuramoto network (5-6)*, controlled using the synchronizing protocol* (5-28) *and the following adaptive laws*

$$\begin{aligned} \dot{k}_i &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(\varepsilon_i - \varepsilon_j) \right) p\beta_0 \dot{\theta}_i \\ \dot{l}_i &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(\varepsilon_i - \varepsilon_j) \right) p\beta_0 \phi \varepsilon_i \\ \dot{c}_i &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(\varepsilon_i - \varepsilon_j) \right) p\beta_0 \\ \dot{g}_{ij} &= -\gamma \left(\sum_{j=1}^{N} a_{ij}(\varepsilon_i - \varepsilon_j) \right) p\beta_0 \sin(\theta_i - \theta_j) \end{aligned} \tag{5-31}$$

with adaptive gain $\gamma > 0$, reaches frequency synchronization ($\dot{\theta}_i - \dot{\theta}_j \to 0$, $\forall i, j$) provided that the scalars p and ϕ are chosen such that

$$p(\alpha_0 + \lambda_i \beta_0 \phi) + (\alpha_0 + \lambda_i \beta_0 \phi)^T p < 0, \ \forall i \in \mathcal{V}/\{1\} \tag{5-32}$$

holds.

Proof: The proof follows from noticing that the transfer function between $\dot{\theta}_i$ and u in (3-3) is a first order stable filter. Therefore, one can choose a homogeneous structure, and consequently a disagreement-based error as already done in Section 3-3-1. One derives convergence of ε to zero, which implies frequency synchronization $\dot{\theta}_i - \dot{\theta}_j \to 0$, $\forall i, j$, but not necessarily phase synchronization.

5-5 Numerical examples

Simulations using protocol (5-14) are carried out in the following, considering the weighted graph shown in Figure 5-2. Furthermore, the protocol (5-27) is used for comparison with a distributed observer approach. The parameters and initial conditions for each heterogeneous Kuramoto agent (5-6) are reported in Table 5-1. Please recall that the agent parameters are unknown to the designer, i.e. the values of Table 5-1 are used for simulations but not for control design.

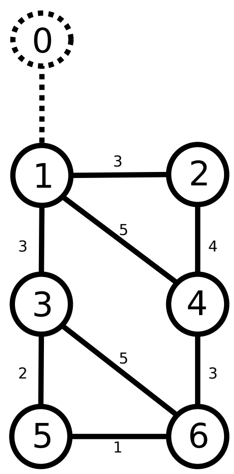

Figure 5-2: The undirected weighted communication graph.

Table 5-1: Parameters and initial conditions for the Kuramoto agents

	m_i	d_i	ω_i	$\theta_i(0)$	$\dot{\theta}_i(0)$
agent #1	1.1	0.1	5	0	0.6
agent #2	1.3	0.15	10	π	0.5
agent #3	1.2	0.2	15	$\pi/2$	0.4
agent #4	1.8	0.21	20	$(5/4)\pi$	0.3
agent #5	1.5	0.25	25	$\pi/4$	0.2
agent #6	1	0.3	30	$(3/2)\pi$	0.1

The reference model is chosen as

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_0} x_m + \underbrace{\begin{bmatrix} 0 \\ 0.8 \end{bmatrix}}_{b_0} u \,, \qquad x_m = \begin{bmatrix} \theta_m \\ \dot{\theta}_m \end{bmatrix} \tag{5-33}$$

which also represents agent 0 in Figure 5-2, with chosen initial conditions $x_m(0) = [0, 1]$. For the distributed observer protocol (5-27), we also need the virtual network of homogeneous agents with dynamics as in (5-33) and initial conditions (leader included) $[0, 1]$, $[(3/2\pi), 0.1]$, $[\pi/4, 0.2]$, $[(5/4)\pi, 0.3]$, $[\pi/2, 0.4]$, $[\pi, 0.5]$, $[0, 0.6]$. The vector f and the matrix P are taken as

$$P = \begin{bmatrix} 1.5824 & 0.5824 \\ 0.5824 & 1.2607 \end{bmatrix}, \qquad f^T = \begin{bmatrix} -1 & -1 \end{bmatrix}. \tag{5-34}$$

which satisfy condition (5-12). Finally, the adaptive gain is taken $\gamma = 1$, the observer gain $\mu = 8$, and all estimated control gains k_i, l_i, c_i, g_{ij}, are initialized to 0.

The adaptive synchronization resulting from (5-14) is shown in Figure 5-3, while the adaptive synchronization resulting from (5-27) is shown in Figure 5-4 (for the virtual homogeneous network) and in Figure 5-5 (for the actual heterogeneous network). Synchronization is achieved in both cases and, due to heterogeneity, notice that each agent has different control inputs u_i that reach different steady-state values.

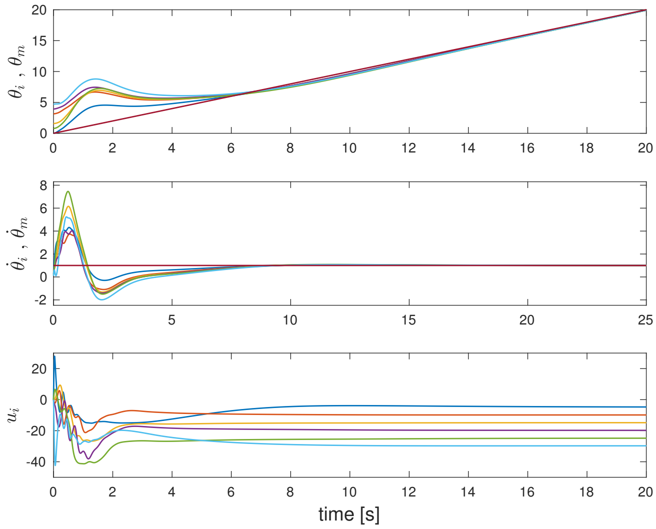

Figure 5-3: Protocol (5-14): synchronization of the states of each agent i to the leader reference state $[\theta_m, \dot{\theta}_m]$. The control inputs u_i are shown at the bottom.

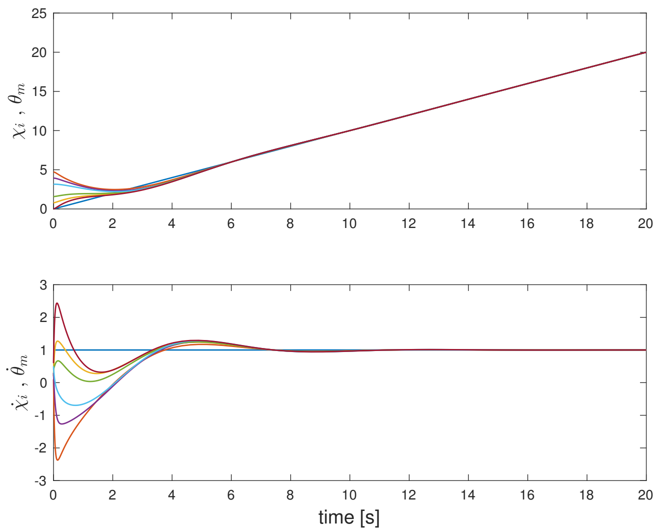

Figure 5-4: Protocol (5-27): synchronization of the observer states.

Finally, Figure 5-6 and Figure 5-7 show the adaptive control gains of (5-14) and (5-27) respectively, for all the systems. Overall, the protocol (5-14) shows synchronization capabilities in the presence of both uncertainty and heterogeneity, and without the need to construct a distributed observer.

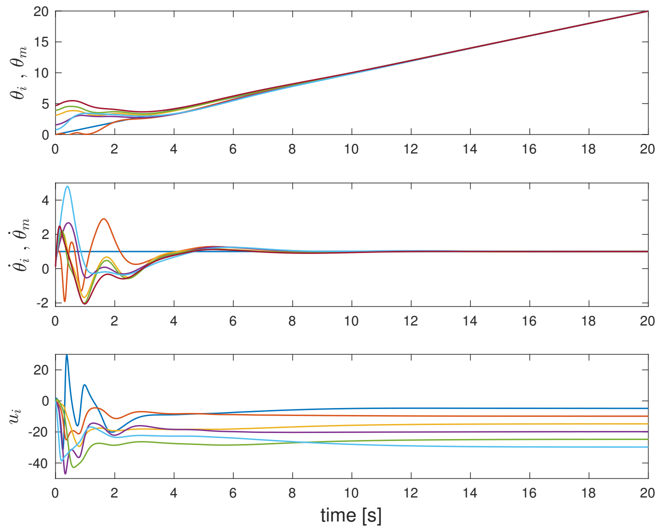

Figure 5-5: Protocol (5-27): synchronization of the states of each agent i to the reference state $[\theta_m, \dot{\theta}_m]$. The control inputs u_i are shown at the bottom.

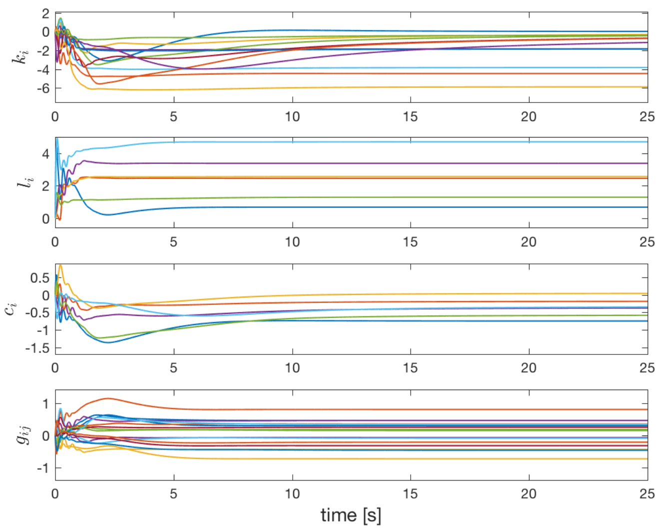

Figure 5-6: Protocol (5-14): adaptive gains.

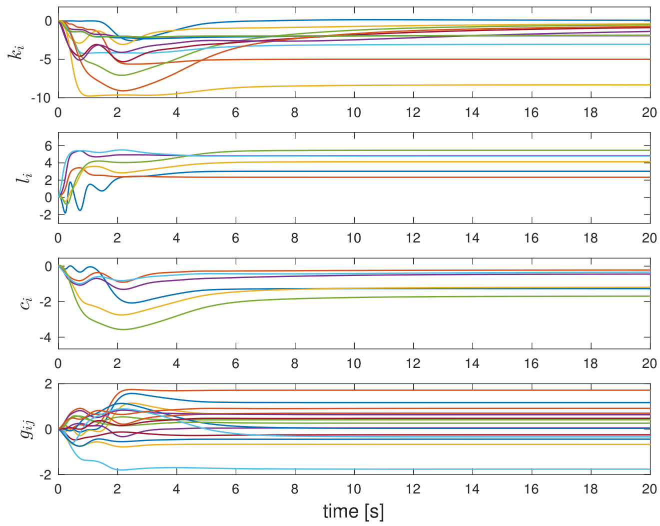

Figure 5-7: Protocol (5-27): adaptive gains.

In order to emphasize that the proposed protocols can guarantee leaderless synchronization (even though the synchronizing behavior is a priori unknown in this case), let us remove the leader agent from the network in Figure 5-2, change the initial conditions of the second states to $\dot{\theta}_i(0) = [6, 0.5, 0.4, 0.3, 0.2, 10]$, and simulate the network using the protocol (5-14). The results are shown in Figure 5-8 and Figure 5-9. Please notice that, for certain initial conditions, the a priori unknown synchronization state could also lead to instability of the single subsystems. This phenomenon was first observed by Narendra in his papers [Narendra and Harshangi, 2014, Narendra and Harshangi, 2015].

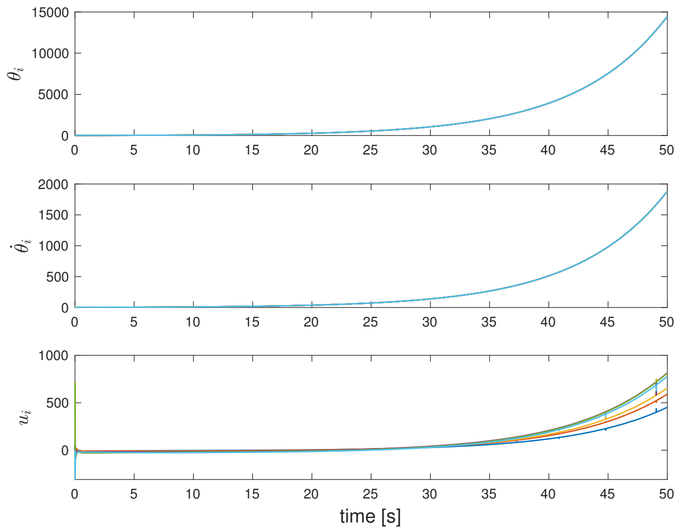

Figure 5-8: Leaderless synchronization of the states of each agent i using protocol (5-14). The control inputs u_i are shown at the bottom.

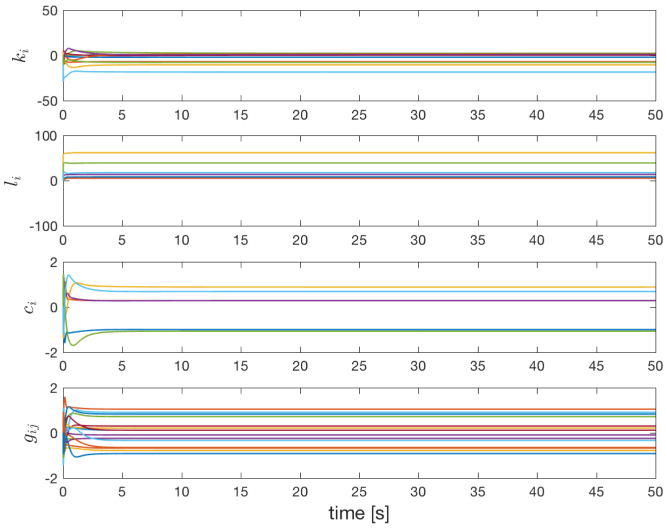

Figure 5-9: Leaderless synchronization using protocol (5-14): adaptive gains.

It is curious to notice that the adaptive gains in Figure 5-9 are converging because the system do actually synchronize, even though they synchronize to an unstable behavior.

5-6 Concluding remarks

In this chapter we extended the results obtained in Chapter 3 over heterogeneous uncertain Kuramoto-like networks. We concluded that the adaptive homogenization-based protocols presented in this thesis show great potential also in the context of synchronization of nonlinear oscillators.

The extension of the protocols of Chapter 4 to the same class of Kuramoto-like systems would be now trivial. In fact, by using the leader-follower topology matrix one would derive a more convenient Lyapunov analysis, always resulting in adaptive laws with simpler communication architecture.

Chapter 6

Conclusions

In this work we have proposed adaptive synchronization protocols for uncertain heterogeneous networks (made of linear agents or Kuramoto-like agents), based on a distributed disagreement reasoning. In particular, we first defined ideal gains (both feedback and coupling gains) that could lead all agents to a homogeneous behavior and, as a consequence, synchronization. However, since these gains were unknown in view of the unknown dynamics, we have designed adaptive laws that can guarantee synchronization. The adaptive laws are always driven by a disagreement error which is calculated among neighbors, thus the algorithms are distributed. The convergence of the synchronization error to zero is always shown via Lyapunov analysis, both for the state and the output synchronization. A distinguishing feature of this work was to achieve synchronization, in the presence of heterogeneity, without any need for a distributed observer. Therefore we have simplified the architecture by removing any local communication except from the neighbors' states (or outputs). Moreover, having adaptive in place of fixed gains is particularly relevant because the restriction of having fixed-gain control implies that synchronization can be achieved only for small parametric uncertainties. Instead, in our setting, the same adaptive distributed protocol (with the same parameters) is locally applied to each agent of the network.

Future work could go in the following directions. First, it is important to generalize the output synchronization protocol to relative degree greater than one. This should be possible by using SPR filters in the spirit of [Ioannou and Sun, 2012, Sect. 6.4]. Another relevant topic would be to study the effects of delays in the computation of the protocols, that could probably lead to bounded synchronization errors using tools as in [Lymperopoulos and Ioannou, 2016]. Another direction could be to extend the results in the switching topology scenario, which can be possibly done by using adaptive switching tools [Yuan et al., 2017, Yuan et al., 2018a, Yuan et al., 2018b]. It could be also relevant to consider networked-induced constraints [Moustakis et al., 2018, Baldi et al., 2018b]

Finally, considering Chapter 5, the proposed adaptive protocols achieve synchronization by "cancelling-out" nonlinearities in a sort of adaptive feedback linearization scheme. However, it has been shown that feedback linearization does not lead in general to optimal control inputs [Sontag, 1989]. It would be of interest to develop new adaptive protocols that, while still achieving synchronization, exploit the nonlinearities instead of cancelling them.

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